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General Fashoda Meet Theorem for Unit Circle and Square

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Summary. Here we will prove Fashoda meet theorem for the unit circle and for a square, when 4 points on the boundary are ordered cyclically. Also, the concepts of general rectangle and general circle are defined.

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The articles [8], [22], [26], [3], [4], [25], [1], [9], [2], [6], [13], [23], [19], [18], [16], [17], [11], [24], [7], [14], [15], [21], [20], [10], [5], and [12] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For all real numbers a, b, r such that $0 \leqslant r$ and $r \leqslant 1$ and $a \leqslant b$ holds $a \leqslant (1-r) \cdot a + r \cdot b$ and $(1-r) \cdot a + r \cdot b \leqslant b$.

(3) For all real numbers a, b such that $a \geqslant 0$ and $b > 0$ or $a > 0$ and $b \geqslant 0$ holds $a + b > 0$.

(4) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $a^{\mathbf{2}} \cdot b^{\mathbf{2}} \leqslant 1$.

(5) For all real numbers a, b such that $a \geqslant 0$ and $b \geqslant 0$ holds $a \cdot \sqrt{b} = \sqrt{a^{\mathbf{2}} \cdot b}$.

(6) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $(-b) \cdot \sqrt{1 + a^{\mathbf{2}}} \leqslant \sqrt{1 + b^{\mathbf{2}}}$ and $-\sqrt{1 + b^{\mathbf{2}}} \leqslant b \cdot \sqrt{1 + a^{\mathbf{2}}}$.

[1]The proposition (1) has been removed.

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(7) For all real numbers a, b such that $-1 \leqslant a$ and $a \leqslant 1$ and $-1 \leqslant b$ and $b \leqslant 1$ holds $b \cdot \sqrt{1+a^{\mathbf{2}}} \leqslant \sqrt{1+b^{\mathbf{2}}}$.

(8) For all real numbers a, b such that $a \geqslant b$ holds $a \cdot \sqrt{1+b^{\mathbf{2}}} \geqslant b \cdot \sqrt{1+a^{\mathbf{2}}}$.

(9) Let a, c, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $c \leqslant d$ and $p \in \mathcal{L}([a, c], [a, d])$, then $p_{\mathbf{1}} = a$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$.

(10) For all real numbers a, c, d and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $c < d$ and $p_{\mathbf{1}} = a$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$ holds $p \in \mathcal{L}([a, c], [a, d])$.

(11) Let a, b, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $a \leqslant b$ and $p \in \mathcal{L}([a, d], [b, d])$, then $p_{\mathbf{2}} = d$ and $a \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant b$.

(12) For all real numbers a, b and for every subset B of $\mathbb{I}$ such that $B = [a, b]$ holds B is closed.

(13) Let X be a topological structure, Y, Z be non empty topological structures, f be a map from X into Y, and g be a map from X into Z. Then $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{dom} f =$ the carrier of X and $\operatorname{dom} f = \Omega_X$.

(14) Let X be a non empty topological space and B be a non empty subset of X. Then there exists a map f from $X \restriction B$ into X such that for every point p of $X \restriction B$ holds $f(p) = p$ and f is continuous.

(15) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = r_1 - a$ and g is continuous.

(16) Let X be a non empty topological space, f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$, and a be a real number. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = a - r_1$ and g is continuous.

(17) Let X be a non empty topological space, n be a natural number, p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f(r) \cdot p$ and g is continuous.

(18) $\operatorname{SqCirc}([-1, 0]) = [-1, 0]$.

(19) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ holds $\operatorname{SqCirc}([-1, 0]) = \text{W-min}\, P$.

(20) Let X be a non empty topological space, n be a natural number, and g_1, g_2 be maps from X into $\mathcal{E}^n_{\mathrm{T}}$. Suppose g_1 is continuous and g_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = g_1(r) + g_2(r)$ and g is continuous.

(21) Let X be a non empty topological space, n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$, and f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous and f_2 is continuous. Then there exists a map g from X into $\mathcal{E}^n_{\mathrm{T}}$ such that for every point r of X holds $g(r) = f_1(r) \cdot p_1 + f_2(r) \cdot p_2$ and

g is continuous.

(22) For every function f and for every set A such that f is one-to-one and $A \subseteq \operatorname{dom} f$ holds $(f^{-1})^{\circ} f^{\circ} A = A$.

2. General Fashoda Theorem for Unit Circle

In the sequel p, p_1, p_2, p_3, q, q_1, q_2 are points of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following propositions:

(23) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_3$ and $g(I) \in K_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(24) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, C_0, K_1, K_2, K_3, K_4 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p : |p| \leqslant 1\}$ and $K_1 = \{q_1; q_1$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_1| = 1 \wedge (q_1)_{\mathbf{2}} \leqslant (q_1)_{\mathbf{1}} \wedge (q_1)_{\mathbf{2}} \geqslant -(q_1)_{\mathbf{1}}\}$ and $K_2 = \{q_2; q_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_2| = 1 \wedge (q_2)_{\mathbf{2}} \geqslant (q_2)_{\mathbf{1}} \wedge (q_2)_{\mathbf{2}} \leqslant -(q_2)_{\mathbf{1}}\}$ and $K_3 = \{q_3; q_3$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_3| = 1 \wedge (q_3)_{\mathbf{2}} \geqslant (q_3)_{\mathbf{1}} \wedge (q_3)_{\mathbf{2}} \geqslant -(q_3)_{\mathbf{1}}\}$ and $K_4 = \{q_4; q_4$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|q_4| = 1 \wedge (q_4)_{\mathbf{2}} \leqslant (q_4)_{\mathbf{1}} \wedge (q_4)_{\mathbf{2}} \leqslant -(q_4)_{\mathbf{1}}\}$ and $f(O) \in K_1$ and $f(I) \in K_2$ and $g(O) \in K_4$ and $g(I) \in K_3$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(25) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_8| \leqslant 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(26) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and $\mathrm{LE}(p_1, p_2, P)$ and $\mathrm{LE}(p_2, p_3, P)$ and $\mathrm{LE}(p_3, p_4, P)$. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$:

$|p_8| \leqslant 1\}$ and $f(0) = p_3$ and $f(1) = p_1$ and $g(0) = p_4$ and $g(1) = p_2$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(27) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and C_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p| = 1\}$ and p_1, p_2, p_3, p_4 are in this order on P. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $C_0 = \{p_8; p_8$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_8| \leqslant 1\}$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\operatorname{rng} f \subseteq C_0$ and $\operatorname{rng} g \subseteq C_0$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

3. General Rectangles and Circles

Let a, b, c, d be real numbers. The functor $\mathrm{Rectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def. 1).

(Def. 1) $\mathrm{Rectangle}(a, b, c, d) = \{p : p_{\mathbf{1}} = a \ \wedge \ c \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant d \ \vee \ p_{\mathbf{2}} = d \ \wedge \ a \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant b \ \vee \ p_{\mathbf{1}} = b \ \wedge \ c \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant d \ \vee \ p_{\mathbf{2}} = c \ \wedge \ a \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant b\}$.

The following proposition is true

(28) Let a, b, c, d be real numbers and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $a \leqslant b$ and $c \leqslant d$ and $p \in \mathrm{Rectangle}(a, b, c, d)$, then $a \leqslant p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leqslant b$ and $c \leqslant p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leqslant d$.

Let a, b, c, d be real numbers. The functor $\mathrm{InsideOfRectangle}(a, b, c, d)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\mathrm{InsideOfRectangle}(a, b, c, d) = \{p : a < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < b \wedge c < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 3) $\mathrm{ClosedInsideOfRectangle}(a, b, c, d) = \{p : a \leqslant p_{\mathbf{1}} \ \wedge \ p_{\mathbf{1}} \leqslant b \ \wedge \ c \leqslant p_{\mathbf{2}} \ \wedge \ p_{\mathbf{2}} \leqslant d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{OutsideOfRectangle}(a, b, c, d)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $\mathrm{OutsideOfRectangle}(a, b, c, d) = \{p : a \not\leqslant p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \not\leqslant b \ \vee \ c \not\leqslant p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \not\leqslant d\}$.

Let a, b, c, d be real numbers. The functor $\mathrm{ClosedOutsideOfRectangle}(a, b, c, d)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 5) $\mathrm{ClosedOutsideOfRectangle}(a, b, c, d) = \{p : a \not< p_{\mathbf{1}} \ \vee \ p_{\mathbf{1}} \not< b \ \vee \ c \not< p_{\mathbf{2}} \ \vee \ p_{\mathbf{2}} \not< d\}$.

Next we state four propositions:

(29) Let a, b, r be real numbers and K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $r \geqslant 0$ and $K_5 = \{q : |q| = 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2 - [a, b]| = r\}$. Then $(\mathrm{AffineMap}(r, a, r, b))^{\circ} K_5 = C_1$.

(30) Let P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose there exists a map from $\mathcal{E}^2_{\rm T}\restriction P$ into $\mathcal{E}^2_{\rm T}\restriction Q$ which is a homeomorphism and P is a simple closed curve. Then Q is a simple closed curve.

(31) For every subset P of $\mathcal{E}^2_{\rm T}$ such that P satisfies conditions of simple closed curve holds P is compact.

(32) Let a, b, r be real numbers and C_1 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $r > 0$ and $C_1 = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| = r\}$. Then C_1 is a simple closed curve.

Let a, b, r be real numbers. Let us assume that $r > 0$. The functor $\mathrm{Circle}(a, b, r)$ yielding a compact non empty subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 6) $\mathrm{Circle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| = r\}$.

Let a, b, r be real numbers. The functor $\mathrm{InsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 7) $\mathrm{InsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| < r\}$.

Let a, b, r be real numbers. The functor $\mathrm{ClosedInsideOfCircle}(a, b, r)$ yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 8) $\mathrm{ClosedInsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| \leqslant r\}$.

Let a, b, r be real numbers. The functor $\mathrm{OutsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 9) $\mathrm{OutsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| > r\}$.

Let a, b, r be real numbers. The functor $\mathrm{ClosedOutsideOfCircle}(a, b, r)$ yielding a subset of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 10) $\mathrm{ClosedOutsideOfCircle}(a, b, r) = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p - [a, b]| \geqslant r\}$.

One can prove the following propositions:

(33) Let r be a real number. Then $\mathrm{InsideOfCircle}(0, 0, r) = \{p : |p| < r\}$ and if $r > 0$, then $\mathrm{Circle}(0, 0, r) = \{p_2 : |p_2| = r\}$ and $\mathrm{OutsideOfCircle}(0, 0, r) = \{p_3 : |p_3| > r\}$ and $\mathrm{ClosedInsideOfCircle}(0, 0, r) = \{q : |q| \leqslant r\}$ and $\mathrm{ClosedOutsideOfCircle}(0, 0, r) = \{q_2 : |q_2| \geqslant r\}$.

(34) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 < p_{\mathbf{1}} \wedge p_{\mathbf{1}} < 1 \wedge -1 < p_{\mathbf{2}} \wedge p_{\mathbf{2}} < 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| < 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(35) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \nless p_{\mathbf{1}} \vee p_{\mathbf{1}} \nless 1 \vee -1 \nless p_{\mathbf{2}} \vee p_{\mathbf{2}} \nless 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| > 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(36) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $K_5 = \{p : -1 \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant 1 \wedge -1 \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\rm T}$: $|p_2| \leqslant 1\}$. Then $\mathrm{SqCirc}^\circ K_5 = C_1$.

(37) Let K_5, C_1 be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K_5 = \{p : -1 \not< p_{\mathbf{1}} \vee p_{\mathbf{1}} \not< 1 \vee -1 \not< p_{\mathbf{2}} \vee p_{\mathbf{2}} \not< 1\}$ and $C_1 = \{p_2; p_2$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $|p_2| \geqslant 1\}$. Then $\mathrm{SqCirc}^{\circ} K_5 = C_1$.

(38) Let P_0, P_1, P_2, P_{11}, K_0, K_6, K_7, K_{11} be subsets of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $P = \mathrm{Circle}(0,0,1)$ and $P_0 = \mathrm{InsideOfCircle}(0,0,1)$ and $P_1 = \mathrm{OutsideOfCircle}(0,0,1)$ and $P_2 = \mathrm{ClosedInsideOfCircle}(0,0,1)$ and $P_{11} = \mathrm{ClosedOutsideOfCircle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $K_0 = \mathrm{InsideOfRectangle}(-1,1,-1,1)$ and $K_6 = \mathrm{OutsideOfRectangle}(-1,1,-1,1)$ and $K_7 = \mathrm{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $K_{11} = \mathrm{ClosedOutsideOfRectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Then $f^{\circ}K = P$ and $(f^{-1})^{\circ}P = K$ and $f^{\circ}K_0 = P_0$ and $(f^{-1})^{\circ}P_0 = K_0$ and $f^{\circ}K_6 = P_1$ and $(f^{-1})^{\circ}P_1 = K_6$ and $f^{\circ}K_7 = P_2$ and $f^{\circ}K_{11} = P_{11}$ and $(f^{-1})^{\circ}P_2 = K_7$ and $(f^{-1})^{\circ}P_{11} = K_{11}$.

4. Order of Points on Rectangle

The following propositions are true:

(39) Let a, b, c, d be real numbers. Suppose $a \leqslant b$ and $c \leqslant d$. Then

(i) $\mathcal{L}([a,c],[a,d]) = \{p_1 : (p_1)_{\mathbf{1}} = a \wedge (p_1)_{\mathbf{2}} \leqslant d \wedge (p_1)_{\mathbf{2}} \geqslant c\}$,

(ii) $\mathcal{L}([a,d],[b,d]) = \{p_2 : (p_2)_{\mathbf{1}} \leqslant b \wedge (p_2)_{\mathbf{1}} \geqslant a \wedge (p_2)_{\mathbf{2}} = d\}$,

(iii) $\mathcal{L}([a,c],[b,c]) = \{q_1 : (q_1)_{\mathbf{1}} \leqslant b \wedge (q_1)_{\mathbf{1}} \geqslant a \wedge (q_1)_{\mathbf{2}} = c\}$, and

(iv) $\mathcal{L}([b,c],[b,d]) = \{q_2 : (q_2)_{\mathbf{1}} = b \wedge (q_2)_{\mathbf{2}} \leqslant d \wedge (q_2)_{\mathbf{2}} \geqslant c\}$.

(40) Let a, b, c, d be real numbers. Suppose $a \leqslant b$ and $c \leqslant d$. Then $\{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\} = \mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d]) \cup (\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d]))$.

(41) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,c],[b,c]) = \{[a,c]\}$.

(42) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[b,c]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,c]\}$.

(43) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,d],[b,d]) \cap \mathcal{L}([b,c],[b,d]) = \{[b,d]\}$.

(44) For all real numbers a, b, c, d such that $a \leqslant b$ and $c \leqslant d$ holds $\mathcal{L}([a,c],[a,d]) \cap \mathcal{L}([a,d],[b,d]) = \{[a,d]\}$.

(45) $\{q : -1 = q_{\mathbf{1}} \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee q_{\mathbf{1}} = 1 \wedge -1 \leqslant q_{\mathbf{2}} \wedge q_{\mathbf{2}} \leqslant 1 \vee -1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1 \vee 1 = q_{\mathbf{2}} \wedge -1 \leqslant q_{\mathbf{1}} \wedge q_{\mathbf{1}} \leqslant 1\} = \{p : p_{\mathbf{1}} = -1 \wedge -1 \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant 1 \vee p_{\mathbf{2}} = 1 \wedge -1 \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant 1 \vee p_{\mathbf{1}} = 1 \wedge -1 \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant 1 \vee p_{\mathbf{2}} = -1 \wedge -1 \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant 1\}$.

(46) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{W-bound}\,K = a$.

(47) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{N-bound}\,K = d$.

(48) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{E-bound}\,K = b$.

(49) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{S-bound}\,K = c$.

(50) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{NW-corner}\,K = [a,d]$.

(51) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{NE-corner}\,K = [b,d]$.

(52) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{SW-corner}\,K = [a,c]$.

(53) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{SE-corner}\,K = [b,c]$.

(54) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{W-most}\,K = \mathcal{L}([a,c],[a,d])$.

(55) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{E-most}\,K = \mathcal{L}([b,c],[b,d])$.

(56) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a,b,c,d)$ and $a \leqslant b$ and $c \leqslant d$, then $\text{W-min}\,K = [a, c]$ and $\text{E-max}\,K = [b,d]$.

(57) Let K be a non empty compact subset of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $\mathcal{L}([a,c],[a,d]) \cup \mathcal{L}([a,d],[b,d])$ is an arc from $\text{W-min}\,K$ to $\text{E-max}\,K$ and $\mathcal{L}([a,c],[b,c]) \cup \mathcal{L}([b,c],[b,d])$ is an arc from $\text{E-max}\,K$ to $\text{W-min}\,K$.

(58) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, and p_0, p_1, p_5, p_{10} be points of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\}$ and $p_0 = [a,c]$ and $p_1 = [b,d]$ and $p_5 = [a,$

$d]$ and $p_{10} = [b, c]$ and $f_1 = \langle p_0, p_5, p_1\rangle$ and $f_2 = \langle p_0, p_{10}, p_1\rangle$. Then f_1 is a special sequence and $\widetilde{\mathcal{L}}(f_1) = \mathcal{L}(p_0, p_5) \cup \mathcal{L}(p_5, p_1)$ and f_2 is a special sequence and $\widetilde{\mathcal{L}}(f_2) = \mathcal{L}(p_0, p_{10}) \cup \mathcal{L}(p_{10}, p_1)$ and $P = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{p_0, p_1\}$ and $(f_1)_1 = p_0$ and $(f_1)_{\mathrm{len}\, f_1} = p_1$ and $(f_2)_1 = p_0$ and $(f_2)_{\mathrm{len}\, f_2} = p_1$.

(59) Let P, P_1, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $P = \{p : p_{\mathbf{1}} = a \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = d \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b \vee p_{\mathbf{1}} = b \wedge c \leqslant p_{\mathbf{2}} \wedge p_{\mathbf{2}} \leqslant d \vee p_{\mathbf{2}} = c \wedge a \leqslant p_{\mathbf{1}} \wedge p_{\mathbf{1}} \leqslant b\}$ and $p_1 = [a, c]$ and $p_2 = [b, d]$ and $f_1 = \langle [a, c], [a, d], [b, d]\rangle$ and $f_2 = \langle [a, c], [b, c], [b, d]\rangle$ and $P_1 = \widetilde{\mathcal{L}}(f_1)$ and $P_3 = \widetilde{\mathcal{L}}(f_2)$. Then P_1 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and P_1 is non empty and P_3 is non empty and $P = P_1 \cup P_3$ and $P_1 \cap P_3 = \{p_1, p_2\}$.

(60) For all real numbers a, b, c, d such that $a < b$ and $c < d$ holds $\mathrm{Rectangle}(a, b, c, d)$ is a simple closed curve.

(61) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$, then $\mathrm{UpperArc}\, K = \mathcal{L}([a, c], [a, d]) \cup \mathcal{L}([a, d], [b, d])$.

(62) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. If $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$, then $\mathrm{LowerArc}\, K = \mathcal{L}([a, c], [b, c]) \cup \mathcal{L}([b, c], [b, d])$.

(63) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{UpperArc}\, K$ such that
f is a homeomorphism and $f(0) = \text{W-min}\, K$ and $f(1) = \text{E-max}\, K$ and $\mathrm{rng}\, f = \mathrm{UpperArc}\, K$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1 - 2 \cdot r) \cdot [a, c] + 2 \cdot r \cdot [a, d]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1 - (2 \cdot r - 1)) \cdot [a, d] + (2 \cdot r - 1) \cdot [b, d]$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a, c], [a, d])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}$ and $\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{2}} - c}{d - c}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([a, d], [b, d])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{1}} - a}{b - a}}{2} + \frac{1}{2}) = p$.

(64) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a, b, c, d)$ and $a < b$ and $c < d$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathrm{LowerArc}\, K$ such that
f is a homeomorphism and $f(0) = \text{E-max}\, K$ and $f(1) = \text{W-min}\, K$ and $\mathrm{rng}\, f = \mathrm{LowerArc}\, K$ and for every real number r such that $r \in [0, \frac{1}{2}]$ holds $f(r) = (1 - 2 \cdot r) \cdot [b, d] + 2 \cdot r \cdot [b, c]$ and for every real number r such that $r \in [\frac{1}{2}, 1]$ holds $f(r) = (1 - (2 \cdot r - 1)) \cdot [b, c] + (2 \cdot r - 1) \cdot [a, c]$ and for every

point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,d],[b,c])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}$ and $\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{2}}-d}{c-d}}{2}) = p$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \mathcal{L}([b,c],[a,c])$ holds $0 \leqslant \frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2}$ and $\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2} \leqslant 1$ and $f(\frac{\frac{p_{\mathbf{1}}-b}{a-b}}{2} + \frac{1}{2}) = p$.

(65) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$ and $p_2 \in \mathcal{L}([a,c],[a,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$.

(66) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$ and $p_2 \in \mathcal{L}([a,d],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$.

(67) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[b,d])$ and $p_2 \in \mathcal{L}([b,c],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$.

(68) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[b,c])$ and $p_2 \in \mathcal{L}([a,c],[b,c])$. Then $\mathrm{LE}(p_1,p_2,K)$ and $p_1 \neq \text{W-min}\,K$ if and only if $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$ and $p_2 \neq \text{W-min}\,K$.

(69) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,c],[a,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,c],[a,d])$ and $(p_1)_{\mathbf{2}} \leqslant (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([a,d],[b,d])$, or

(iii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iv) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \text{W-min}\,K$.

(70) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([a,d],[b,d])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([a,d],[b,d])$ and $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$, or

(ii) $p_2 \in \mathcal{L}([b,d],[b,c])$, or

(iii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \text{W-min}\,K$.

(71) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,d],[b,c])$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if one of the following conditions is satisfied:

(i) $p_2 \in \mathcal{L}([b,d],[b,c])$ and $(p_1)_{\mathbf{2}} \geqslant (p_2)_{\mathbf{2}}$, or

(ii) $p_2 \in \mathcal{L}([b,c],[a,c])$ and $p_2 \neq \text{W-min}\,K$.

(72) Let K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$ and $p_1 \in \mathcal{L}([b,c],[a,c])$ and $p_1 \neq \text{W-min}\,K$. Then $\mathrm{LE}(p_1,p_2,K)$ if and only if the following conditions are satisfied:

(i) $p_2 \in \mathcal{L}([b,c],[a,c])$,

(ii) $(p_1)_{\mathbf{1}} \geqslant (p_2)_{\mathbf{1}}$, and

(iii) $p_2 \neq \text{W-min}\,K$.

(73) Let x be a set and a, b, c, d be real numbers. Suppose $x \in \mathrm{Rectangle}(a,b,c,d)$ and $a < b$ and $c < d$. Then $x \in \mathcal{L}([a,c],[a,d])$ or $x \in \mathcal{L}([a,d],[b,d])$ or $x \in \mathcal{L}([b,d],[b,c])$ or $x \in \mathcal{L}([b,c],[a,c])$.

5. General Fashoda Theorem for Square

The following propositions are true:

(74) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and K be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $\mathrm{LE}(p_1,p_2,K)$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$. Then $p_2 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_2)_{\mathbf{2}} \geqslant (p_1)_{\mathbf{2}}$ or $p_2 \in \mathcal{L}([-1,1],[1,1])$ or $p_2 \in \mathcal{L}([1,1],[1,-1])$ or $p_2 \in \mathcal{L}([1,-1],[-1,-1])$ and $p_2 \neq [-1,-1]$.

(75) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $\mathrm{LE}(p_1,p_2,K)$. Then $\mathrm{LE}(f(p_1),f(p_2),P)$.

(76) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $p_1 \in \mathcal{L}([-1,-1],[-1,1])$ and $(p_1)_{\mathbf{2}} \geqslant 0$ and $\mathrm{LE}(p_1,p_2,K)$ and $\mathrm{LE}(p_2,p_3,K)$. Then $\mathrm{LE}(f(p_2),f(p_3),P)$.

(77) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $f = \mathrm{SqCirc}$ and $p_{\mathbf{1}} = -1$ and $p_{\mathbf{2}} < 0$, then $f(p)_{\mathbf{1}} < 0$ and $f(p)_{\mathbf{2}} < 0$.

(78) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$, then $f(p)_{\mathbf{1}} \geqslant 0$ iff $p_{\mathbf{1}} \geqslant 0$.

(79) Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$, then $f(p)_{\mathbf{2}} \geqslant 0$ iff $p_{\mathbf{2}} \geqslant 0$.

(80) Let p, q be points of $\mathcal{E}^2_{\mathrm{T}}$ and f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. If $f = \mathrm{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([1,-1],[-1,-1])$, then $f(p)_{\mathbf{1}} \leqslant f(q)_{\mathbf{1}}$.

(81) Let p, q be points of $\mathcal{E}^2_{\rm T}$ and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $f = \mathrm{SqCirc}$ and $p \in \mathcal{L}([-1,-1],[-1,1])$ and $q \in \mathcal{L}([-1,-1],[-1,1])$ and $p_{\mathbf{2}} \geqslant q_{\mathbf{2}}$ and $p_{\mathbf{2}} < 0$. Then $f(p)_{\mathbf{2}} \geqslant f(q)_{\mathbf{2}}$.

(82) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Suppose $\mathrm{LE}(p_1,p_2,K)$ and $\mathrm{LE}(p_2,p_3,K)$ and $\mathrm{LE}(p_3,p_4,K)$. Then $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(83) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and P be a non empty compact subset of $\mathcal{E}^2_{\rm T}$. If P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and not $\mathrm{LE}(p_1,p_2,P)$, then $\mathrm{LE}(p_2,p_1,P)$.

(84) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\rm T}$ and P be a non empty compact subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$. Then $\mathrm{LE}(p_1,p_2,P)$ and $\mathrm{LE}(p_2,p_3,P)$ or $\mathrm{LE}(p_1,p_3,P)$ and $\mathrm{LE}(p_3,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_2,p_3,P)$ and $\mathrm{LE}(p_3,p_1,P)$ or $\mathrm{LE}(p_3,p_1,P)$ and $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_3,p_2,P)$ and $\mathrm{LE}(p_2,p_1,P)$.

(85) Let p_1, p_2, p_3 be points of $\mathcal{E}^2_{\rm T}$ and P be a non empty compact subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $\mathrm{LE}(p_2,p_3,P)$. Then $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_3,p_1,P)$.

(86) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and P be a non empty compact subset of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve and $p_1 \in P$ and $p_2 \in P$ and $p_3 \in P$ and $p_4 \in P$ and $\mathrm{LE}(p_2,p_3,P)$ and $\mathrm{LE}(p_3,p_4,P)$. Then $\mathrm{LE}(p_1,p_2,P)$ or $\mathrm{LE}(p_2,p_1,P)$ and $\mathrm{LE}(p_1,p_3,P)$ or $\mathrm{LE}(p_3,p_1,P)$ and $\mathrm{LE}(p_1,p_4,P)$ or $\mathrm{LE}(p_4,p_1,P)$.

(87) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$ and $\mathrm{LE}(f(p_1),f(p_2),P)$ and $\mathrm{LE}(f(p_2),f(p_3),P)$ and $\mathrm{LE}(f(p_3),f(p_4),P)$. Then p_1, p_2, p_3, p_4 are in this order on K.

(88) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, P, K be non empty compact subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $P = \mathrm{Circle}(0,0,1)$ and $K = \mathrm{Rectangle}(-1,1,-1,1)$ and $f = \mathrm{SqCirc}$. Then p_1, p_2, p_3, p_4 are in this order on K if and only if $f(p_1)$, $f(p_2)$, $f(p_3)$, $f(p_4)$ are in this order on P.

(89) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, K be a compact non empty subset of $\mathcal{E}^2_{\rm T}$, and K_0 be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $K = \mathrm{Rectangle}(-1,1,-1,1)$ and p_1, p_2, p_3, p_4 are in this order on K. Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that f is continuous and one-to-one and g is continuous and one-to-one and $K_0 = \mathrm{ClosedInsideOfRectangle}(-1,1,-1,1)$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and $\mathrm{rng}\, f \subseteq K_0$ and $\mathrm{rng}\, g \subseteq K_0$.

Then rng f meets rng g.

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On Some Properties of Real Hilbert Space. Part I

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Summary. In this paper, we first introduce the notion of summability of an infinite set of vectors of real Hilbert space, without using index sets. Further we introduce the notion of weak summability, which is weaker than that of summability. Then, several statements for summable sets and weakly summable ones are proved. In the last part of the paper, we give a necessary and sufficient condition for summability of an infinite set of vectors of real Hilbert space as our main theorem. The last theorem is due to [8].

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The terminology and notation used here are introduced in the following articles: [18], [21], [6], [1], [16], [9], [22], [4], [5], [7], [12], [20], [13], [14], [15], [3], [10], [17], [11], [2], [19], and [23].

1. Preliminaries

In this paper X is a real unitary space, x is a point of X, and i is a natural number.

Let us consider X. Let us assume that the addition of X is commutative and associative and the addition of X has a unity. Let Y be a finite subset of the carrier of X. The functor $\mathrm{Setsum}(Y)$ yielding an element of the carrier of X is defined by the condition (Def. 1).

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(Def. 1) There exists a finite sequence p of elements of the carrier of X such that p is one-to-one and $\operatorname{rng} p = Y$ and $\operatorname{Setsum}(Y) =$ the addition of $X \odot p$.

We now state two propositions:

(1) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a finite subset of the carrier of X and I be a function from the carrier of X into the carrier of X. Suppose $Y \subseteq \operatorname{dom} I$ and for every set x such that $x \in \operatorname{dom} I$ holds $I(x) = x$. Then $\operatorname{Setsum}(Y) = \operatorname{setopfunc}(Y,$ the carrier of X, the carrier of X, $I,$ the addition of $X)$.

(2) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y_1, Y_2 be finite subsets of the carrier of X. Suppose Y_1 misses Y_2. Let Z be a finite subset of the carrier of X. If $Z = Y_1 \cup Y_2$, then $\operatorname{Setsum}(Z) = \operatorname{Setsum}(Y_1) + \operatorname{Setsum}(Y_2)$.

2. Summability

Let us consider X and let Y be a subset of the carrier of X. We say that Y is summable_set if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists x such that for every real number e if $e > 0$, then there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $\|x - \operatorname{Setsum}(Y_1)\| < e$.

Let us consider X and let Y be a subset of the carrier of X. Let us assume that Y is summable_set. The functor $\operatorname{sum} Y$ yielding a point of X is defined by the condition (Def. 3).

(Def. 3) Let e be a real number. Suppose $e > 0$. Then there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $\|\operatorname{sum} Y - \operatorname{Setsum}(Y_1)\| < e$.

Let us consider X and let L be a linear functional in X. We say that L is Bounded if and only if:

(Def. 4) There exists a real number K such that $K > 0$ and for every x holds $|L(x)| \leqslant K \cdot \|x\|$.

Let us consider X and let Y be a subset of the carrier of X. We say that Y is weakly summable_set if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists x such that for every linear functional L in X if L is Bounded, then for every real number e such that $e > 0$ there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|L(x - \operatorname{Setsum}(Y_1))| < e$.

Let us consider X, let Y be a subset of the carrier of X, and let L be a functional in X. We say that Y is summable set by L if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a real number r such that for every real number e if $e > 0$, then there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|r - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

Let us consider X, let Y be a subset of the carrier of X, and let L be a functional in X. Let us assume that Y is summable set by L. The functor $\mathrm{SumByfunc}(Y, L)$ yielding a real number is defined by the condition (Def. 7).

(Def. 7) Let e be a real number. Suppose $e > 0$. Then there exists a finite subset Y_0 of the carrier of X such that

(i) Y_0 is non empty,

(ii) $Y_0 \subseteq Y$, and

(iii) for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|\mathrm{SumByfunc}(Y, L) - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

The following propositions are true:

(3) For every subset Y of the carrier of X such that Y is summable_set holds Y is weakly summable_set.

(4) Let L be a linear functional in X and p be a finite sequence of elements of the carrier of X. Suppose $\mathrm{len}\, p \geqslant 1$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\mathrm{dom}\, p = \mathrm{dom}\, q$ and for every i such that $i \in \mathrm{dom}\, q$ holds $q(i) = L(p(i))$. Then $L(\text{the addition of } X \odot p) = +_{\mathbb{R}} \odot q$.

(5) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite subset of the carrier of X. Suppose S is non empty. Let L be a linear functional in X. Then $L(\mathrm{Setsum}(S)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})$.

(6) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of the carrier of X. Suppose Y is weakly summable_set. Then there exists x such that for every linear functional L in X if L is Bounded, then for every real number e such that $e > 0$ there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq Y$ holds $|L(x) - \mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

(7) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of the carrier of X. Suppose Y is weakly summable_set. Let L be a linear functional in X. If L is Bounded, then Y is summable set by L.

(8) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let Y be a subset of the carrier of X. Suppose Y is summable_set. Let L be a linear functional in X. If L is Bounded, then Y is summable set by L.

(9) For every finite subset Y of the carrier of X such that Y is non empty holds Y is summable_set.

3. Necessary and Sufficient Condition for Summability

One can prove the following proposition

(10) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let Y be a subset of the carrier of X. Then Y is summable_set if and only if for every real number e such that $e > 0$ there exists a finite subset Y_0 of the carrier of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that Y_1 is non empty and $Y_1 \subseteq Y$ and Y_0 misses Y_1 holds $\|\mathrm{Setsum}(Y_1)\| < e$.

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Full Subtracter Circuit. Part II

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Summary. In this article we continue investigations from [22] of verification of a design of subtracter circuit. We define it as a combination of multi cell circuit using schemes from [6]. As the main result we prove the stability of the circuit.

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The articles [17], [16], [21], [15], [3], [18], [25], [1], [9], [10], [4], [8], [2], [19], [24], [14], [20], [13], [12], [11], [23], [5], [7], and [22] provide the terminology and notation for this paper.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterStr(x, y) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by the condition (Def. 1).

(Def. 1) There exist many sorted sets f, g indexed by $\mathbb{N}$ such that

(i) n-BitSubtracterStr$(x, y) = f(n)$,

(ii) $f(0) = $ 1GateCircStr$(\varepsilon, Boolean^0 \longmapsto true)$,

(iii) $g(0) = \langle \varepsilon, Boolean^0 \longmapsto true \rangle$, and

(iv) for every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = g(n)$ holds $f(n + 1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n + 1), y(n + 1), z)$ and $g(n + 1) = $ BorrowOutput$(x(n + 1), y(n + 1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterCirc(x, y) yielding a Boolean strict circuit of

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n-BitSubtracterStr(x, y) with denotation held in gates is defined by the condition (Def. 2).

(Def. 2) There exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that

(i) n-BitSubtracterStr$(x, y) = f(n)$,

(ii) n-BitSubtracterCirc$(x, y) = g(n)$,

(iii) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(iv) $g(0) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(v) $h(0) = \langle\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}\rangle$, and

(vi) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) = $ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitBorrowOutput(x, y) yields an element of InnerVertices(n-BitSubtracterStr (x, y)) and is defined by the condition (Def. 3).

(Def. 3) There exists a many sorted set h indexed by $\mathbb{N}$ such that

(i) n-BitBorrowOutput$(x, y) = h(n)$,

(ii) $h(0) = \langle\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}\rangle$, and

(iii) for every natural number n and for every set z such that $z = h(n)$ holds $h(n+1) = $ BorrowOutput$(x(n+1), y(n+1), z)$.

One can prove the following propositions:

(1) Let x, y be finite sequences and f, g, h be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(ii) $g(0) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(iii) $h(0) = \langle\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}\rangle$, and

(iv) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) = $ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number. Then n-BitSubtracterStr$(x, y) = f(n)$ and n-BitSubtracterCirc$(x, y) = g(n)$ and n-BitBorrowOutput$(x, y) = h(n)$.

(2) For all finite sequences a, b holds 0-BitSubtracterStr$(a, b) = $ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitSubtracterCirc$(a, b) = $ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitBorrowOutput$(a, b) = \langle\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true}\rangle$.

(3) Let a, b be finite sequences and c be a set. Suppose $c = \langle\varepsilon,\ Boolean^0 \longmapsto true\rangle$. Then 1-BitSubtracterStr(a, b) = 1GateCircStr$(\varepsilon, Boolean^0 \longmapsto true)$+· BitSubtracterWithBorrowStr$(a(1), b(1), c)$ and 1-BitSubtracterCirc(a, b) = 1GateCircuit$(\varepsilon, Boolean^0 \longmapsto true)$+· BitSubtracterWithBorrowCirc$(a(1), b(1), c)$ and 1-BitBorrowOutput(a, b) = BorrowOutput$(a(1), b(1), c)$.

(4) For all sets a, b, c such that $c = \langle\varepsilon,\ Boolean^0 \longmapsto true\rangle$ holds 1-BitSubtracterStr$(\langle a\rangle, \langle b\rangle)$ = 1GateCircStr$(\varepsilon, Boolean^0 \longmapsto true)$+· BitSubtracterWithBorrowStr(a, b, c) and 1-BitSubtracterCirc$(\langle a\rangle, \langle b\rangle)$ = 1GateCircuit$(\varepsilon, Boolean^0 \longmapsto true)$+· BitSubtracterWithBorrowCirc(a, b, c) and 1-BitBorrowOutput$(\langle a\rangle, \langle b\rangle)$ = BorrowOutput(a, b, c).

(5) Let n be a natural number, p, q be finite sequences with length n, and p_1, p_2, q_1, q_2 be finite sequences. Then n-BitSubtracterStr$(p ^\frown p_1, q ^\frown q_1)$ = n-BitSubtracterStr$(p ^\frown p_2, q ^\frown q_2)$ and n-BitSubtracterCirc$(p ^\frown p_1, q ^\frown q_1)$ = n-BitSubtracterCirc$(p ^\frown p_2, q ^\frown q_2)$ and n-BitBorrowOutput$(p ^\frown p_1, q ^\frown q_1)$ = n-BitBorrowOutput$(p ^\frown p_2, q ^\frown q_2)$.

(6) Let n be a natural number, x, y be finite sequences with length n, and a, b be sets.
Then $(n+1)$-BitSubtracterStr$(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle)$ = (n-BitSubtracterStr(x, y))+· BitSubtracterWithBorrowStr$(a, b, n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitSubtracterCirc$(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle)$ = (n-BitSubtracterCirc(x, y))+· BitSubtracterWithBorrowCirc$(a, b, n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitBorrowOutput$(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle)$ = BorrowOutput$(a, b, n$-BitBorrowOutput$(x, y))$.

(7) Let n be a natural number and x, y be finite sequences. Then $(n+1)$-BitSubtracterStr(x, y) = (n-BitSubtracterStr(x, y))+· BitSubtracterWithBorrowStr$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitSubtracterCirc(x, y) = (n-BitSubtracterCirc(x, y))+· BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitBorrowOutput(x, y) = BorrowOutput$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$.

(8) For all natural numbers n, m such that $n \leqslant m$ and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x, y)) $\subseteq$ InnerVertices(m-BitSubtracterStr(x, y)).

(9) For every natural number n and for all finite sequences x, y holds InnerVertices($(n+1)$-BitSubtracterStr(x, y)) = InnerVertices(n-BitSubtracterStr(x, y)) $\cup$ InnerVertices (BitSubtracterWithBorrowStr$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$).

Let k, n be natural numbers. Let us assume that $k \geqslant 1$ and $k \leqslant n$. Let x, y be finite sequences. The functor (k, n)-BitSubtracterOutput(x, y) yielding an element of InnerVertices(n-BitSubtracterStr(x, y)) is defined by:

(Def. 4) There exists a natural number i such that $k = i + 1$ and (k, n)-BitSubtracterOutput$(x, y) =$ BitSubtracterOutput$(x(k), y(k),$ i-BitBorrowOutput$(x, y))$.

One can prove the following propositions:

(10) For all natural numbers n, k such that $k < n$ and for all finite sequences x, y holds $(k + 1, n)$-BitSubtracterOutput$(x, y) =$ BitSubtracterOutput$(x(k + 1), y(k + 1), k$-BitBorrowOutput$(x, y))$.

(11) For every natural number n and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x, y)) is a binary relation.

(12) For all sets x, y, c holds InnerVertices(BorrowIStr(x, y, c)) $= \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\}$.

(13) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowIStr(x, y, c)) $= \{x, y, c\}$.

(14) For all sets x, y, c holds InnerVertices(BorrowStr(x, y, c)) $= \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x, y, c)\}$.

(15) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowStr(x, y, c)) $= \{x, y, c\}$.

(16) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$ holds InputVertices(BitSubtracterWithBorrowStr(x, y, c)) $= \{x, y, c\}$.

(17) For all sets x, y, c holds InnerVertices(BitSubtracterWithBorrowStr(x, y, c)) $= \{\langle\langle x, y\rangle, \mathrm{xor}\rangle,$ 2GatesCircOutput$(x, y, c, \mathrm{xor})\} \cup \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x, y, c)\}$.

Let n be a natural number and let x, y be finite sequences. Observe that n-BitBorrowOutput(x, y) is pair.

The following propositions are true:

(18) Let x, y be finite sequences and n be a natural number. Then $(n$-BitBorrowOutput$(x, y))_{\mathbf{1}} = \varepsilon$ and $(n$-BitBorrowOutput$(x, y))_{\mathbf{2}} =$ $\underline{\underline{\mathit{Boolean}^0 \longmapsto \mathit{true}}}$ and $\pi_1((n$-BitBorrowOutput$(x, y))_{\mathbf{2}}) = \mathit{Boolean}^0$ or $\overline{(n\text{-BitBorrowOutput}(x, y))_{\mathbf{1}}} = 3$ and $(n$-BitBorrowOutput$(x, y))_{\mathbf{2}} = \mathrm{or}_3$ and $\pi_1((n$-BitBorrowOutput$(x, y))_{\mathbf{2}}) = \mathit{Boolean}^3$.

(19) Let n be a natural number, x, y be finite sequences, and p be a set. Then n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{and}_2\rangle$ and n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{and}_{2a}\rangle$ and n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{xor}\rangle$.

(20) Let f, g be nonpair yielding finite sequences and n be a natural number. Then InputVertices($(n + 1)$-BitSubtracterStr(f, g)) $=$ InputVertices(n-BitSubtracterStr(f, g)) $\cup$ (InputVertices

(BitSubtracterWithBorrowStr($f(n+1), g(n+1)$, n-BitBorrowOutput(f, g)))$\setminus$ {n-BitBorrowOutput(f, g)}) and InnerVertices(n-BitSubtracterStr(f, g)) is a binary relation and InputVertices(n-BitSubtracterStr(f, g)) has no pairs.

(21) For every natural number n and for all nonpair yielding finite sequences x, y with length n holds InputVertices(n-BitSubtracterStr(x, y)) $=$ rng $x \cup$ rng y.

(22) Let x, y, c be sets, s be a state of BorrowCirc(x, y, c), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle)$ and $a_2 = s(\langle\langle y, c\rangle, \mathrm{and}_2\rangle)$ and $a_3 = s(\langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle)$, then (Following($s$))(BorrowOutput($x, y, c$)) $=$ $a_1 \vee a_2 \vee a_3$.

(23) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$. Let s be a state of BorrowCirc(x, y, c). Then Following($s, 2$) is stable.

(24) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$. Let s be a state of BitSubtracterWithBorrowCirc(x, y, c) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then (Following($s, 2$))(BitSubtracterOutput(x, y, c)) $= a_1 \oplus a_2 \oplus a_3$ and (Following($s, 2$))(BorrowOutput(x, y, c)) $= \neg a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee \neg a_1 \wedge a_3$.

(25) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$. Let s be a state of BitSubtracterWithBorrowCirc(x, y, c). Then Following($s, 2$) is stable.

(26) Let n be a natural number, x, y be nonpair yielding finite sequences with length n, and s be a state of n-BitSubtracterCirc(x, y). Then Following($s, 1 + 2 \cdot n$) is stable.

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Dijkstra's Shortest Path Algorithm

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Summary. The article formalizes Dijkstra's shortest path algorithm [11]. A path from a source vertex v to a target vertex u is said to be the shortest path if its total cost is minimum among all v-to-u paths. Dijkstra's algorithm is based on the following assumptions:

- All edge costs are non-negative.
- The number of vertices is finite.
- The source is a single vertex, but the target may be all other vertices.

The underlying principle of the algorithm may be described as follows: the algorithm starts with the source; it visits the vertices in order of increasing cost, and maintains a set V of visited vertices (denoted by UsedVx in the article) whose cost from the source has been computed, and a tentative cost $D(u)$ to each unvisited vertex u. In the article, the set of all unvisited vertices is denoted by UnusedVx. $D(u)$ is the cost of the shortest path from the source to u in the subgraph induced by $V \cup \{u\}$. We denote the set of all unvisited vertices whose D-values are not infinite (i.e. in the subgraph each of which has a path from the source to itself) by OuterVx. Dijkstra's algorithm repeatedly searches OuterVx for the vertex with minimum tentative cost (this procedure is called findmin in the article), adds it to the set V and modifies D-values by a procedure, called Relax. Suppose the unvisited vertex with minimum tentative cost is x, the procedure Relax replaces $D(u)$ with $\min\{D(u), D(u) + cost(x,u)\}$ where u is a vertex in UnusedVx, and cost(x,u) is the cost of edge (x,u). In the Mizar library, there are several computer models, e.g. SCMFSA and SCMPDS etc. However, it is extremely difficult to use these models to formalize the algorithm. Instead, we adopt functions in the Mizar library, which seem to be pseudo-codes, and are similar to those in the functional programming language, e.g. Lisp. To date, there is no rigorous justification with respect to the correctness of Dijkstra's algorithm. The article presents first the rigorous justification.

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The papers [12], [2], [20], [19], [22], [23], [6], [3], [5], [21], [1], [10], [13], [7], [15], [9], [16], [18], [8], [14], [17], and [4] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: X denotes a set, i, j, k, m, n denote natural numbers, p denotes a finite sequence of elements of X, and i_1 denotes an integer.

We now state three propositions:

(1) For every finite sequence p and for every set x holds $x \notin \operatorname{rng} p$ and p is one-to-one iff $p \frown \langle x \rangle$ is one-to-one.

(2) If $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} p$, then $p(i_1) \in X$.

(3) If $1 \leqslant i_1$ and $i_1 \leqslant \operatorname{len} p$, then $p_{i_1} = p(i_1)$.

For simplicity, we adopt the following rules: G denotes a graph, p_1, q_1 denote finite sequences of elements of the edges of G, p, q denote oriented chains of G, W denotes a function, U, V, e, e_1 denote sets, and v_1, v_2, v_3, v_4 denote vertices of G.

We now state three propositions:

(4) If W is weight of G and $\operatorname{len} p_1 = 1$, then $\operatorname{cost}(p_1, W) = W(p_1(1))$.

(5) If $e \in$ the edges of G, then $\langle e \rangle$ is a Simple oriented chain of G.

(6) Let p be a Simple oriented chain of G. Suppose $p = p_1 \frown q_1$ and $\operatorname{len} p_1 \geqslant 1$ and $\operatorname{len} q_1 \geqslant 1$. Then (the target of G)$(p(\operatorname{len} p)) \neq$ (the target of G)$(p_1(\operatorname{len} p_1))$ and (the source of G)$(p(1)) \neq$ (the source of G)$(q_1(1))$.

2. The Fundamental Properties of Directed Paths and Shortest Paths

We now state several propositions:

(7) p is oriented path from v_1 to v_2 in V iff p is oriented path from v_1 to v_2 in $V \cup \{v_2\}$.

(8) p is shortest path from v_1 to v_2 in V w.r.t. W iff p is shortest path from v_1 to v_2 in $V \cup \{v_2\}$ w.r.t. W.

(9) Suppose p is shortest path from v_1 to v_2 in V w.r.t. W and q is shortest path from v_1 to v_2 in V w.r.t. W. Then $\operatorname{cost}(p, W) = \operatorname{cost}(q, W)$.

(10) Let G be an oriented graph, v_1, v_2 be vertices of G, and e_2, e_3 be sets. Suppose $e_2 \in$ the edges of G and $e_3 \in$ the edges of G and e_2 orientedly joins v_1, v_2 and e_3 orientedly joins v_1, v_2. Then $e_2 = e_3$.

(11) Suppose that
(i) the vertices of $G = U \cup V$,
(ii) $v_1 \in U$,
(iii) $v_2 \in V$, and
(iv) for all v_3, v_4 such that $v_3 \in U$ and $v_4 \in V$ it is not true that there exists e such that $e \in$ the edges of G and e orientedly joins v_3, v_4.
Then there exists no p which is oriented path from v_1 to v_2.

(12) Suppose that
(i) the vertices of $G = U \cup V$,
(ii) $v_1 \in U$,
(iii) for all v_3, v_4 such that $v_3 \in U$ and $v_4 \in V$ it is not true that there exists e such that $e \in$ the edges of G and e orientedly joins v_3, v_4, and
(iv) p is oriented path from v_1 to v_2.
Then p is oriented path from v_1 to v_2 in U.

3. The Basic Theorems for Dijkstra's Shortest Path Algorithm (continue)

We adopt the following convention: G is a finite graph, P, Q are oriented chains of G, and v_1, v_2, v_3 are vertices of G.

Next we state the proposition

(13) Suppose that W is nonnegative weight of G and P is shortest path from v_1 to v_2 in V w.r.t. W and $v_1 \neq v_2$ and $v_1 \neq v_3$ and Q is shortest path from v_1 to v_3 in V w.r.t. W and it is not true that there exists e such that $e \in$ the edges of G and e orientedly joins v_2, v_3 and P is longest in shortest path from v_1 in V w.r.t. W. Then Q is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W.

For simplicity, we adopt the following rules: G is a finite oriented graph, P, Q are oriented chains of G, W is a function from the edges of G into $\mathbb{R}_{\geqslant 0}$, and v_1, v_2, v_3, v_4 are vertices of G.

One can prove the following three propositions:

(14) Suppose $e \in$ the edges of G and $v_1 \neq v_2$ and $P = \langle e \rangle$ and e orientedly joins v_1, v_2. Then P is shortest path from v_1 to v_2 in $\{v_1\}$ w.r.t. W.

(15) Suppose that $e \in$ the edges of G and P is shortest path from v_1 to v_2 in V w.r.t. W and $v_1 \neq v_3$ and $Q = P \frown \langle e \rangle$ and e orientedly joins v_2, v_3 and $v_1 \in V$ and for every v_4 such that $v_4 \in V$ it is not true that there exists e_1 such that $e_1 \in$ the edges of G and e_1 orientedly joins v_4, v_3. Then Q is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W.

(16) Suppose that
(i) the vertices of $G = U \cup V$,

(ii) $v_1 \in U$, and

(iii) for all v_3, v_4 such that $v_3 \in U$ and $v_4 \in V$ it is not true that there exists e such that $e \in$ the edges of G and e orientedly joins v_3, v_4.

Then P is shortest path from v_1 to v_2 in U w.r.t. W if and only if P is shortest path from v_1 to v_2 in W.

4. The Definition of Assignment Statement

Let f be a function and let i, x be sets. We introduce $f_i{:=}x$ as a synonym of $f +\cdot (i, x)$.

We now state the proposition

(17) For all sets x, y and for every function f holds $\mathrm{rng}(f_x{:=}y) \subseteq \mathrm{rng}\, f \cup \{y\}$.

Let f be a finite sequence of elements of $\mathbb{R}$, let x be a set, and let r be a real number. Then $f_x{:=}r$ is a finite sequence of elements of $\mathbb{R}$.

Let i, k be natural numbers, let f be a finite sequence of elements of $\mathbb{R}$, and let r be a real number. The functor $(f, i) := (k, r)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 1) $(f, i) := (k, r) = f_i{:=}k_k{:=}r$.

In the sequel f, g, h denote elements of $\mathbb{R}^*$ and r denotes a real number.

One can prove the following propositions:

(18) If $i \neq k$ and $i \in \mathrm{dom}\, f$, then $((f, i) := (k, r))(i) = k$.

(19) If $m \neq i$ and $m \neq k$ and $m \in \mathrm{dom}\, f$, then $((f, i) := (k, r))(m) = f(m)$.

(20) If $k \in \mathrm{dom}\, f$, then $((f, i) := (k, r))(k) = r$.

(21) $\mathrm{dom}((f, i) := (k, r)) = \mathrm{dom}\, f$.

5. The Definition of Pascal–Like "while" - "do" Statement

Let X be a set. Then id_X is an element of X^X.

Let X be a set and let f, g be functions from X into X. Then $g \cdot f$ is a function from X into X.

Let X be a set and let f, g be elements of X^X. Then $g \cdot f$ is an element of X^X.

Let X be a set, let f be an element of X^X, and let g be an element of X. Then $f(g)$ is an element of X.

Let X be a set and let f be an element of X^X. The functor repeat f yields a function from $\mathbb{N}$ into X^X and is defined by:

(Def. 2) $(\mathrm{repeat}\, f)(0) = \mathrm{id}_X$ and for every natural number i and for every element x of X^X such that $x = (\mathrm{repeat}\, f)(i)$ holds $(\mathrm{repeat}\, f)(i + 1) = f \cdot x$.

Next we state two propositions:

(22) For every element F of $(\mathbb{R}^*)^{\mathbb{R}^*}$ and for every element f of $\mathbb{R}^*$ and for all natural numbers n, i holds $(\operatorname{repeat} F)(0)(f) = f$.

(23) Let F, G be elements of $(\mathbb{R}^*)^{\mathbb{R}^*}$, f be an element of $\mathbb{R}^*$, and i be a natural number. Then $(\operatorname{repeat}(F \cdot G))(i+1)(f) = F(G((\operatorname{repeat}(F \cdot G))(i)(f)))$.

Let g be an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ and let f be an element of $\mathbb{R}^*$. Then $g(f)$ is an element of $\mathbb{R}^*$.

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. The functor $\operatorname{OuterVx}(f, n)$ yielding a subset of $\mathbb{N}$ is defined by:

(Def. 3) $\operatorname{OuterVx}(f, n) = \{i : i \in \operatorname{dom} f \wedge 1 \leqslant i \wedge i \leqslant n \wedge f(i) \neq -1 \wedge f(n+i) \neq -1\}$.

Let f be an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$, let g be an element of $\mathbb{R}^*$, and let n be a natural number. Let us assume that there exists i such that $\operatorname{OuterVx}((\operatorname{repeat} f)(i)(g), n) = \emptyset$. The functor $\operatorname{LifeSpan}(f, g, n)$ yielding a natural number is defined by:

(Def. 4) $\operatorname{OuterVx}((\operatorname{repeat} f)(\operatorname{LifeSpan}(f, g, n))(g), n) = \emptyset$ and for every natural number k such that $\operatorname{OuterVx}((\operatorname{repeat} f)(k)(g), n) = \emptyset$ holds $\operatorname{LifeSpan}(f, g, n) \leqslant k$.

Let f be an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ and let n be a natural number. The functor $\operatorname{WhileDo}(f, n)$ yielding an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ is defined as follows:

(Def. 5) $\operatorname{dom} \operatorname{WhileDo}(f, n) = \mathbb{R}^*$ and for every element h of $\mathbb{R}^*$ holds $(\operatorname{WhileDo}(f, n))(h) = (\operatorname{repeat} f)(\operatorname{LifeSpan}(f, h, n))(h)$.

6. Defining a Weight Function for an Oriented Graph

Let G be an oriented graph and let v_1, v_2 be vertices of G. Let us assume that there exists a set e such that $e \in$ the edges of G and e orientedly joins v_1, v_2. The functor $\operatorname{Edge}(v_1, v_2)$ is defined as follows:

(Def. 6) There exists a set e such that $\operatorname{Edge}(v_1, v_2) = e$ and $e \in$ the edges of G and e orientedly joins v_1, v_2.

Let G be an oriented graph, let v_1, v_2 be vertices of G, and let W be a function. The functor $\operatorname{Weight}(v_1, v_2, W)$ is defined as follows:

(Def. 7) $$\operatorname{Weight}(v_1, v_2, W) = \begin{cases} W(\operatorname{Edge}(v_1, v_2)), & \text{if there exists a set } e \text{ such that } e \in \text{the edges of } G \text{ and } e \text{ orientedly joins } v_1,\ v_2, \\ -1, & \text{otherwise.} \end{cases}$$

Let G be an oriented graph, let v_1, v_2 be vertices of G, and let W be a function from the edges of G into $\mathbb{R}_{\geqslant 0}$. Then $\operatorname{Weight}(v_1, v_2, W)$ is a real number.

In the sequel G is an oriented graph, v_1, v_2 are vertices of G, and W is a function from the edges of G into $\mathbb{R}_{\geqslant 0}$.

We now state three propositions:

(24) Weight$(v_1, v_2, W) \geqslant 0$ iff there exists a set e such that $e \in$ the edges of G and e orientedly joins v_1, v_2.

(25) Weight$(v_1, v_2, W) = -1$ iff it is not true that there exists a set e such that $e \in$ the edges of G and e orientedly joins v_1, v_2.

(26) If $e \in$ the edges of G and e orientedly joins v_1, v_2, then Weight$(v_1, v_2, W) = W(e)$.

7. Basic Operations for Dijkstra's Shortest Path Algorithm

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. The functor UnusedVx(f, n) yields a subset of $\mathbb{N}$ and is defined as follows:

(Def. 8) UnusedVx$(f, n) = \{i : i \in \operatorname{dom} f \ \wedge \ 1 \leqslant i \ \wedge \ i \leqslant n \ \wedge \ f(i) \neq -1\}$.

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. The functor UsedVx(f, n) yielding a subset of $\mathbb{N}$ is defined as follows:

(Def. 9) UsedVx$(f, n) = \{i : i \in \operatorname{dom} f \ \wedge \ 1 \leqslant i \ \wedge \ i \leqslant n \ \wedge \ f(i) = -1\}$.

The following proposition is true

(27) UnusedVx$(f, n) \subseteq \operatorname{Seg} n$.

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. One can verify that UnusedVx(f, n) is finite.

Next we state two propositions:

(28) OuterVx$(f, n) \subseteq$ UnusedVx(f, n).

(29) OuterVx$(f, n) \subseteq \operatorname{Seg} n$.

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. Observe that OuterVx(f, n) is finite.

Let X be a finite subset of $\mathbb{N}$, let f be an element of $\mathbb{R}^*$, and let us consider n. The functor Argmin(X, f, n) yielding a natural number is defined by the conditions (Def. 10).

(Def. 10)(i) If $X \neq \emptyset$, then there exists i such that $i =$ Argmin(X, f, n) and $i \in X$ and for every k such that $k \in X$ holds $f_{2 \cdot n+i} \leqslant f_{2 \cdot n+k}$ and for every k such that $k \in X$ and $f_{2 \cdot n+i} = f_{2 \cdot n+k}$ holds $i \leqslant k$, and

(ii) if $X = \emptyset$, then Argmin$(X, f, n) = 0$.

We now state two propositions:

(30) If OuterVx$(f, n) \neq \emptyset$ and $j =$ Argmin(OuterVx$(f, n), f, n)$, then $j \in \operatorname{dom} f$ and $1 \leqslant j$ and $j \leqslant n$ and $f(j) \neq -1$ and $f(n + j) \neq -1$.

(31) Argmin(OuterVx$(f, n), f, n) \leqslant n$.

Let n be a natural number. The functor findmin n yields an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ and is defined as follows:

(Def. 11) dom findmin $n = \mathbb{R}^*$ and for every element f of $\mathbb{R}^*$ holds (findmin $n)(f) = (f, n \cdot n + 3 \cdot n + 1) :=$ (Argmin(OuterVx$(f, n), f, n), -1)$.

Next we state four propositions:

(32) If $i \in \operatorname{dom} f$ and $i > n$ and $i \neq n \cdot n + 3 \cdot n + 1$, then $(\operatorname{findmin} n)(f)(i) = f(i)$.

(33) If $i \in \operatorname{dom} f$ and $f(i) = -1$ and $i \neq n \cdot n + 3 \cdot n + 1$, then $(\operatorname{findmin} n)(f)(i) = -1$.

(34) $\operatorname{dom}(\operatorname{findmin} n)(f) = \operatorname{dom} f$.

(35) If $\operatorname{OuterVx}(f, n) \neq \emptyset$, then there exists j such that $j \in \operatorname{OuterVx}(f, n)$ and $1 \leqslant j$ and $j \leqslant n$ and $(\operatorname{findmin} n)(f)(j) = -1$.

Let f be an element of $\mathbb{R}^*$ and let n, k be natural numbers. The functor $\operatorname{newpathcost}(f, n, k)$ yielding a real number is defined as follows:

(Def. 12) $\operatorname{newpathcost}(f, n, k) = f_{2 \cdot n + f_{n \cdot n + 3 \cdot n + 1}} + f_{2 \cdot n + n \cdot f_{n \cdot n + 3 \cdot n + 1} + k}$.

Let n, k be natural numbers and let f be an element of $\mathbb{R}^*$. We say that f has better path at n, k if and only if:

(Def. 13) $f(n + k) = -1$ or $f_{2 \cdot n + k} > \operatorname{newpathcost}(f, n, k)$ but $f_{2 \cdot n + n \cdot f_{n \cdot n + 3 \cdot n + 1} + k} \geqslant 0$ but $f(k) \neq -1$.

Let f be an element of $\mathbb{R}^*$ and let n be a natural number. The functor $\operatorname{Relax}(f, n)$ yields an element of $\mathbb{R}^*$ and is defined by the conditions (Def. 14).

(Def. 14)(i) $\operatorname{dom} \operatorname{Relax}(f, n) = \operatorname{dom} f$, and

(ii) for every natural number k such that $k \in \operatorname{dom} f$ holds if $n < k$ and $k \leqslant 2 \cdot n$, then if f has better path at n, $k -' n$, then $(\operatorname{Relax}(f, n))(k) = f_{n \cdot n + 3 \cdot n + 1}$ and if f does not have better path at n, $k -' n$, then $(\operatorname{Relax}(f, n))(k) = f(k)$ and if $2 \cdot n < k$ and $k \leqslant 3 \cdot n$, then if f has better path at n, $k -' 2 \cdot n$, then $(\operatorname{Relax}(f, n))(k) = \operatorname{newpathcost}(f, n, k -' 2 \cdot n)$ and if f does not have better path at n, $k -' 2 \cdot n$, then $(\operatorname{Relax}(f, n))(k) = f(k)$ and if $k \leqslant n$ or $k > 3 \cdot n$, then $(\operatorname{Relax}(f, n))(k) = f(k)$.

Let n be a natural number. The functor $\operatorname{Relax} n$ yields an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ and is defined by:

(Def. 15) $\operatorname{dom} \operatorname{Relax} n = \mathbb{R}^*$ and for every element f of $\mathbb{R}^*$ holds $(\operatorname{Relax} n)(f) = \operatorname{Relax}(f, n)$.

One can prove the following propositions:

(36) $\operatorname{dom}(\operatorname{Relax} n)(f) = \operatorname{dom} f$.

(37) If $i \leqslant n$ or $i > 3 \cdot n$ and if $i \in \operatorname{dom} f$, then $(\operatorname{Relax} n)(f)(i) = f(i)$.

(38) $\operatorname{dom}(\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i)(f) = \operatorname{dom}(\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i + 1)(f)$.

(39) If $\operatorname{OuterVx}((\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i)(f), n) \neq \emptyset$, then $\operatorname{UnusedVx}((\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i + 1)(f), n) \subset \operatorname{UnusedVx}((\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i)(f), n)$.

(40) If $g = (\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i)(f)$ and $h = (\operatorname{repeat}(\operatorname{Relax} n \cdot \operatorname{findmin} n))(i + 1)(f)$ and $k = \operatorname{Argmin}(\operatorname{OuterVx}(g, n), g, n)$ and

OuterVx$(g, n) \neq \emptyset$, then UsedVx$(h, n) =$ UsedVx$(g, n) \cup \{k\}$ and $k \notin$ UsedVx(g, n).

(41) There exists i such that $i \leqslant n$ and OuterVx((repeat(Relax $n \cdot$ findmin n))$(i)(f), n) = \emptyset$.

(42) dom $f =$ dom(repeat(Relax $n \cdot$ findmin n))$(i)(f)$.

Let f, g be elements of $\mathbb{R}^*$ and let us consider m, n. We say that f, g are equal at m, n if and only if:

(Def. 16) dom $f =$ dom g and for every k such that $k \in$ dom f and $m \leqslant k$ and $k \leqslant n$ holds $f(k) = g(k)$.

One can prove the following propositions:

(43) f, f are equal at m, n.

(44) If f, g are equal at m, n and g, h are equal at m, n, then f, h are equal at m, n.

(45) (repeat(Relax $n \cdot$ findmin n))$(i)(f)$, (repeat(Relax $n \cdot$ findmin n))$(i+1)(f)$ are equal at $3 \cdot n + 1$, $n \cdot n + 3 \cdot n$.

(46) Let F be an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$, f be an element of $\mathbb{R}^*$, and n, i be natural numbers. If $i <$ LifeSpan(F, f, n), then OuterVx((repeat F)$(i)(f), n) \neq \emptyset$.

(47) f, (repeat(Relax $n \cdot$ findmin n))$(i)(f)$ are equal at $3 \cdot n + 1$, $n \cdot n + 3 \cdot n$.

(48) Suppose that
 (i) $1 \leqslant n$,
 (ii) $1 \in$ dom f,
 (iii) $f(n+1) \neq -1$,
 (iv) for every i such that $1 \leqslant i$ and $i \leqslant n$ holds $f(i) = 1$, and
 (v) for every i such that $2 \leqslant i$ and $i \leqslant n$ holds $f(n+i) = -1$.

Then $1 =$ Argmin(OuterVx$(f, n), f, n)$ and UsedVx$(f, n) = \emptyset$ and $\{1\} =$ UsedVx((repeat(Relax $n \cdot$ findmin n))$(1)(f), n)$.

(49) If $g =$ (repeat(Relax $n \cdot$ findmin n))$(1)(f)$ and $h =$ (repeat(Relax $n \cdot$ findmin n))$(i)(f)$ and $1 \leqslant i$ and $i \leqslant$ LifeSpan(Relax $n \cdot$ findmin $n, f, n)$ and $m \in$ UsedVx(g, n), then $m \in$ UsedVx(h, n).

Let p be a finite sequence of elements of $\mathbb{N}$, let f be an element of $\mathbb{R}^*$, and let i, n be natural numbers. We say that p is vertex sequence at f, i, n if and only if:

(Def. 17) $p(\operatorname{len} p) = i$ and for every k such that $1 \leqslant k$ and $k < \operatorname{len} p$ holds $p(\operatorname{len} p - k) = f(n + p_{(\operatorname{len} p - k)+1})$.

Let p be a finite sequence of elements of $\mathbb{N}$, let f be an element of $\mathbb{R}^*$, and let i, n be natural numbers. We say that p is simple vertex sequence at f, i, n if and only if:

(Def. 18) $p(1) = 1$ and $\operatorname{len} p > 1$ and p is vertex sequence at f, i, n and one-to-one.

Next we state the proposition

(50) Let p, q be finite sequences of elements of $\mathbb{N}$, f be an element of $\mathbb{R}^*$, and i, n be natural numbers. Suppose p is simple vertex sequence at f, i, n and q is simple vertex sequence at f, i, n. Then $p = q$.

Let G be a graph, let p be a finite sequence of elements of the edges of G, and let v_5 be a finite sequence. We say that p is oriented edge sequence at v_5 if and only if:

(Def. 19) $\operatorname{len} v_5 = \operatorname{len} p + 1$ and for every n such that $1 \leqslant n$ and $n \leqslant \operatorname{len} p$ holds (the source of $G)(p(n)) = v_5(n)$ and (the target of $G)(p(n)) = v_5(n+1)$.

One can prove the following two propositions:

(51) Let G be an oriented graph, v_5 be a finite sequence, and p, q be oriented chains of G. Suppose p is oriented edge sequence at v_5 and q is oriented edge sequence at v_5. Then $p = q$.

(52) Let G be a graph, v_6, v_7 be finite sequences, and p be an oriented chain of G. Suppose p is oriented edge sequence at v_6 and oriented edge sequence at v_7 and $\operatorname{len} p \geqslant 1$. Then $v_6 = v_7$.

8. Data Structure for Dijkstra's Shortest Path Algorithm

Let f be an element of $\mathbb{R}^*$, let G be an oriented graph, let n be a natural number, and let W be a function from the edges of G into $\mathbb{R}_{\geqslant 0}$. We say that f is input of Dijkstra algorithm G to n in W if and only if the conditions (Def. 20) are satisfied.

(Def. 20)(i) $\operatorname{len} f = n \cdot n + 3 \cdot n + 1$,

(ii) $\operatorname{Seg} n =$ the vertices of G,

(iii) for every i such that $1 \leqslant i$ and $i \leqslant n$ holds $f(i) = 1$ and $f(2 \cdot n + i) = 0$,

(iv) $f(n+1) = 0$,

(v) for every i such that $2 \leqslant i$ and $i \leqslant n$ holds $f(n+i) = -1$, and

(vi) for all vertices i, j of G and for all k, m such that $k = i$ and $m = j$ holds $f(2 \cdot n + n \cdot k + m) = \mathrm{Weight}(i, j, W)$.

9. The Definition of Dijkstra's Shortest Path Algorithm

Let n be a natural number. The functor $\mathrm{DijkstraAlgorithm}\, n$ yielding an element of $(\mathbb{R}^*)^{\mathbb{R}^*}$ is defined as follows:

(Def. 21) $\mathrm{DijkstraAlgorithm}\, n = \mathrm{WhileDo}(\mathrm{Relax}\, n \cdot \mathrm{findmin}\, n, n)$.

10. Justifying the Correctness of Dijkstra's Shortest Path Algorithm

For simplicity, we adopt the following rules: p is a finite sequence of elements of $\mathbb{N}$, G is a finite oriented graph, P, Q are oriented chains of G, W is a function from the edges of G into $\mathbb{R}_{\geqslant 0}$, and v_1, v_2 are vertices of G.

We now state the proposition

(53) Suppose f is input of Dijkstra algorithm G to n in W and $v_1 = 1$ and $1 \neq v_2$ and $v_2 = i$ and $n \geqslant 1$ and $g = (\mathrm{DijkstraAlgorithm}\, n)(f)$. Then

(i) the vertices of $G = \mathrm{UsedVx}(g, n) \cup \mathrm{UnusedVx}(g, n)$,

(ii) if $v_2 \in \mathrm{UsedVx}(g, n)$, then there exist p, P such that p is simple vertex sequence at g, i, n and P is oriented edge sequence at p and shortest path from v_1 to v_2 in W and $\mathrm{cost}(P, W) = g(2 \cdot n + i)$, and

(iii) if $v_2 \in \mathrm{UnusedVx}(g, n)$, then there exists no Q which is oriented path from v_1 to v_2.

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Real Linear Space of Real Sequences

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Summary. The article is a continuation of [14]. As the example of real linear spaces, we introduce the arithmetic addition in the set of real sequences and also introduce the multiplication. This set has the arithmetic structure which depends on these arithmetic operations.

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The notation and terminology used here are introduced in the following papers: [12], [15], [5], [11], [6], [16], [2], [4], [3], [14], [13], [9], [8], [7], [10], and [1].

The non empty set the set of real sequences is defined by:

(Def. 1) For every set x holds $x \in$ the set of real sequences iff x is a sequence of real numbers.

Let a be a set. Let us assume that $a \in$ the set of real sequences. The functor $\mathrm{id}_{\mathrm{seq}}(a)$ yields a sequence of real numbers and is defined by:

(Def. 2) $\mathrm{id}_{\mathrm{seq}}(a) = a$.

Let a be a set. Let us assume that $a \in \mathbb{R}$. The functor $\mathrm{id}_{\mathbb{R}}(a)$ yielding a real number is defined by:

(Def. 3) $\mathrm{id}_{\mathbb{R}}(a) = a$.

We now state two propositions:

(1) There exists a binary operation A_1 on the set of real sequences such that for all elements a, b of the set of real sequences holds $A_1(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$ and A_1 is commutative and associative.

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(2) There exists a function f from $[:\mathbb{R}, \text{the set of real sequences}:]$ into the set of real sequences such that for all sets r, x if $r \in \mathbb{R}$ and $x \in$ the set of real sequences, then $f(\langle r, x \rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The binary operation $\mathrm{add}_{\mathrm{seq}}$ on the set of real sequences is defined as follows:

(Def. 4) For all elements a, b of the set of real sequences holds $\mathrm{add}_{\mathrm{seq}}(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$.

The function $\mathrm{mult}_{\mathrm{seq}}$ from $[:\mathbb{R}, \text{the set of real sequences}:]$ into the set of real sequences is defined by:

(Def. 5) For all sets r, x such that $r \in \mathbb{R}$ and $x \in$ the set of real sequences holds $\mathrm{mult}_{\mathrm{seq}}(\langle r, x \rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The element Zeroseq of the set of real sequences is defined by:

(Def. 6) For every natural number n holds $(\mathrm{id}_{\mathrm{seq}}(\mathrm{Zeroseq}))(n) = 0$.

One can prove the following propositions:

(3) For every sequence x of real numbers holds $\mathrm{id}_{\mathrm{seq}}(x) = x$.

(4) For all vectors v, w of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $v + w = \mathrm{id}_{\mathrm{seq}}(v) + \mathrm{id}_{\mathrm{seq}}(w)$.

(5) For every real number r and for every vector v of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $r \cdot v = r\ \mathrm{id}_{\mathrm{seq}}(v)$.

One can verify that $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ is Abelian.

We now state several propositions:

(6) For all vectors u, v, w of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $(u + v) + w = u + (v + w)$.

(7) For every vector v of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $v + 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle} = v$.

(8) Let v be a vector of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$. Then there exists a vector w of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ such that $v + w = 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle}$.

(9) For every real number a and for all vectors v, w of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $a \cdot (v + w) = a \cdot v + a \cdot w$.

(10) For all real numbers a, b and for every vector v of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $(a + b) \cdot v = a \cdot v + b \cdot v$.

(11) For all real numbers a, b and for every vector v of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

(12) For every vector v of $\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$ holds $1 \cdot v = v$.

The real linear space the linear space of real sequences is defined by:

(Def. 7) The linear space of real sequences $= \langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Add\_}(X_1, X)$ yielding a binary operation on X_1 is defined by:

(Def. 8) $\mathrm{Add\_}(X_1, X) = (\text{the addition of } X) \restriction [\!: X_1, X_1 :\!]$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Mult\_}(X_1, X)$ yielding a function from $[\!: \mathbb{R}, X_1 :\!]$ into X_1 is defined as follows:

(Def. 9) $\mathrm{Mult\_}(X_1, X) = (\text{the external multiplication of } X) \restriction [\!: \mathbb{R}, X_1 :\!]$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Zero\_}(X_1, X)$ yields an element of X_1 and is defined by:

(Def. 10) $\mathrm{Zero\_}(X_1, X) = 0_X$.

We now state the proposition

(13) Let V be a real linear space and V_1 be a subset of the carrier of V. Suppose V_1 is linearly closed and non empty. Then $\langle V_1, \mathrm{Zero\_}(V_1, V), \mathrm{Add\_}(V_1, V), \mathrm{Mult\_}(V_1, V)\rangle$ is a subspace of V.

The subset the set of l2-real sequences of the carrier of the linear space of real sequences is defined by the conditions (Def. 11).

(Def. 11)(i) The set of l2-real sequences is non empty, and

(ii) for every set x holds $x \in$ the set of l2-real sequences iff $x \in$ the set of real sequences and $\mathrm{id_{seq}}(x)\ \mathrm{id_{seq}}(x)$ is summable.

Next we state several propositions:

(14) The set of l2-real sequences is linearly closed and the set of l2-real sequences is non empty.

(15) $\langle$the set of l2-real sequences, $\mathrm{Zero\_}$(the set of l2-real sequences, the linear space of real sequences), $\mathrm{Add\_}$(the set of l2-real sequences, the linear space of real sequences), $\mathrm{Mult\_}$(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a subspace of the linear space of real sequences.

(16) $\langle$the set of l2-real sequences, $\mathrm{Zero\_}$(the set of l2-real sequences, the linear space of real sequences), $\mathrm{Add\_}$(the set of l2-real sequences, the linear space of real sequences), $\mathrm{Mult\_}$(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a real linear space.

(17)(i) The carrier of the linear space of real sequences $=$ the set of real sequences,

(ii) for every set x holds x is an element of the carrier of the linear space of real sequences iff x is a sequence of real numbers,

(iii) for every set x holds x is a vector of the linear space of real sequences iff x is a sequence of real numbers,

(iv) for every vector u of the linear space of real sequences holds $u = \mathrm{id_{seq}}(u)$,

(v) for all vectors u, v of the linear space of real sequences holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$, and

(vi) for every real number r and for every vector u of the linear space of real sequences holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$.

(18) There exists a function f from $[\!:$ the set of l2-real sequences, the set of l2-real sequences $:\!]$ into $\mathbb{R}$ such that for all sets x, y if $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences, then $f(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

The function $\mathrm{scalar}_{\mathrm{seq}}$ from $[\!:$ the set of l2-real sequences, the set of l2-real sequences $:\!]$ into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x, y be sets. Suppose $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences. Then $\mathrm{scalar}_{\mathrm{seq}}(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

One can check that $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}\rangle$ is non empty.

The non empty unitary space structure l2-Space is defined by the condition (Def. 13).

(Def. 13) l2-Space $=$ $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}\rangle$.

One can prove the following propositions:

(19) Let l be a unitary space structure. Suppose $\langle$the carrier of l, the zero of l, the addition of l, the external multiplication of $l\rangle$ is a real linear space. Then l is a real linear space.

(20) Let r_1 be a sequence of real numbers. If for every natural number n holds $r_1(n) = 0$, then r_1 is summable and $\sum r_1 = 0$.

(21) Let r_1 be a sequence of real numbers. Suppose for every natural number n holds $0 \leqslant r_1(n)$ and r_1 is summable and $\sum r_1 = 0$. Let n be a natural number. Then $r_1(n) = 0$.

Let us observe that l2-Space is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

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Hilbert Space of Real Sequences

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Summary. A continuation of [16]. As the example of real unitary spaces, we introduce the arithmetic addition and multiplication in the set of square sum able real sequences and introduce the scaler products also. This set has the structure of the Hilbert space.

MML Identifier: `RSSPACE2`.

The articles [15], [17], [3], [14], [5], [18], [1], [2], [16], [12], [9], [10], [11], [8], [6], [7], [13], and [4] provide the terminology and notation for this paper.

1. Hilbert Space of Real Sequences

One can prove the following two propositions:

(1) The carrier of l2-Space = the set of l2-real sequences and for every set x holds x is an element of the carrier of l2-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(x)$ is summable and for every set x holds x is a vector of l2-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(x)$ is summable and $0_{\text{l2-Space}} = \mathrm{Zeroseq}$ and for every vector u of l2-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l2-Space holds $u+v = \mathrm{id}_{\mathrm{seq}}(u)+\mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of l2-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of l2-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l2-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for all vectors v, w of

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12-Space holds $\mathrm{id}_{\mathrm{seq}}(v)\ \mathrm{id}_{\mathrm{seq}}(w)$ is summable and for all vectors v, w of 12-Space holds $(v|w) = \sum(\mathrm{id}_{\mathrm{seq}}(v)\ \mathrm{id}_{\mathrm{seq}}(w))$.

(2) Let x, y, z be points of 12-Space and a be a real number. Then $(x|x) = 0$ iff $x = 0_{\text{l2-Space}}$ and $0 \leqslant (x|x)$ and $(x|y) = (y|x)$ and $((x + y)|z) = (x|z) + (y|z)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

Let us note that 12-Space is real unitary space-like.

One can prove the following proposition

(3) For every sequence v_1 of 12-Space such that v_1 is a Cauchy sequence holds v_1 is convergent.

Let us mention that 12-Space is Hilbert and complete.

2. Miscellaneous

We now state several propositions:

(4) Let r_1 be a sequence of real numbers. Suppose for every natural number n holds $0 \leqslant r_1(n)$ and r_1 is summable. Then

(i) for every natural number n holds $r_1(n) \leqslant (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$,

(ii) for every natural number n holds $0 \leqslant (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$,

(iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant \sum r_1$, and

(iv) for every natural number n holds $r_1(n) \leqslant \sum r_1$.

(5) For all real numbers x, y holds $(x + y) \cdot (x + y) \leqslant 2 \cdot x \cdot x + 2 \cdot y \cdot y$ and for all real numbers x, y holds $x \cdot x \leqslant 2 \cdot (x - y) \cdot (x - y) + 2 \cdot y \cdot y$.

(6) Let e be a real number and s_1 be a sequence of real numbers. Suppose s_1 is convergent and there exists a natural number k such that for every natural number i such that $k \leqslant i$ holds $s_1(i) \leqslant e$. Then $\lim s_1 \leqslant e$.

(7) Let c be a real number and s_1 be a sequence of real numbers. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = (s_1(i) - c) \cdot (s_1(i) - c)$. Then r_1 is convergent and $\lim r_1 = (\lim s_1 - c) \cdot (\lim s_1 - c)$.

(8) Let c be a real number and s_1, s_2 be sequences of real numbers. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = (s_1(i) - c) \cdot (s_1(i) - c) + s_2(i)$. Then r_1 is convergent and $\lim r_1 = (\lim s_1 - c) \cdot (\lim s_1 - c) + \lim s_2$.

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Intuitionistic Propositional Calculus in the Extended Framework with Modal Operator. Part I

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Summary. In this paper, we develop intuitionistic propositional calculus IPC in the extended language with single modal operator. The formulation that we adopt in this paper is very useful not only to formalize the calculus but also to do a number of logics with essentially propositional character. In addition, it is much simpler than the past formalization for modal logic. In the first section, we give the mentioned formulation which the author heavily owes to the formalism of Adam Grabowski's [4]. After the theoretical development of the logic, we prove a number of valid formulas of IPC in the sections 2–4. The last two sections are devoted to present classical propositional calculus and modal calculus S4 in our framework, as a preparation for future study. In the forthcoming Part II of this paper, we shall prove, among others, a number of intuitionistically valid formulas with negation.

MML Identifier: `INTPRO_1`.

The articles [6], [7], [5], [8], [3], [1], and [2] provide the notation and terminology for this paper.

1. Intuitionistic Propositional Calculus IPC in the Extended Language with Modal Operator

Let E be a set. We say that E has FALSUM if and only if:

(Def. 1) $\langle 0 \rangle \in E$.

Let E be a set. We say that E has intuitionistic implication if and only if:

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(Def. 2) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 1 \rangle ^\frown p ^\frown q \in E$.

Let E be a set. We say that E has intuitionistic conjunction if and only if:

(Def. 3) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 2 \rangle ^\frown p ^\frown q \in E$.

Let E be a set. We say that E has intuitionistic disjunction if and only if:

(Def. 4) For all finite sequences p, q such that $p \in E$ and $q \in E$ holds $\langle 3 \rangle ^\frown p ^\frown q \in E$.

Let E be a set. We say that E has intuitionistic propositional variables if and only if:

(Def. 5) For every natural number n holds $\langle 5 + 2 \cdot n \rangle \in E$.

Let E be a set. We say that E has intuitionistic modal operator if and only if:

(Def. 6) For every finite sequence p such that $p \in E$ holds $\langle 6 \rangle ^\frown p \in E$.

Let E be a set. We say that E is MC-closed if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) $E \subseteq \mathbb{N}^*$, and

(ii) E has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator.

One can check that every set which is MC-closed is also non empty and has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator and every subset of $\mathbb{N}^*$ which has FALSUM, intuitionistic implication, intuitionistic conjunction, intuitionistic disjunction, intuitionistic propositional variables, and intuitionistic modal operator is also MC-closed.

The set MC-wff is defined by:

(Def. 8) MC-wff is MC-closed and for every set E such that E is MC-closed holds MC-wff $\subseteq E$.

One can verify that MC-wff is MC-closed.

Let us note that there exists a set which is MC-closed and non empty.

One can verify that every element of MC-wff is relation-like and function-like.

Let us note that every element of MC-wff is finite sequence-like.

A MC-formula is an element of MC-wff.

The MC-formula FALSUM is defined as follows:

(Def. 9) FALSUM $= \langle 0 \rangle$.

Let p, q be elements of MC-wff. The functor $p \Rightarrow q$ yields a MC-formula and is defined as follows:

(Def. 10) $p \Rightarrow q = \langle 1 \rangle \frown p \frown q$.

The functor $p \wedge q$ yields a MC-formula and is defined as follows:

(Def. 11) $p \wedge q = \langle 2 \rangle \frown p \frown q$.

The functor $p \vee q$ yielding a MC-formula is defined by:

(Def. 12) $p \vee q = \langle 3 \rangle \frown p \frown q$.

Let p be an element of MC-wff. The functor $\mathrm{Nes}(p)$ yielding a MC-formula is defined by:

(Def. 13) $\mathrm{Nes}(p) = \langle 6 \rangle \frown p$.

We use the following convention: T, X, Y denote subsets of MC-wff and p, q, r, s denote elements of MC-wff.

Let T be a subset of MC-wff. We say that T is IPC theory if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and $\mathrm{FALSUM} \Rightarrow p \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

Let us consider X. The functor $\mathrm{CnIPC}(X)$ yielding a subset of MC-wff is defined as follows:

(Def. 15) $r \in \mathrm{CnIPC}(X)$ iff for every T such that T is IPC theory and $X \subseteq T$ holds $r \in T$.

The subset IPC-Taut of MC-wff is defined as follows:

(Def. 16) $\text{IPC-Taut} = \mathrm{CnIPC}(\emptyset_{\text{MC-wff}})$.

Let p be an element of MC-wff. The functor $\mathrm{neg}(p)$ yields a MC-formula and is defined as follows:

(Def. 17) $\mathrm{neg}(p) = p \Rightarrow \mathrm{FALSUM}$.

The MC-formula IVERUM is defined by:

(Def. 18) $\mathrm{IVERUM} = \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM}$.

The following propositions are true:

(1) $p \Rightarrow (q \Rightarrow p) \in \mathrm{CnIPC}(X)$.

(2) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{CnIPC}(X)$.

(3) $p \wedge q \Rightarrow p \in \mathrm{CnIPC}(X)$.

(4) $p \wedge q \Rightarrow q \in \mathrm{CnIPC}(X)$.

(5) $p \Rightarrow (q \Rightarrow p \wedge q) \in \mathrm{CnIPC}(X)$.

(6) $p \Rightarrow p \vee q \in \mathrm{CnIPC}(X)$.

(7) $q \Rightarrow p \vee q \in \mathrm{CnIPC}(X)$.

(8) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \mathrm{CnIPC}(X)$.

(9) $\mathrm{FALSUM} \Rightarrow p \in \mathrm{CnIPC}(X)$.

(10) If $p \in \mathrm{CnIPC}(X)$ and $p \Rightarrow q \in \mathrm{CnIPC}(X)$, then $q \in \mathrm{CnIPC}(X)$.

(11) If T is IPC theory and $X \subseteq T$, then $\mathrm{CnIPC}(X) \subseteq T$.

(12) $X \subseteq \mathrm{CnIPC}(X)$.

(13) If $X \subseteq Y$, then $\mathrm{CnIPC}(X) \subseteq \mathrm{CnIPC}(Y)$.

(14) $\mathrm{CnIPC}(\mathrm{CnIPC}(X)) = \mathrm{CnIPC}(X)$.

Let X be a subset of MC-wff. Observe that $\mathrm{CnIPC}(X)$ is IPC theory.
The following propositions are true:

(15) T is IPC theory iff $\mathrm{CnIPC}(T) = T$.

(16) If T is IPC theory, then IPC-Taut $\subseteq T$.

One can verify that IPC-Taut is IPC theory.

2. Formulas Provable in IPC: Implication

We now state a number of propositions:

(17) $p \Rightarrow p \in$ IPC-Taut.

(18) If $q \in$ IPC-Taut, then $p \Rightarrow q \in$ IPC-Taut.

(19) IVERUM $\in$ IPC-Taut.

(20) $(p \Rightarrow q) \Rightarrow (p \Rightarrow p) \in$ IPC-Taut.

(21) $(q \Rightarrow p) \Rightarrow (p \Rightarrow p) \in$ IPC-Taut.

(22) $(q \Rightarrow r) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(23) If $p \Rightarrow (q \Rightarrow r) \in$ IPC-Taut, then $q \Rightarrow (p \Rightarrow r) \in$ IPC-Taut.

(24) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(25) If $p \Rightarrow q \in$ IPC-Taut, then $(q \Rightarrow r) \Rightarrow (p \Rightarrow r) \in$ IPC-Taut.

(26) If $p \Rightarrow q \in$ IPC-Taut and $q \Rightarrow r \in$ IPC-Taut, then $p \Rightarrow r \in$ IPC-Taut.

(27) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((s \Rightarrow q) \Rightarrow (p \Rightarrow (s \Rightarrow r))) \in$ IPC-Taut.

(28) $((p \Rightarrow q) \Rightarrow r) \Rightarrow (q \Rightarrow r) \in$ IPC-Taut.

(29) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)) \in$ IPC-Taut.

(30) $(p \Rightarrow (p \Rightarrow q)) \Rightarrow (p \Rightarrow q) \in$ IPC-Taut.

(31) $q \Rightarrow ((q \Rightarrow p) \Rightarrow p) \in$ IPC-Taut.

(32) If $s \Rightarrow (q \Rightarrow p) \in$ IPC-Taut and $q \in$ IPC-Taut, then $s \Rightarrow p \in$ IPC-Taut.

3. Formulas Provable in IPC: Conjunction

The following propositions are true:

(33) $p \Rightarrow p \wedge p \in$ IPC-Taut.

(34) $p \wedge q \in$ IPC-Taut iff $p \in$ IPC-Taut and $q \in$ IPC-Taut.

(35) $p \wedge q \in$ IPC-Taut iff $q \wedge p \in$ IPC-Taut.

(36) $(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r)) \in$ IPC-Taut.

(37) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \wedge q \Rightarrow r) \in$ IPC-Taut.

(38) $(r \Rightarrow p) \Rightarrow ((r \Rightarrow q) \Rightarrow (r \Rightarrow p \wedge q)) \in$ IPC-Taut.

(39) $(p \Rightarrow q) \wedge p \Rightarrow q \in$ IPC-Taut.

(40) $(p \Rightarrow q) \wedge p \wedge s \Rightarrow q \in$ IPC-Taut.

(41) $(q \Rightarrow s) \Rightarrow (p \wedge q \Rightarrow s) \in$ IPC-Taut.

(42) $(q \Rightarrow s) \Rightarrow (q \wedge p \Rightarrow s) \in$ IPC-Taut.

(43) $(p \wedge s \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ IPC-Taut.

(44) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow q \wedge s) \in$ IPC-Taut.

(45) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow q \wedge s \in$ IPC-Taut.

(46) $p \wedge q \Rightarrow q \wedge p \in$ IPC-Taut.

(47) $(p \Rightarrow q) \wedge (p \wedge s) \Rightarrow s \wedge q \in$ IPC-Taut.

(48) $(p \Rightarrow q) \Rightarrow (p \wedge s \Rightarrow s \wedge q) \in$ IPC-Taut.

(49) $(p \Rightarrow q) \Rightarrow (s \wedge p \Rightarrow s \wedge q) \in$ IPC-Taut.

(50) $p \wedge (s \wedge q) \Rightarrow p \wedge (q \wedge s) \in$ IPC-Taut.

(51) $(p \Rightarrow q) \wedge (p \Rightarrow s) \Rightarrow (p \Rightarrow q \wedge s) \in$ IPC-Taut.

(52) $p \wedge q \wedge s \Rightarrow p \wedge (q \wedge s) \in$ IPC-Taut.

(53) $p \wedge (q \wedge s) \Rightarrow p \wedge q \wedge s \in$ IPC-Taut.

4. Formulas Provable in IPC: Disjunction

We now state a number of propositions:

(54) $p \vee p \Rightarrow p \in$ IPC-Taut.

(55) If $p \in$ IPC-Taut or $q \in$ IPC-Taut, then $p \vee q \in$ IPC-Taut.

(56) $p \vee q \Rightarrow q \vee p \in$ IPC-Taut.

(57) $p \vee q \in$ IPC-Taut iff $q \vee p \in$ IPC-Taut.

(58) $(p \Rightarrow q) \Rightarrow (p \Rightarrow q \vee s) \in$ IPC-Taut.

(59) $(p \Rightarrow q) \Rightarrow (p \Rightarrow s \vee q) \in$ IPC-Taut.

(60) $(p \Rightarrow q) \Rightarrow (p \vee s \Rightarrow q \vee s) \in$ IPC-Taut.

(61) If $p \Rightarrow q \in$ IPC-Taut, then $p \vee s \Rightarrow q \vee s \in$ IPC-Taut.

(62) $(p \Rightarrow q) \Rightarrow (s \vee p \Rightarrow s \vee q) \in$ IPC-Taut.

(63) If $p \Rightarrow q \in$ IPC-Taut, then $s \vee p \Rightarrow s \vee q \in$ IPC-Taut.

(64) $p \vee (q \vee s) \Rightarrow q \vee (p \vee s) \in$ IPC-Taut.

(65) $p \vee (q \vee s) \Rightarrow p \vee q \vee s \in$ IPC-Taut.

(66) $p \vee q \vee s \Rightarrow p \vee (q \vee s) \in$ IPC-Taut.

5. Classical Propositional Calculus CPC

We use the following convention: T, X, Y are subsets of MC-wff and p, q, r are elements of MC-wff.

Let T be a subset of MC-wff. We say that T is CPC theory if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and FALSUM $\Rightarrow p \in T$ and $p \vee (p \Rightarrow \text{FALSUM}) \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$.

One can prove the following proposition

(67) If T is CPC theory, then T is IPC theory.

Let us consider X. The functor CnCPC(X) yielding a subset of MC-wff is defined by:

(Def. 20) $r \in \text{CnCPC}(X)$ iff for every T such that T is CPC theory and $X \subseteq T$ holds $r \in T$.

The subset CPC-Taut of MC-wff is defined by:

(Def. 21) $\text{CPC-Taut} = \text{CnCPC}(\emptyset_{\text{MC-wff}})$.

Next we state several propositions:

(68) $\text{CnIPC}(X) \subseteq \text{CnCPC}(X)$.

(69) $p \Rightarrow (q \Rightarrow p) \in \text{CnCPC}(X)$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \text{CnCPC}(X)$ and $p \wedge q \Rightarrow p \in \text{CnCPC}(X)$ and $p \wedge q \Rightarrow q \in \text{CnCPC}(X)$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in \text{CnCPC}(X)$ and $p \Rightarrow p \vee q \in \text{CnCPC}(X)$ and $q \Rightarrow p \vee q \in \text{CnCPC}(X)$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \text{CnCPC}(X)$ and FALSUM $\Rightarrow p \in \text{CnCPC}(X)$ and $p \vee (p \Rightarrow \text{FALSUM}) \in \text{CnCPC}(X)$.

(70) If $p \in \text{CnCPC}(X)$ and $p \Rightarrow q \in \text{CnCPC}(X)$, then $q \in \text{CnCPC}(X)$.

(71) If T is CPC theory and $X \subseteq T$, then $\text{CnCPC}(X) \subseteq T$.

(72) $X \subseteq \text{CnCPC}(X)$.

(73) If $X \subseteq Y$, then $\text{CnCPC}(X) \subseteq \text{CnCPC}(Y)$.

(74) $\text{CnCPC}(\text{CnCPC}(X)) = \text{CnCPC}(X)$.

Let X be a subset of MC-wff. Note that CnCPC(X) is CPC theory.

Next we state two propositions:

(75) T is CPC theory iff $\text{CnCPC}(T) = T$.

(76) If T is CPC theory, then $\text{CPC-Taut} \subseteq T$.

Let us note that CPC-Taut is CPC theory.

The following proposition is true

(77) $\text{IPC-Taut} \subseteq \text{CPC-Taut}$.

6. Modal Calculus S4

We use the following convention: T, X, Y are subsets of MC-wff and p, q, r are elements of MC-wff.

Let T be a subset of MC-wff. We say that T is S4 theory if and only if the condition (Def. 22) is satisfied.

(Def. 22) Let p, q, r be elements of MC-wff. Then $p \Rightarrow (q \Rightarrow p) \in T$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in T$ and $p \wedge q \Rightarrow p \in T$ and $p \wedge q \Rightarrow q \in T$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in T$ and $p \Rightarrow p \vee q \in T$ and $q \Rightarrow p \vee q \in T$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in T$ and $\mathrm{FALSUM} \Rightarrow p \in T$ and $p \vee (p \Rightarrow \mathrm{FALSUM}) \in T$ and $\mathrm{Nes}(p \Rightarrow q) \Rightarrow (\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(q)) \in T$ and $\mathrm{Nes}(p) \Rightarrow p \in T$ and $\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(\mathrm{Nes}(p)) \in T$ and if $p \in T$ and $p \Rightarrow q \in T$, then $q \in T$ and if $p \in T$, then $\mathrm{Nes}(p) \in T$.

Next we state two propositions:

(78) If T is S4 theory, then T is CPC theory.

(79) If T is S4 theory, then T is IPC theory.

Let us consider X. The functor $\mathrm{CnS4}(X)$ yielding a subset of MC-wff is defined by:

(Def. 23) $r \in \mathrm{CnS4}(X)$ iff for every T such that T is S4 theory and $X \subseteq T$ holds $r \in T$.

The subset S4-Taut of MC-wff is defined by:

(Def. 24) $\text{S4-Taut} = \mathrm{CnS4}(\emptyset_{\text{MC-wff}})$.

Next we state a number of propositions:

(80) $\mathrm{CnCPC}(X) \subseteq \mathrm{CnS4}(X)$.

(81) $\mathrm{CnIPC}(X) \subseteq \mathrm{CnS4}(X)$.

(82) $p \Rightarrow (q \Rightarrow p) \in \mathrm{CnS4}(X)$ and $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in \mathrm{CnS4}(X)$ and $p \wedge q \Rightarrow p \in \mathrm{CnS4}(X)$ and $p \wedge q \Rightarrow q \in \mathrm{CnS4}(X)$ and $p \Rightarrow (q \Rightarrow p \wedge q) \in \mathrm{CnS4}(X)$ and $p \Rightarrow p \vee q \in \mathrm{CnS4}(X)$ and $q \Rightarrow p \vee q \in \mathrm{CnS4}(X)$ and $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)) \in \mathrm{CnS4}(X)$ and $\mathrm{FALSUM} \Rightarrow p \in \mathrm{CnS4}(X)$ and $p \vee (p \Rightarrow \mathrm{FALSUM}) \in \mathrm{CnS4}(X)$.

(83) If $p \in \mathrm{CnS4}(X)$ and $p \Rightarrow q \in \mathrm{CnS4}(X)$, then $q \in \mathrm{CnS4}(X)$.

(84) $\mathrm{Nes}(p \Rightarrow q) \Rightarrow (\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(q)) \in \mathrm{CnS4}(X)$.

(85) $\mathrm{Nes}(p) \Rightarrow p \in \mathrm{CnS4}(X)$.

(86) $\mathrm{Nes}(p) \Rightarrow \mathrm{Nes}(\mathrm{Nes}(p)) \in \mathrm{CnS4}(X)$.

(87) If $p \in \mathrm{CnS4}(X)$, then $\mathrm{Nes}(p) \in \mathrm{CnS4}(X)$.

(88) If T is S4 theory and $X \subseteq T$, then $\mathrm{CnS4}(X) \subseteq T$.

(89) $X \subseteq \mathrm{CnS4}(X)$.

(90) If $X \subseteq Y$, then $\mathrm{CnS4}(X) \subseteq \mathrm{CnS4}(Y)$.

(91) $\mathrm{CnS4}(\mathrm{CnS4}(X)) = \mathrm{CnS4}(X)$.

Let X be a subset of MC-wff. One can verify that CnS4(X) is S4 theory.

Next we state two propositions:

(92) T is S4 theory iff $\mathrm{CnS4}(T) = T$.

(93) If T is S4 theory, then S4-Taut $\subseteq T$.

Let us note that S4-Taut is S4 theory.

The following propositions are true:

(94) CPC-Taut $\subseteq$ S4-Taut.

(95) IPC-Taut $\subseteq$ S4-Taut.

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Some Properties for Convex Combinations

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Summary. This is a continuation of [6]. In this article, we proved that convex combination on convex family is convex.

MML Identifier: CONVEX2.

The notation and terminology used in this paper are introduced in the following articles: [13], [18], [12], [8], [2], [19], [3], [5], [1], [10], [4], [17], [16], [15], [14], [11], [7], [6], and [9].

1. Convex Combinations on Convex Family

The following propositions are true:

(1) For every non empty RLS structure V and for all convex subsets M, N of V holds $M \cap N$ is convex.

(2) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, F be a finite sequence of elements of the carrier of V, and B be a finite sequence of elements of $\mathbb{R}$. Suppose $M = \{u; u$ ranges over vectors of V: $\bigwedge_{i\,:\,\text{natural number}} (i \in \operatorname{dom} F \cap \operatorname{dom} B \Rightarrow \bigvee_{v\,:\,\text{vector of } V} (v = F(i) \wedge (u|v) \leqslant B(i)))\}$. Then M is convex.

(3) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, F be a finite sequence of elements of the carrier of V, and B be a finite sequence of elements of $\mathbb{R}$. Suppose $M = \{u; u$ ranges over vectors of V: $\bigwedge_{i\,:\,\text{natural number}} (i \in \operatorname{dom} F \cap \operatorname{dom} B \Rightarrow \bigvee_{v\,:\,\text{vector of } V} (v = F(i) \wedge (u|v) < B(i)))\}$. Then M is convex.

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(4) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, F be a finite sequence of elements of the carrier of V, and B be a finite sequence of elements of $\mathbb{R}$. Suppose $M = \{u; u$ ranges over vectors of V: $\bigwedge_{i\,:\,\text{natural number}} (i \in \operatorname{dom} F \cap \operatorname{dom} B \Rightarrow \bigvee_{v\,:\,\text{vector of } V} (v = F(i) \wedge (u|v) \geqslant B(i)))\}$. Then M is convex.

(5) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, F be a finite sequence of elements of the carrier of V, and B be a finite sequence of elements of $\mathbb{R}$. Suppose $M = \{u; u$ ranges over vectors of V: $\bigwedge_{i\,:\,\text{natural number}} (i \in \operatorname{dom} F \cap \operatorname{dom} B \Rightarrow \bigvee_{v\,:\,\text{vector of } V} (v = F(i) \wedge (u|v) > B(i)))\}$. Then M is convex.

(6) Let V be a real linear space and M be a subset of V. Then for every subset N of V and for every linear combination L of N such that L is convex and $N \subseteq M$ holds $\sum L \in M$ if and only if M is convex.

Let V be a real linear space and let M be a subset of V. The functor LC_M yielding a set is defined as follows:

(Def. 1) For every set L holds $L \in \mathrm{LC}_M$ iff L is a linear combination of M.

Let V be a real linear space. Observe that there exists a linear combination of V which is convex.

Let V be a real linear space. A convex combination of V is a convex linear combination of V.

Let V be a real linear space and let M be a non empty subset of V. One can verify that there exists a linear combination of M which is convex.

Let V be a real linear space and let M be a non empty subset of V. A convex combination of M is a convex linear combination of M.

The following propositions are true:

(7) For every real linear space V and for every subset M of V holds Convex-Family $M \neq \emptyset$.

(8) For every real linear space V and for every subset M of V holds $M \subseteq \operatorname{conv} M$.

(9) Let V be a real linear space, L_1, L_2 be convex combinations of V, and r be a real number. If $0 < r$ and $r < 1$, then $r \cdot L_1 + (1 - r) \cdot L_2$ is a convex combination of V.

(10) Let V be a real linear space, M be a non empty subset of V, L_1, L_2 be convex combinations of M, and r be a real number. If $0 < r$ and $r < 1$, then $r \cdot L_1 + (1 - r) \cdot L_2$ is a convex combination of M.

(11) For every real linear space V holds there exists a linear combination of V which is convex.

(12) For every real linear space V and for every non empty subset M of V holds there exists a linear combination of M which is convex.

2. Miscellaneous

We now state several propositions:

(13) For every real linear space V and for all subspaces W_1, W_2 of V holds $\mathrm{Up}(W_1 + W_2) = \mathrm{Up}(W_1) + \mathrm{Up}(W_2)$.

(14) For every real linear space V and for all subspaces W_1, W_2 of V holds $\mathrm{Up}(W_1 \cap W_2) = \mathrm{Up}(W_1) \cap \mathrm{Up}(W_2)$.

(15) Let V be a real linear space, L_1, L_2 be convex combinations of V, and a, b be real numbers. Suppose $a \cdot b > 0$. Then the support of $a \cdot L_1 + b \cdot L_2 =$ (the support of $a \cdot L_1$) $\cup$ (the support of $b \cdot L_2$).

(16) Let F, G be functions. Suppose F and G are fiberwise equipotent. Let x_1, x_2 be sets. Suppose $x_1 \in \mathrm{dom}\, F$ and $x_2 \in \mathrm{dom}\, F$ and $x_1 \neq x_2$. Then there exist sets z_1, z_2 such that $z_1 \in \mathrm{dom}\, G$ and $z_2 \in \mathrm{dom}\, G$ and $z_1 \neq z_2$ and $F(x_1) = G(z_1)$ and $F(x_2) = G(z_2)$.

(17) Let V be a real linear space, L be a linear combination of V, and A be a subset of V. Suppose $A \subseteq$ the support of L. Then there exists a linear combination L_1 of V such that the support of $L_1 = A$.

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On Some Properties of Real Hilbert Space. Part II

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Summary. This paper is a continuation of our paper [21]. We give an analogue of the necessary and sufficient condition for summable set (i.e. the main theorem of [21]) with respect to summable set by a functional L in real Hilbert space. After presenting certain useful lemmas, we prove our main theorem that the summability for an orthonormal infinite set in real Hilbert space is equivalent to its summability with respect to the square of norm, say $H(x) = (x, x)$. Then we show that the square of norm H commutes with infinite sum operation if the orthonormal set under our consideration is summable. Our main theorem is due to [7].

MML Identifier: BHSP_7.

The articles [16], [18], [5], [14], [8], [3], [4], [19], [17], [11], [12], [13], [2], [6], [9], [15], [10], [1], [20], and [21] provide the notation and terminology for this paper.

1. Necessary and Sufficient Condition for Summable Set

In this paper X is a real unitary space and x, y are points of X.

The following propositions are true:

(1) Let Y be a subset of the carrier of X and L be a functional in X. Then Y is summable set by L if and only if for every real number e such that $0 < e$ there exists a finite subset Y_0 of the carrier of X such that Y_0 is non

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empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that Y_1 is non empty and $Y_1 \subseteq Y$ and Y_0 misses Y_1 holds $|\mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

(2) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let I be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} I$ and for every y such that $y \in S$ holds $I(y) = y$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (y|y)$. Then $(\mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X)|\,\mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(3) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let H be a functional in X. Suppose $S \subseteq \operatorname{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then $(\mathrm{Setsum}(S)|\,\mathrm{Setsum}(S)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(4) Let Y be an orthogonal family of X and Z be a subset of the carrier of X. If Z is a subset of Y, then Z is an orthogonal family of X.

(5) Let Y be an orthonormal family of X and Z be a subset of the carrier of X. If Z is a subset of Y, then Z is an orthonormal family of X.

2. Equivalence of Summability

Next we state three propositions:

(6) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X and H be a functional in X. Suppose $S \subseteq \operatorname{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then S is summable_set if and only if S is summable set by H.

(7) Let S be a subset of the carrier of X. Suppose S is non empty and summable_set. Let e be a real number. Suppose $0 < e$. Then there exists a finite subset Y_0 of the carrier of X such that

(i) Y_0 is non empty,

(ii) $Y_0 \subseteq S$, and

(iii) for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq S$ holds $|(\operatorname{sum} S|\operatorname{sum} S) - (\mathrm{Setsum}(Y_1)|\,\mathrm{Setsum}(Y_1))| < e$.

(8) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X. Suppose S is non empty. Let H be a functional

in X. Suppose $S \subseteq \mathrm{dom}\, H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. If S is summable_set, then $(\mathrm{sum}\, S|\, \mathrm{sum}\, S) = \mathrm{SumByfunc}(S, H)$.

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Inner Products and Angles of Complex Numbers

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Summary. An inner product of complex numbers is defined and used to characterize the (counter-clockwise) angle between (a,0) and (0,b) in the complex plane. For complex a, b and c we then define the (counter-clockwise) angle between (a,c) and (c, b) and prove theorems about the sum of internal and external angles of a triangle.

MML Identifier: `COMPLEX2`.

The papers [9], [13], [10], [12], [14], [3], [7], [15], [5], [6], [8], [11], [2], [1], and [4] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers a, b holds $-(a + bi) = -a + (-b)i$.

(2) For all real numbers a, b such that $b > 0$ there exists a real number r such that $r = b \cdot -\lfloor \frac{a}{b} \rfloor + a$ and $0 \leqslant r$ and $r < b$.

(3) Let a, b, c be real numbers. Suppose $a > 0$ and $b \geqslant 0$ and $c \geqslant 0$ and $b < a$ and $c < a$. Let i be an integer. If $b = c + a \cdot i$, then $b = c$.

(4) For all real numbers a, b holds $\sin(a - b) = \sin a \cdot \cos b - \cos a \cdot \sin b$ and $\cos(a - b) = \cos a \cdot \cos b + \sin a \cdot \sin b$.

(5) For every real number a holds $\sin(a - \pi) = -\sin(a)$ and $\cos(a - \pi) = -\cos(a)$.

(6) For every real number a holds $\sin(a - \pi) = -\sin a$ and $\cos(a - \pi) = -\cos a$.

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(7) For all real numbers a, b such that $a \in]0, \frac{\pi}{2}[$ and $b \in]0, \frac{\pi}{2}[$ holds $a < b$ iff $\sin a < \sin b$.

(8) For all real numbers a, b such that $a \in]\frac{\pi}{2}, \pi[$ and $b \in]\frac{\pi}{2}, \pi[$ holds $a < b$ iff $\sin a > \sin b$.

(9) For every real number a and for every integer i holds $\sin a = \sin(2 \cdot \pi \cdot i + a)$.

(10) For every real number a and for every integer i holds $\cos a = \cos(2 \cdot \pi \cdot i + a)$.

(11) For every real number a such that $\sin a = 0$ holds $\cos a \neq 0$.

(12) For all real numbers a, b such that $0 \leqslant a$ and $a < 2 \cdot \pi$ and $0 \leqslant b$ and $b < 2 \cdot \pi$ and $\sin a = \sin b$ and $\cos a = \cos b$ holds $a = b$.

2. More on the Argument of a Complex Number

Let us observe that $\mathbb{C}_{\mathrm{F}}$ is non empty.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Ftize}(z)$ yields an element of the carrier of $\mathbb{C}_{\mathrm{F}}$ and is defined as follows:

(Def. 1) $\mathrm{Ftize}(z) = z$.

We now state four propositions:

(13) For every element z of $\mathbb{C}$ holds $\Re(z) = \Re(\mathrm{Ftize}(z))$ and $\Im(z) = \Im(\mathrm{Ftize}(z))$.

(14) For all elements x, y of $\mathbb{C}$ holds $\mathrm{Ftize}(x + y) = \mathrm{Ftize}(x) + \mathrm{Ftize}(y)$.

(15) For every element z of $\mathbb{C}$ holds $z = 0_{\mathbb{C}}$ iff $\mathrm{Ftize}(z) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element z of $\mathbb{C}$ holds $|z| = |\,\mathrm{Ftize}(z)|$.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{Arg}\, z$ yields a real number and is defined as follows:

(Def. 2) $\mathrm{Arg}\, z = \mathrm{Arg}\, \mathrm{Ftize}(z)$.

One can prove the following propositions:

(17) For every element z of $\mathbb{C}$ and for every element u of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $z = u$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, u$.

(18) For every element z of $\mathbb{C}$ holds $0 \leqslant \mathrm{Arg}\, z$ and $\mathrm{Arg}\, z < 2 \cdot \pi$.

(19) For every element z of $\mathbb{C}$ holds $z = |z| \cdot \cos \mathrm{Arg}\, z + (|z| \cdot \sin \mathrm{Arg}\, z)i$.

(20) $\mathrm{Arg}(0_{\mathbb{C}}) = 0$.

(21) Let z be an element of $\mathbb{C}$ and r be a real number. If $z \neq 0$ and $z = |z| \cdot \cos r + (|z| \cdot \sin r)i$ and $0 \leqslant r$ and $r < 2 \cdot \pi$, then $r = \mathrm{Arg}\, z$.

(22) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds if $\mathrm{Arg}\, z < \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z + \pi$ and if $\mathrm{Arg}\, z \geqslant \pi$, then $\mathrm{Arg}(-z) = \mathrm{Arg}\, z - \pi$.

(23) For every real number r such that $r \geqslant 0$ holds $\mathrm{Arg}(r + 0i) = 0$.

(24) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $z = |z| + 0i$.

(25) For every element z of $\mathbb{C}$ such that $z \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}\, z < \pi$ iff $\mathrm{Arg}(-z) \geqslant \pi$.

(26) For all elements x, y of $\mathbb{C}$ such that $x \neq y$ or $x-y \neq 0_{\mathbb{C}}$ holds $\mathrm{Arg}(x-y) < \pi$ iff $\mathrm{Arg}(y-x) \geqslant \pi$.

(27) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \pi[$ iff $\Im(z) > 0$.

(28) For every element z of $\mathbb{C}$ such that $\mathrm{Arg}\, z \neq 0$ holds $\mathrm{Arg}\, z < \pi$ iff $\sin \mathrm{Arg}\, z > 0$.

(29) For all elements x, y of $\mathbb{C}$ such that $\mathrm{Arg}\, x < \pi$ and $\mathrm{Arg}\, y < \pi$ holds $\mathrm{Arg}(x+y) < \pi$.

(30) For every real number x such that $x > 0$ holds $\mathrm{Arg}(0 + xi) = \frac{\pi}{2}$.

(31) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]0, \frac{\pi}{2}[$ iff $\Re(z) > 0$ and $\Im(z) > 0$.

(32) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z \in]\frac{\pi}{2}, \pi[$ iff $\Re(z) < 0$ and $\Im(z) > 0$.

(33) For every element z of $\mathbb{C}$ such that $\Im(z) > 0$ holds $\sin \mathrm{Arg}\, z > 0$.

(34) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = 0$ iff $\Re(z) \geqslant 0$ and $\Im(z) = 0$.

(35) For every element z of $\mathbb{C}$ holds $\mathrm{Arg}\, z = \pi$ iff $\Re(z) < 0$ and $\Im(z) = 0$.

(36) For every element z of $\mathbb{C}$ holds $\Im(z) = 0$ iff $\mathrm{Arg}\, z = 0$ or $\mathrm{Arg}\, z = \pi$.

(37) For every element z of $\mathbb{C}$ such that $\mathrm{Arg}\, z \leqslant \pi$ holds $\Im(z) \geqslant 0$.

(38) For every element z of $\mathbb{C}$ such that $z \neq 0$ holds $\cos \mathrm{Arg}(-z) = -\cos \mathrm{Arg}\, z$ and $\sin \mathrm{Arg}(-z) = -\sin \mathrm{Arg}\, z$.

(39) For every element a of $\mathbb{C}$ such that $a \neq 0$ holds $\cos \mathrm{Arg}\, a = \frac{\Re(a)}{|a|}$ and $\sin \mathrm{Arg}\, a = \frac{\Im(a)}{|a|}$.

(40) For every element a of $\mathbb{C}$ and for every real number r such that $r > 0$ holds $\mathrm{Arg}(a \cdot (r + 0i)) = \mathrm{Arg}\, a$.

(41) For every element a of $\mathbb{C}$ and for every real number r such that $r < 0$ holds $\mathrm{Arg}(a \cdot (r + 0i)) = \mathrm{Arg}(-a)$.

3. Inner Product

Let x, y be elements of $\mathbb{C}$. The functor $(x|y)$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 3) $(x|y) = x \cdot \overline{y}$.

In the sequel a, b, c, d, x, y, z are elements of $\mathbb{C}$.

The following propositions are true:

(42) $(x|y) = (\Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)) + (-\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y))i$.

(43) $(z|z) = (\Re(z) \cdot \Re(z) + \Im(z) \cdot \Im(z)) + 0i$ and $(z|z) = (\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}) + 0i$.

(44) $(z|z) = |z|^{\mathbf{2}} + 0i$ and $|z|^{\mathbf{2}} = \Re((z|z))$.

(45) $|(x|y)| = |x| \cdot |y|$.

(46) If $(x|x) = 0$, then $x = 0$.

(47) $(y|x) = \overline{(x|y)}$.
(48) $((x+y)|z) = (x|z) + (y|z)$.
(49) $(x|(y+z)) = (x|y) + (x|z)$.
(50) $((a \cdot x)|y) = a \cdot (x|y)$.
(51) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.
(52) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.
(53) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.
(54) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.
(55) $((-x)|y) = (x|-y)$.
(56) $((-x)|y) = -(x|y)$.
(57) $-(x|y) = (x|-y)$.
(58) $((-x)|-y) = (x|y)$.
(59) $((x-y)|z) = (x|z) - (y|z)$.
(60) $(x|(y-z)) = (x|y) - (x|z)$.
(61) $(0_{\mathbb{C}}|x) = 0_{\mathbb{C}}$ and $(x|0_{\mathbb{C}}) = 0_{\mathbb{C}}$.
(62) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.
(63) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.
(64) $\Re((x|y)) = 0$ iff $\Im((x|y)) = |x| \cdot |y|$ or $\Im((x|y)) = -|x| \cdot |y|$.

4. Rotation

Let a be an element of $\mathbb{C}$ and let r be a real number. The functor $a \circlearrowleft r$ yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 4) $a \circlearrowleft r = |a| \cdot \cos(r + \operatorname{Arg} a) + (|a| \cdot \sin(r + \operatorname{Arg} a))i$.

In the sequel r denotes a real number.

We now state a number of propositions:

(65) $a \circlearrowleft 0 = a$.
(66) $a \circlearrowleft r = 0_{\mathbb{C}}$ iff $a = 0_{\mathbb{C}}$.
(67) $|a \circlearrowleft r| = |a|$.
(68) If $a \neq 0_{\mathbb{C}}$, then there exists an integer i such that $\operatorname{Arg}(a \circlearrowleft r) = 2 \cdot \pi \cdot i + (r + \operatorname{Arg} a)$.
(69) $a \circlearrowleft -\operatorname{Arg} a = |a| + 0i$.
(70) $\Re(a \circlearrowleft r) = \Re(a) \cdot \cos r - \Im(a) \cdot \sin r$ and $\Im(a \circlearrowleft r) = \Re(a) \cdot \sin r + \Im(a) \cdot \cos r$.
(71) $a + b \circlearrowleft r = (a \circlearrowleft r) + (b \circlearrowleft r)$.
(72) $-a \circlearrowleft r = -(a \circlearrowleft r)$.
(73) $a - b \circlearrowleft r = (a \circlearrowleft r) - (b \circlearrowleft r)$.
(74) $a \circlearrowleft \pi = -a$.

5. Angles

Let a, b be elements of $\mathbb{C}$. The functor $\measuredangle(a, b)$ yielding a real number is defined by:

(Def. 5) $\measuredangle(a, b) = \begin{cases} \operatorname{Arg}(b \circlearrowleft -\operatorname{Arg} a), & \text{if } \operatorname{Arg} a = 0 \text{ or } b \neq 0, \\ 2 \cdot \pi - \operatorname{Arg} a, & \text{otherwise.} \end{cases}$

Next we state several propositions:

(75) If $r \geqslant 0$, then $\measuredangle(r + 0i, a) = \operatorname{Arg} a$.

(76) If $\operatorname{Arg} a = \operatorname{Arg} b$ and $a \neq 0$ and $b \neq 0$, then $\operatorname{Arg}(a \circlearrowleft r) = \operatorname{Arg}(b \circlearrowleft r)$.

(77) If $r > 0$, then $\measuredangle(a, b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(78) If $a \neq 0$ and $b \neq 0$ and $\operatorname{Arg} a = \operatorname{Arg} b$, then $\operatorname{Arg}(-a) = \operatorname{Arg}(-b)$.

(79) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a, b) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r)$.

(80) If $r < 0$ and $a \neq 0$ and $b \neq 0$, then $\measuredangle(a, b) = \measuredangle(a \cdot (r + 0i), b \cdot (r + 0i))$.

(81) If $a \neq 0$ and $b \neq 0$, then $\measuredangle(a, b) = \measuredangle(-a, -b)$.

(82) If $b \neq 0$ and $\measuredangle(a, b) = 0$, then $\measuredangle(a, -b) = \pi$.

(83) If $a \neq 0$ and $b \neq 0$, then $\cos \measuredangle(a, b) = \frac{\Re((a|b))}{|a| \cdot |b|}$ and $\sin \measuredangle(a, b) = -\frac{\Im((a|b))}{|a| \cdot |b|}$.

Let x, y, z be elements of $\mathbb{C}$. The functor $\measuredangle(x, y, z)$ yielding a real number is defined as follows:

(Def. 6) $\measuredangle(x, y, z) = \begin{cases} \operatorname{Arg}(z - y) - \operatorname{Arg}(x - y), & \text{if } \operatorname{Arg}(z - y) - \operatorname{Arg}(x - y) \geqslant 0, \\ 2 \cdot \pi + (\operatorname{Arg}(z - y) - \operatorname{Arg}(x - y)), & \text{otherwise.} \end{cases}$

One can prove the following propositions:

(84) $0 \leqslant \measuredangle(x, y, z)$ and $\measuredangle(x, y, z) < 2 \cdot \pi$.

(85) $\measuredangle(x, y, z) = \measuredangle(x - y, 0_{\mathbb{C}}, z - y)$.

(86) $\measuredangle(a, b, c) = \measuredangle(a + d, b + d, c + d)$.

(87) $\measuredangle(a, b) = \measuredangle(a, 0_{\mathbb{C}}, b)$.

(88) If $\measuredangle(x, y, z) = 0$, then $\operatorname{Arg}(x - y) = \operatorname{Arg}(z - y)$ and $\measuredangle(z, y, x) = 0$.

(89) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Re((a|b)) = 0$ iff $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{\pi}{2}$ or $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{3}{2} \cdot \pi$.

(90) If $a \neq 0_{\mathbb{C}}$ and $b \neq 0_{\mathbb{C}}$, then $\Im((a|b)) = |a| \cdot |b|$ or $\Im((a|b)) = -|a| \cdot |b|$ iff $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{\pi}{2}$ or $\measuredangle(a, 0_{\mathbb{C}}, b) = \frac{3}{2} \cdot \pi$.

(91) If $x \neq y$ and if $z \neq y$ and if $\measuredangle(x, y, z) = \frac{\pi}{2}$ or $\measuredangle(x, y, z) = \frac{3}{2} \cdot \pi$, then $|x - y|^{\mathbf{2}} + |z - y|^{\mathbf{2}} = |x - z|^{\mathbf{2}}$.

(92) If $a \neq b$ and $b \neq c$, then $\measuredangle(a, b, c) = \measuredangle(a \circlearrowleft r, b \circlearrowleft r, c \circlearrowleft r)$.

(93) $\measuredangle(a, b, a) = 0$.

(94) $\measuredangle(a, b, c) \neq 0$ iff $\measuredangle(a, b, c) + \measuredangle(c, b, a) = 2 \cdot \pi$.

(95) If $\measuredangle(a, b, c) \neq 0$, then $\measuredangle(c, b, a) \neq 0$.

(96) If $\measuredangle(a, b, c) = \pi$, then $\measuredangle(c, b, a) = \pi$.

(97) If $a \neq b$ and $a \neq c$ and $b \neq c$, then $\measuredangle(a,b,c) \neq 0$ or $\measuredangle(b,c,a) \neq 0$ or $\measuredangle(c,a,b) \neq 0$.

(98) If $a \neq b$ and $b \neq c$ and $0 < \measuredangle(a,b,c)$ and $\measuredangle(a,b,c) < \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ and $0 < \measuredangle(b,c,a)$ and $0 < \measuredangle(c,a,b)$.

(99) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) > \pi$, then $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5 \cdot \pi$ and $\measuredangle(b,c,a) > \pi$ and $\measuredangle(c,a,b) > \pi$.

(100) If $a \neq b$ and $b \neq c$ and $\measuredangle(a,b,c) = \pi$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = 0$.

(101) If $a \neq b$ and $a \neq c$ and $b \neq c$ and $\measuredangle(a,b,c) = 0$, then $\measuredangle(b,c,a) = 0$ and $\measuredangle(c,a,b) = \pi$ or $\measuredangle(b,c,a) = \pi$ and $\measuredangle(c,a,b) = 0$.

(102) $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = \pi$ or $\measuredangle(a,b,c) + \measuredangle(b,c,a) + \measuredangle(c,a,b) = 5 \cdot \pi$ iff $a \neq b$ and $a \neq c$ and $b \neq c$.

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Angle and Triangle in Euclidean Topological Space

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Summary. Two transformations between the complex space and 2-dimensional Euclidean topological space are defined. By them, the concept of argument is induced to 2-dimensional vectors using argument of complex number. Similarly, the concept of an angle is introduced using the angle of two complex numbers. The concept of a triangle and related concepts are also defined in n-dimensional Euclidean topological spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [17], [19], [18], [20], [4], [12], [21], [5], [16], [11], [3], [13], [15], [8], [2], [6], [7], [1], [10], [9], and [14].

We follow the rules: z, z_1, z_2 are elements of $\mathbb{C}$, r, r_1, r_2, x_1, x_2 are real numbers, and p, p_1, p_2, p_3, q are points of $\mathcal{E}^2_{\rm T}$.

Let z be an element of $\mathbb{C}$. The functor $\mathrm{cpx2euc}(z)$ yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) $\mathrm{cpx2euc}(z) = [\Re(z), \Im(z)]$.

Let p be a point of $\mathcal{E}^2_{\rm T}$. The functor $\mathrm{euc2cpx}(p)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 2) $\mathrm{euc2cpx}(p) = p_{\mathbf{1}} + p_{\mathbf{2}} i$.

One can prove the following propositions:

(1) $\mathrm{euc2cpx}(\mathrm{cpx2euc}(z)) = z$.

(2) $\mathrm{cpx2euc}(\mathrm{euc2cpx}(p)) = p$.

(3) For every p there exists z such that $p = \mathrm{cpx2euc}(z)$.

(4) For every z there exists p such that $z = \mathrm{euc2cpx}(p)$.

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(5) For all z_1, z_2 such that $\mathrm{cpx2euc}(z_1) = \mathrm{cpx2euc}(z_2)$ holds $z_1 = z_2$.

(6) For all p_1, p_2 such that $\mathrm{euc2cpx}(p_1) = \mathrm{euc2cpx}(p_2)$ holds $p_1 = p_2$.

(7) $(\mathrm{cpx2euc}(z))_{\mathbf{1}} = \Re(z)$ and $(\mathrm{cpx2euc}(z))_{\mathbf{2}} = \Im(z)$.

(8) $\Re(\mathrm{euc2cpx}(p)) = p_{\mathbf{1}}$ and $\Im(\mathrm{euc2cpx}(p)) = p_{\mathbf{2}}$.

(9) $\mathrm{cpx2euc}(x_1 + x_2 i) = [x_1, x_2]$.

(10) $[\Re(z_1 + z_2), \Im(z_1 + z_2)] = [\Re(z_1) + \Re(z_2), \Im(z_1) + \Im(z_2)]$.

(11) $\mathrm{cpx2euc}(z_1 + z_2) = \mathrm{cpx2euc}(z_1) + \mathrm{cpx2euc}(z_2)$.

(12) $(p_1 + p_2)_{\mathbf{1}} + (p_1 + p_2)_{\mathbf{2}} i = ((p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}) + ((p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}})i$.

(13) $\mathrm{euc2cpx}(p_1 + p_2) = \mathrm{euc2cpx}(p_1) + \mathrm{euc2cpx}(p_2)$.

(14) $[\Re(-z), \Im(-z)] = [-\Re(z), -\Im(z)]$.

(15) $\mathrm{cpx2euc}(-z) = -\mathrm{cpx2euc}(z)$.

(16) $(-p)_{\mathbf{1}} + (-p)_{\mathbf{2}} i = -p_{\mathbf{1}} + (-p_{\mathbf{2}})i$.

(17) $\mathrm{euc2cpx}(-p) = -\mathrm{euc2cpx}(p)$.

(18) $\mathrm{cpx2euc}(z_1 - z_2) = \mathrm{cpx2euc}(z_1) - \mathrm{cpx2euc}(z_2)$.

(19) $\mathrm{euc2cpx}(p_1 - p_2) = \mathrm{euc2cpx}(p_1) - \mathrm{euc2cpx}(p_2)$.

(20) $\mathrm{cpx2euc}(0_{\mathbb{C}}) = 0_{\mathcal{E}^2_{\mathrm{T}}}$.

(21) $\mathrm{euc2cpx}(0_{\mathcal{E}^2_{\mathrm{T}}}) = 0_{\mathbb{C}}$.

(22) If $\mathrm{euc2cpx}(p) = 0_{\mathbb{C}}$, then $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$.

(23) $\mathrm{cpx2euc}((r + 0i) \cdot z) = r \cdot \mathrm{cpx2euc}(z)$.

(24) $(r + 0i) \cdot (r_1 + r_2 i) = r \cdot r_1 + (r \cdot r_2)i$.

(25) $\mathrm{euc2cpx}(r \cdot p) = (r + 0i) \cdot \mathrm{euc2cpx}(p)$.

(26) $|\mathrm{euc2cpx}(p)| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$.

(27) For every finite sequence f of elements of $\mathbb{R}$ such that $\mathrm{len}\, f = 2$ holds $|f| = \sqrt{f(1)^{\mathbf{2}} + f(2)^{\mathbf{2}}}$.

(28) For every finite sequence f of elements of $\mathbb{R}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{len}\, f = 2$ and $p = f$ holds $|p| = |f|$.

(29) $|\mathrm{cpx2euc}(z)| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

(30) $|\mathrm{cpx2euc}(z)| = |z|$.

(31) $|\mathrm{euc2cpx}(p)| = |p|$.

Let us consider p. The functor $\mathrm{Arg}\, p$ yields a real number and is defined as follows:

(Def. 3) $\mathrm{Arg}\, p = \mathrm{Arg}\, \mathrm{euc2cpx}(p)$.

We now state a number of propositions:

(32) For every element z of $\mathbb{C}$ and for every p such that $z = \mathrm{euc2cpx}(p)$ or $p = \mathrm{cpx2euc}(z)$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, p$.

(33) For every p holds $0 \leqslant \mathrm{Arg}\, p$ and $\mathrm{Arg}\, p < 2 \cdot \pi$.

(34) For all real numbers x_1, x_2 and for every p such that $x_1 = |p| \cdot \cos \operatorname{Arg} p$ and $x_2 = |p| \cdot \sin \operatorname{Arg} p$ holds $p = [x_1, x_2]$.

(35) $\operatorname{Arg}(0_{\mathcal{E}^2_{\mathrm{T}}}) = 0$.

(36) For every p such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds if $\operatorname{Arg} p < \pi$, then $\operatorname{Arg}(-p) = \operatorname{Arg} p + \pi$ and if $\operatorname{Arg} p \geqslant \pi$, then $\operatorname{Arg}(-p) = \operatorname{Arg} p - \pi$.

(37) For every p such that $\operatorname{Arg} p = 0$ holds $p = [|p|, 0]$ and $p_{\mathbf{2}} = 0$.

(38) For every p such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $\operatorname{Arg} p < \pi$ iff $\operatorname{Arg}(-p) \geqslant \pi$.

(39) For all p_1, p_2 such that $p_1 \neq p_2$ or $p_1 - p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $\operatorname{Arg}(p_1 - p_2) < \pi$ iff $\operatorname{Arg}(p_2 - p_1) \geqslant \pi$.

(40) For every p holds $\operatorname{Arg} p \in]0, \pi[$ iff $p_{\mathbf{2}} > 0$.

(41) For every p such that $\operatorname{Arg} p \neq 0$ holds $\operatorname{Arg} p < \pi$ iff $\sin \operatorname{Arg} p > 0$.

(42) For all p_1, p_2 such that $\operatorname{Arg} p_1 < \pi$ and $\operatorname{Arg} p_2 < \pi$ holds $\operatorname{Arg}(p_1 + p_2) < \pi$.

Let us consider p_1, p_2, p_3. The functor $\measuredangle(p_1, p_2, p_3)$ yielding a real number is defined as follows:

(Def. 4) $\measuredangle(p_1, p_2, p_3) = \measuredangle(\mathrm{euc2cpx}(p_1), \mathrm{euc2cpx}(p_2), \mathrm{euc2cpx}(p_3))$.

The following propositions are true:

(43) For all p_1, p_2, p_3 holds $0 \leqslant \measuredangle(p_1, p_2, p_3)$ and $\measuredangle(p_1, p_2, p_3) < 2 \cdot \pi$.

(44) For all p_1, p_2, p_3 holds $\measuredangle(p_1, p_2, p_3) = \measuredangle(p_1 - p_2, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_3 - p_2)$.

(45) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) = 0$ holds $\operatorname{Arg}(p_1 - p_2) = \operatorname{Arg}(p_3 - p_2)$ and $\measuredangle(p_3, p_2, p_1) = 0$.

(46) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(47) For all p_1, p_2, p_3 such that $\measuredangle(p_3, p_2, p_1) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(48) For all elements x, y of $\mathbb{C}$ holds $\Re((x|y)) = \Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)$.

(49) For all elements x, y of $\mathbb{C}$ holds $\Im((x|y)) = -\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y)$.

(50) For all p, q holds $|(p, q)| = p_{\mathbf{1}} \cdot q_{\mathbf{1}} + p_{\mathbf{2}} \cdot q_{\mathbf{2}}$.

(51) For all p_1, p_2 holds $|(p_1, p_2)| = \Re((\mathrm{euc2cpx}(p_1) | \mathrm{euc2cpx}(p_2)))$.

(52) For all p_1, p_2, p_3 such that $p_1 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $|(p_1, p_2)| = 0$ iff $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{3}{2} \cdot \pi$.

(53) Let given p_1, p_2. Suppose $p_1 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = |p_1| \cdot |p_2|$ or $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = -|p_1| \cdot |p_2|$ if and only if $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{3}{2} \cdot \pi$.

(54) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ and $p_3 \neq p_2$ holds $|(p_1 - p_2, p_3 - p_2)| = 0$ iff $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$.

(55) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ but $p_3 \neq p_2$ but $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$ holds $|p_1 - p_2|^{\mathbf{2}} + |p_3 - p_2|^{\mathbf{2}} = |p_1 - p_3|^{\mathbf{2}}$.

(56) For all p_1, p_2, p_3 such that $p_2 \neq p_1$ and $p_1 \neq p_3$ and $p_3 \neq p_2$ and $\measuredangle(p_2, p_1, p_3) < \pi$ and $\measuredangle(p_1, p_3, p_2) < \pi$ and $\measuredangle(p_3, p_2, p_1) < \pi$ holds $\measuredangle(p_2, p_1, p_3) + \measuredangle(p_1, p_3, p_2) + \measuredangle(p_3, p_2, p_1) = \pi$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{Triangle}(p_1, p_2, p_3)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 5) $\mathrm{Triangle}(p_1, p_2, p_3) = \mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, p_3) \cup \mathcal{L}(p_3, p_1)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 6) $\mathrm{ClInsideOfTriangle}(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} (0 \leqslant a_1 \wedge 0 \leqslant a_2 \wedge 0 \leqslant a_3 \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 7) $\mathrm{InsideOfTriangle}(p_1, p_2, p_3) = \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3) \setminus \mathrm{Triangle}(p_1, p_2, p_3)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{OutsideOfTriangle}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by the condition (Def. 8).

(Def. 8) $\mathrm{OutsideOfTriangle}(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\mathrm{T}}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} ((0 > a_1 \vee 0 > a_2 \vee 0 > a_3) \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{plane}(p_1, p_2, p_3)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 9) $\mathrm{plane}(p_1, p_2, p_3) = \mathrm{OutsideOfTriangle}(p_1, p_2, p_3) \cup \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$.

One can prove the following propositions:

(57) Let n be a natural number and p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p \in \mathrm{plane}(p_1, p_2, p_3)$. Then there exist real numbers a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

(58) For every natural number n and for all points p_1, p_2, p_3 of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{Triangle}(p_1, p_2, p_3) \subseteq \mathrm{ClInsideOfTriangle}(p_1, p_2, p_3)$.

Let n be a natural number and let q_1, q_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. We say that q_1, q_2 are lindependent2 if and only if:

(Def. 10) For all real numbers a_1, a_2 such that $a_1 \cdot q_1 + a_2 \cdot q_2 = 0_{\mathcal{E}^n_{\mathrm{T}}}$ holds $a_1 = 0$ and $a_2 = 0$.

We introduce q_1, q_2 are ldependent2 as an antonym of q_1, q_2 are lindependent2.

One can prove the following propositions:

(59) Let n be a natural number and q_1, q_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. If q_1, q_2 are lindependent2, then $q_1 \neq q_2$ and $q_1 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$ and $q_2 \neq 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(60) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p_0 \in \mathrm{plane}(p_1, p_2, p_3)$. Then there exist real numbers a_1, a_2, a_3 such that

(i) $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$,

(ii) $a_1 + a_2 + a_3 = 1$, and

(iii) for all real numbers b_1, b_2, b_3 such that $p_0 = b_1 \cdot p_1 + b_2 \cdot p_2 + b_3 \cdot p_3$ and $b_1 + b_2 + b_3 = 1$ holds $b_1 = a_1$ and $b_2 = a_2$ and $b_3 = a_3$.

(61) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\mathrm{T}}$. Given real numbers a_1, a_2, a_3 such that $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$ and $a_1 + a_2 + a_3 = 1$. Then $p_0 \in \mathrm{plane}(p_1, p_2, p_3)$.

(62) Let n be a natural number and p_1, p_2, p_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. Then $\mathrm{plane}(p_1, p_2, p_3) = \{p; p \text{ ranges over points of } \mathcal{E}^n_{\mathrm{T}}\colon \bigvee_{a_1, a_2, a_3 \,:\, \text{real number}} (a_1 + a_2 + a_3 = 1 \ \wedge\ p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

(63) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds $\mathrm{plane}(p_1, p_2, p_3) = \mathcal{R}^2$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor $\mathrm{tricord1}(p_1, p_2, p_3, p)$ yields a real number and is defined as follows:

(Def. 11) There exist real numbers a_2, a_3 such that $\mathrm{tricord1}(p_1, p_2, p_3, p) + a_2 + a_3 = 1$ and $p = \mathrm{tricord1}(p_1, p_2, p_3, p) \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor $\mathrm{tricord2}(p_1, p_2, p_3, p)$ yielding a real number is defined as follows:

(Def. 12) There exist real numbers a_1, a_3 such that $a_1 + \mathrm{tricord2}(p_1, p_2, p_3, p) + a_3 = 1$ and $p = a_1 \cdot p_1 + \mathrm{tricord2}(p_1, p_2, p_3, p) \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in \mathrm{plane}(p_1, p_2, p_3)$. The functor $\mathrm{tricord2}(p_1, p_2, p_3, p)$ yielding a real number is defined as follows:

(Def. 13) There exist real numbers a_1, a_2 such that $a_1 + a_2 + \mathrm{tricord2}(p_1, p_2, p_3, p) = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 + \mathrm{tricord2}(p_1, p_2, p_3, p) \cdot p_3$.

Let us consider p_1, p_2, p_3. The functor $\mathrm{trcmap1}(p_1, p_2, p_3)$ yielding a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 14) For every p holds $(\mathrm{trcmap1}(p_1, p_2, p_3))(p) = \mathrm{tricord1}(p_1, p_2, p_3, p)$.

Let us consider p_1, p_2, p_3. The functor $\mathrm{trcmap2}(p_1, p_2, p_3)$ yields a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined as follows:

(Def. 15) For every p holds $(\mathrm{trcmap2}(p_1, p_2, p_3))(p) = \mathrm{tricord2}(p_1, p_2, p_3, p)$.

Let us consider p_1, p_2, p_3. The functor $\mathrm{trcmap3}(p_1, p_2, p_3)$ yielding a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 16) For every p holds $(\mathrm{trcmap3}(p_1, p_2, p_3))(p) = \mathrm{tricord2}(p_1, p_2, p_3, p)$.

Next we state several propositions:

(64) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{OutsideOfTriangle}(p_1, p_2, p_3)$ if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) < 0$, or
(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$, or
(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$.

(65) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{Triangle}(p_1, p_2, p_3)$ if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$,
(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$,
(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, and
(iv) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(66) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{Triangle}(p_1, p_2, p_3)$ if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, or
(ii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, or
(iii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(67) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) > 0$,
(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$, and
(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$.

(68) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds $\mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ is non empty.

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The Class of Series-Parallel Graphs. Part II[1]

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Summary. In this paper we introduce two new operations on graphs: sum and union corresponding to parallel and series operation respectively. We determine N-free graph as the graph that does not embed Necklace 4. We define "fin_RelStr" as the set of all graphs with finite carriers. We also define the smallest class of graphs which contains the one-element graph and which is closed under parallel and series operations. The goal of the article is to prove the theorem that the class of finite series-parallel graphs is the class of finite N-free graphs. This paper formalizes the next part of [12].

MML Identifier: NECKLA_2.

The terminology and notation used in this paper are introduced in the following papers: [15], [14], [18], [7], [20], [8], [1], [2], [3], [13], [16], [4], [17], [19], [11], [5], [6], [9], and [10].

In this paper U denotes a universal class.

Next we state two propositions:

(1) For all sets X, Y such that $X \in U$ and $Y \in U$ and for every relation R between X and Y holds $R \in U$.

(2) The internal relation of Necklace 4 $= \{\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle\}$.

Let n be a natural number. One can check that every element of $\mathbf{R}_n$ is finite.

Next we state the proposition

(3) For every set x such that $x \in \mathbf{U}_0$ holds x is finite.

Let us mention that every element of $\mathbf{U}_0$ is finite.

Let us note that every number which is finite and ordinal is also natural.

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Let G be a non empty relational structure. We say that G is N-free if and only if:

(Def. 1) G does not embed Necklace 4.

Let us mention that there exists a non empty relational structure which is N-free.

Let R, S be relational structures. The functor UnionOf(R, S) yielding a strict relational structure is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of UnionOf(R, S) = (the carrier of R) $\cup$ (the carrier of S), and

(ii) the internal relation of UnionOf(R, S) = (the internal relation of R) $\cup$ (the internal relation of S).

Let R, S be relational structures. The functor SumOf(R, S) yielding a strict relational structure is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of SumOf(R, S) = (the carrier of R) $\cup$ (the carrier of S), and

(ii) the internal relation of SumOf(R, S) = (the internal relation of R) $\cup$ (the internal relation of S) $\cup$ $[\!:$ the carrier of R, the carrier of S $:\!]$ $\cup$ $[\!:$ the carrier of S, the carrier of R $:\!]$.

The functor FinRelStr is defined by the condition (Def. 4).

(Def. 4) Let X be a set. Then $X \in$ FinRelStr if and only if there exists a strict relational structure R such that $X = R$ and the carrier of $R \in \mathbf{U}_0$.

Let us mention that FinRelStr is non empty.

The subset FinRelStrSp of FinRelStr is defined by the conditions (Def. 5).

(Def. 5)(i) For every strict relational structure R such that the carrier of R is non empty and trivial and the carrier of $R \in \mathbf{U}_0$ holds $R \in$ FinRelStrSp,

(ii) for all strict relational structures H_1, H_2 such that the carrier of H_1 misses the carrier of H_2 and $H_1 \in$ FinRelStrSp and $H_2 \in$ FinRelStrSp holds UnionOf$(H_1, H_2) \in$ FinRelStrSp and SumOf$(H_1, H_2) \in$ FinRelStrSp, and

(iii) for every subset M of FinRelStr such that for every strict relational structure R such that the carrier of R is non empty and trivial and the carrier of $R \in \mathbf{U}_0$ holds $R \in M$ and for all strict relational structures H_1, H_2 such that the carrier of H_1 misses the carrier of H_2 and $H_1 \in M$ and $H_2 \in M$ holds UnionOf$(H_1, H_2) \in M$ and SumOf$(H_1, H_2) \in M$ holds FinRelStrSp $\subseteq M$.

One can verify that FinRelStrSp is non empty.

We now state four propositions:

(4) For every set X such that $X \in$ FinRelStrSp holds X is a finite strict non empty relational structure.

(5) For every relational structure R such that $R \in$ FinRelStrSp holds the carrier of $R \in \mathbf{U}_0$.

(6) Let X be a set. Suppose $X \in \mathrm{FinRelStrSp}$. Then

(i) X is a strict non empty trivial relational structure, or

(ii) there exist strict relational structures H_1, H_2 such that the carrier of H_1 misses the carrier of H_2 and $H_1 \in \mathrm{FinRelStrSp}$ and $H_2 \in \mathrm{FinRelStrSp}$ and $X = \mathrm{UnionOf}(H_1, H_2)$ or $X = \mathrm{SumOf}(H_1, H_2)$.

(7) For every strict non empty relational structure R such that $R \in \mathrm{FinRelStrSp}$ holds R is N-free.

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Characterization and Existence of Gröbner Bases

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Summary. We continue the Mizar formalization of Gröbner bases following [8]. In this article we prove a number of characterizations of Gröbner bases among them that Gröbner bases are convergent rewriting systems. We also show the existence and uniqueness of reduced Gröbner bases.

MML Identifier: `GROEB_1`.

The papers [24], [31], [33], [32], [10], [5], [17], [29], [28], [11], [13], [4], [2], [30], [9], [7], [15], [16], [12], [20], [19], [25], [27], [18], [1], [6], [14], [22], [26], [23], [3], and [21] provide the terminology and notation for this paper.

1. Preliminaries

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, and let p be a polynomial of n, L. Then $\{p\}$ is a non empty finite subset of Polynom-Ring(n, L).

We now state several propositions:

(1) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and f, p, g be polynomials of n, L. Suppose f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m * p$.

(2) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. Suppose

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f reduces to g, p, T. Then there exists a monomial m of n, L such that $g = f - m*p$ and $\mathrm{HT}(m*p, T) \notin \mathrm{Support}\, g$ and $\mathrm{HT}(m*p, T) \leqslant_T \mathrm{HT}(f, T)$.

(3) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, g be polynomials of n, L, and P, Q be subsets of Polynom-Ring(n, L). If $P \subseteq Q$, then if f reduces to g, P, T, then f reduces to g, Q, T.

(4) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P, Q be subsets of Polynom-Ring(n, L). If $P \subseteq Q$, then PolyRedRel$(P, T) \subseteq$ PolyRedRel(Q, T).

(5) Let n be an ordinal number, L be a right zeroed add-associative right complementable non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{Support}(-p) = \mathrm{Support}\, p$.

(6) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p be a polynomial of n, L. Then $\mathrm{HT}(-p, T) = \mathrm{HT}(p, T)$.

(7) Let n be an ordinal number, T be an admissible connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and p, q be polynomials of n, L. Then $\mathrm{HT}(p - q, T) \leqslant_T \max_T(\mathrm{HT}(p, T), \mathrm{HT}(q, T))$.

(8) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p, q be polynomials of n, L. If $\mathrm{Support}\, q \subseteq \mathrm{Support}\, p$, then $q \leqslant_T p$.

(9) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p be non-zero polynomials of n, L. If f is reducible wrt p, T, then $\mathrm{HT}(p, T) \leqslant_T \mathrm{HT}(f, T)$.

2. Characterization of Gröbner Bases

Next we state two propositions:

(10) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double

loop structure, and p be a polynomial of n, L. Then $\mathrm{PolyRedRel}(\{p\}, T)$ is locally-confluent.

(11) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. Given a polynomial p of n, L such that $p \in P$ and P–ideal $= \{p\}$–ideal. Then $\mathrm{PolyRedRel}(P, T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, and let P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. The functor $\mathrm{HT}(P, T)$ yields a subset of $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 1) $\mathrm{HT}(P, T) = \{\mathrm{HT}(p, T); p$ ranges over polynomials of n, L: $p \in P \ \wedge\ p \neq 0_n L\}$.

Let n be an ordinal number and let S be a subset of $\mathrm{Bags}\, n$. The functor $\mathrm{multiples}(S)$ yields a subset of $\mathrm{Bags}\, n$ and is defined by:

(Def. 2) $\mathrm{multiples}(S) = \{b; b$ ranges over elements of $\mathrm{Bags}\, n : \bigvee_{b' : \text{bag of } n} (b' \in S \ \wedge\ b' \mid b)\}$.

We now state several propositions:

(12) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. If $\mathrm{PolyRedRel}(P, T)$ is locally-confluent, then $\mathrm{PolyRedRel}(P, T)$ is confluent.

(13) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. If $\mathrm{PolyRedRel}(P, T)$ is confluent, then $\mathrm{PolyRedRel}(P, T)$ has unique normal form property.

(14) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. Suppose $\mathrm{PolyRedRel}(P, T)$ has unique normal form property. Then $\mathrm{PolyRedRel}(P, T)$ has Church-Rosser property.

(15) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a non empty subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. Suppose $\mathrm{PolyRedRel}(P, T)$ has Church-

Rosser property. Let f be a polynomial of n, L. If $f \in P$–ideal, then PolyRedRel(P, T) reduces f to $0_n L$.

(16) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n, L). Suppose that for every polynomial f of n, L such that $f \in P$–ideal holds PolyRedRel(P, T) reduces f to $0_n L$. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is reducible wrt P, T.

(17) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n, L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is reducible wrt P, T. Let f be a non-zero polynomial of n, L. If $f \in P$–ideal, then f is top reducible wrt P, T.

(18) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n, L). Suppose that for every non-zero polynomial f of n, L such that $f \in P$–ideal holds f is top reducible wrt P, T. Let b be a bag of n. If $b \in \mathrm{HT}(P\text{–ideal}, T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(P, T)$ and $b' \mid b$.

(19) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and P be a subset of Polynom-Ring(n, L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(P\text{–ideal}, T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(P, T)$ and $b' \mid b$. Then $\mathrm{HT}(P\text{–ideal}, T) \subseteq \mathrm{multiples}(\mathrm{HT}(P, T))$.

(20) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n, L). If $\mathrm{HT}(P\text{–ideal}, T) \subseteq \mathrm{multiples}(\mathrm{HT}(P, T))$, then PolyRedRel$(P, T)$ is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G be a subset of Polynom-Ring(n, L). We say that G is a Groebner basis wrt T if and only if:

(Def. 3) PolyRedRel(G, T) is locally-confluent.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let G, I be subsets of Polynom-Ring(n, L). We say that G is a Groebner basis of I, T if and only if:

(Def. 4) G–ideal $= I$ and PolyRedRel(G, T) is locally-confluent.

One can prove the following propositions:

(21) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and G, P be non empty subsets of Polynom-Ring(n, L). If G is a Groebner basis of P, T, then PolyRedRel(G, T) is a completion of PolyRedRel(P, T).

(22) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, p, q be elements of Polynom-Ring(n, L), and G be a non empty subset of Polynom-Ring(n, L). Suppose G is a Groebner basis wrt T. Then $p \equiv q (\mathrm{mod}\, G\text{–ideal})$ if and only if $\mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(p) = \mathrm{nf}_{\mathrm{PolyRedRel}(G,T)}(q)$.

(23) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f be a polynomial of n, L, and P be a non empty subset of Polynom-Ring(n, L). Suppose P is a Groebner basis wrt T. Then $f \in P$–ideal if and only if PolyRedRel(P, T) reduces f to $0_n L$.

(24) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be a subset of Polynom-Ring(n, L), and G be a non empty subset of Polynom-Ring(n, L). Suppose G is a Groebner basis of I, T. Let f be a polynomial of n, L. If $f \in I$, then PolyRedRel(G, T) reduces f to $0_n L$.

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n, L). Suppose that for every polynomial f of n, L such that $f \in I$ holds PolyRedRel(G, T) reduces f to $0_n L$. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is reducible wrt G, T.

(26) Let n be a natural number, T be an admissible connected term order of

n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n, L), and G be a subset of Polynom-Ring(n, L). Suppose $G \subseteq I$. Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is reducible wrt G, T. Let f be a non-zero polynomial of n, L. If $f \in I$, then f is top reducible wrt G, T.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n, L). Suppose that for every non-zero polynomial f of n, L such that $f \in I$ holds f is top reducible wrt G, T. Let b be a bag of n. If $b \in \mathrm{HT}(I, T)$, then there exists a bag b' of n such that $b' \in \mathrm{HT}(G, T)$ and $b' \mid b$.

(28) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and G, I be subsets of Polynom-Ring(n, L). Suppose that for every bag b of n such that $b \in \mathrm{HT}(I, T)$ there exists a bag b' of n such that $b' \in \mathrm{HT}(G, T)$ and $b' \mid b$. Then $\mathrm{HT}(I, T) \subseteq$ multiples$(\mathrm{HT}(G, T))$.

(29) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n, L), and G be a non empty subset of Polynom-Ring(n, L). If $G \subseteq I$, then if $\mathrm{HT}(I, T) \subseteq$ multiples$(\mathrm{HT}(G, T))$, then G is a Groebner basis of I, T.

3. Existence of Gröbner Bases

Next we state four propositions:

(30) Let n be a natural number, T be a connected admissible term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure. Then $\{0_n L\}$ is a Groebner basis of $\{0_n L\}$, T.

(31) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\{p\}$ is a Groebner basis of $\{p\}$–ideal, T.

(32) Let T be an admissible connected term order of $\emptyset$, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of $\mathrm{Polynom\text{-}Ring}(\emptyset, L)$, and P be a non empty subset of $\mathrm{Polynom\text{-}Ring}(\emptyset, L)$. If $P \subseteq I$ and $P \neq \{0_{\emptyset}L\}$, then P is a Groebner basis of I, T.

(33) Let n be a non empty ordinal number, T be an admissible connected term order of n, and L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure. Then there exists a subset P of $\mathrm{Polynom\text{-}Ring}(n, L)$ such that P is not a Groebner basis wrt T.

Let n be an ordinal number. The functor $\mathrm{DivOrder}(n)$ yields an order in $\mathrm{Bags}\, n$ and is defined by:

(Def. 5) For all bags b_1, b_2 of n holds $\langle b_1, b_2\rangle \in \mathrm{DivOrder}(n)$ iff $b_1 \mid b_2$.

Let n be a natural number. One can check that $\langle \mathrm{Bags}\, n, \mathrm{DivOrder}(n)\rangle$ is Dickson.

The following propositions are true:

(34) For every natural number n and for every subset N of the carrier of $\langle \mathrm{Bags}\, n, \mathrm{DivOrder}(n)\rangle$ holds there exists a finite subset of $\mathrm{Bags}\, n$ which is Dickson basis of N, $\langle \mathrm{Bags}\, n, \mathrm{DivOrder}(n)\rangle$.

(35) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. Then there exists a finite subset of $\mathrm{Polynom\text{-}Ring}(n, L)$ which is a Groebner basis of I, T.

(36) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of $\mathrm{Polynom\text{-}Ring}(n, L)$. Suppose $I \neq \{0_n L\}$. Then there exists a finite subset G of $\mathrm{Polynom\text{-}Ring}(n, L)$ such that G is a Groebner basis of I, T and $0_n L \notin G$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty multiplicative loop with zero structure, and let p be a polynomial of n, L. We say that p is monic wrt T if and only if:

(Def. 6) $\mathrm{HC}(p, T) = \mathbf{1}_L$.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable commutative associative well unital distributive field-like non trivial non empty double loop structure, and let P be a subset of Polynom-Ring(n, L). We say that P is reduced wrt T if and only if:

(Def. 7) For every polynomial p of n, L such that $p \in P$ holds p is monic wrt T and irreducible wrt $P \setminus \{p\}$, T.

We introduce P is autoreduced wrt T as a synonym of P is reduced wrt T.

Next we state four propositions:

(37) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, I be an add closed left ideal subset of Polynom-Ring(n, L), m be a monomial of n, L, and f, g be polynomials of n, L. Suppose $f \in I$ and $g \in I$ and $\mathrm{HM}(f, T) = m$ and $\mathrm{HM}(g, T) = m$. Suppose that

(i) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T f$ and $\mathrm{HM}(p, T) = m$, and

(ii) it is not true that there exists a polynomial p of n, L such that $p \in I$ and $p <_T g$ and $\mathrm{HM}(p, T) = m$.

Then $f = g$.

(38) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n, L), G be a subset of Polynom-Ring(n, L), p be a polynomial of n, L, and q be a non-zero polynomial of n, L. Suppose $p \in G$ and $q \in G$ and $p \neq q$ and $\mathrm{HT}(q, T) \mid \mathrm{HT}(p, T)$. If G is a Groebner basis of I, T, then $G \setminus \{p\}$ is a Groebner basis of I, T.

(39) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and I be an add closed left ideal non empty subset of Polynom-Ring(n, L). If $I \neq \{0_n L\}$, then there exists a finite subset G of Polynom-Ring(n, L) which is a Groebner basis of I, T and reduced wrt T.

(40) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, I be an add closed left ideal non empty subset of Polynom-Ring(n, L), and G_1, G_2 be non empty finite subsets of Polynom-Ring(n, L). Suppose G_1 is a Groebner basis of I, T and reduced wrt T and G_2 is a Groebner basis of I, T and reduced wrt T. Then $G_1 = G_2$.

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Construction of Gröbner bases. S-Polynomials and Standard Representations

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Summary. We continue the Mizar formalization of Gröbner bases following [6]. In this article we introduce S-polynomials and standard representations and show how these notions can be used to characterize Gröbner bases.

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The notation and terminology used here are introduced in the following papers: [24], [31], [32], [34], [33], [8], [3], [15], [30], [29], [9], [7], [5], [14], [12], [19], [18], [25], [28], [17], [1], [4], [13], [22], [21], [27], [26], [16], [10], [23], [2], [20], [11], and [35].

1. Preliminaries

One can prove the following propositions:

(1) For every set X and for every finite sequence p of elements of X such that $p \neq \emptyset$ holds $p \restriction 1 = \langle p_1 \rangle$.

(2) Let L be a non empty loop structure, p be a finite sequence of elements of L, and n, m be natural numbers. If $m \leqslant n$, then $p \restriction n \restriction m = p \restriction m$.

(3) Let L be an add-associative right zeroed right complementable non empty loop structure, p be a finite sequence of elements of L, and n be a natural number. Suppose that for every natural number k such that $k \in \operatorname{dom} p$ and $k > n$ holds $p(k) = 0_L$. Then $\sum p = \sum (p \restriction n)$.

(4) Let L be an add-associative right zeroed Abelian non empty loop structure, f be a finite sequence of elements of L, and i, j be natural numbers. Then $\sum \operatorname{Swap}(f, i, j) = \sum f$.

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(5) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leqslant_T b_3$ and $b_2 \leqslant_T b_3$, then $\max_T(b_1, b_2) \leqslant_T b_3$.

(6) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_3 \leqslant_T b_1$ and $b_3 \leqslant_T b_2$, then $b_3 \leqslant_T \min_T(b_1, b_2)$.

Let X be a set and let b_1, b_2 be bags of X. Let us assume that $b_2 \mid b_1$. The functor $\frac{b_1}{b_2}$ yields a bag of X and is defined by:

(Def. 1) $b_2 + \frac{b_1}{b_2} = b_1$.

Let X be a set and let b_1, b_2 be bags of X. The functor $\mathrm{lcm}(b_1, b_2)$ yields a bag of X and is defined as follows:

(Def. 2) For every set k holds $\mathrm{lcm}(b_1, b_2)(k) = \max(b_1(k), b_2(k))$.

Let us observe that the functor $\mathrm{lcm}(b_1, b_2)$ is commutative and idempotent. We introduce $\mathrm{lcm}(b_1, b_2)$ as a synonym of $\mathrm{lcm}(b_1, b_2)$.

Let X be a set and let b_1, b_2 be bags of X. We say that b_1, b_2 are disjoint if and only if:

(Def. 3) For every set i holds $b_1(i) = 0$ or $b_2(i) = 0$.

We introduce b_1, b_2 are non disjoint as an antonym of b_1, b_2 are disjoint.

We now state several propositions:

(7) For every set X and for all bags b_1, b_2 of X holds $b_1 \mid \mathrm{lcm}(b_1, b_2)$ and $b_2 \mid \mathrm{lcm}(b_1, b_2)$.

(8) For every set X and for all bags b_1, b_2, b_3 of X such that $b_1 \mid b_3$ and $b_2 \mid b_3$ holds $\mathrm{lcm}(b_1, b_2) \mid b_3$.

(9) Let X be a set, T be a term order of X, and b_1, b_2 be bags of X. Then b_1, b_2 are disjoint if and only if $\mathrm{lcm}(b_1, b_2) = b_1 + b_2$.

(10) For every set X and for every bag b of X holds $\frac{b}{b} = \mathrm{EmptyBag}\, X$.

(11) For every set X and for all bags b_1, b_2 of X holds $b_2 \mid b_1$ iff $\mathrm{lcm}(b_1, b_2) = b_1$.

(12) For every set X and for all bags b_1, b_2, b_3 of X such that $b_1 \mid \mathrm{lcm}(b_2, b_3)$ holds $\mathrm{lcm}(b_2, b_1) \mid \mathrm{lcm}(b_2, b_3)$.

(13) For every set X and for all bags b_1, b_2, b_3 of X such that $\mathrm{lcm}(b_2, b_1) \mid \mathrm{lcm}(b_2, b_3)$ holds $\mathrm{lcm}(b_1, b_3) \mid \mathrm{lcm}(b_2, b_3)$.

(14) For every set n and for all bags b_1, b_2, b_3 of n such that $\mathrm{lcm}(b_1, b_3) \mid \mathrm{lcm}(b_2, b_3)$ holds $b_1 \mid \mathrm{lcm}(b_2, b_3)$.

(15) Let n be a natural number, T be a connected admissible term order of n, and P be a non empty subset of $\mathrm{Bags}\, n$. Then there exists a bag b of n such that $b \in P$ and for every bag b' of n such that $b' \in P$ holds $b \leqslant_T b'$.

Let L be an add-associative right zeroed right complementable non trivial loop structure and let a be a non-zero element of L. Note that $-a$ is non-zero.

Let X be a set, let L be a left zeroed right zeroed add-cancelable distributive non empty double loop structure, let m be a monomial of X, L, and let a be an element of L. One can verify that $a \cdot m$ is monomial-like.

Let n be an ordinal number, let L be a left zeroed right zeroed add-cancelable distributive integral domain-like non trivial double loop structure, let p be a non-zero polynomial of n, L, and let a be a non-zero element of L. One can verify that $a \cdot p$ is non-zero.

Next we state several propositions:

(16) Let n be an ordinal number, T be a term order of n, L be a right zeroed right distributive non empty double loop structure, p, q be series of n, L, and b be a bag of n. Then $b * (p + q) = b * p + b * q$.

(17) Let n be an ordinal number, T be a term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a series of n, L, and b be a bag of n. Then $b * -p = -b * p$.

(18) Let n be an ordinal number, T be a term order of n, L be a left zeroed add-right-cancelable right distributive non empty double loop structure, p be a series of n, L, b be a bag of n, and a be an element of L. Then $b * (a \cdot p) = a \cdot (b * p)$.

(19) Let n be an ordinal number, T be a term order of n, L be a right distributive non empty double loop structure, p, q be series of n, L, and a be an element of L. Then $a \cdot (p + q) = a \cdot p + a \cdot q$.

(20) Let X be a set, L be an add-associative right zeroed right complementable non empty double loop structure, and a be an element of L. Then $-(a\,\_(X, L)) = -a\,\_(X, L)$.

2. S-Polynomials

The following proposition is true

(21) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and P be a subset of Polynom-Ring(n, L). Suppose $0_n L \notin P$. Suppose that for all polynomials p_1, p_2 of n, L such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ and for all monomials m_1, m_2 of n, L such that $\mathrm{HM}(m_1 * p_1, T) = \mathrm{HM}(m_2 * p_2, T)$ holds PolyRedRel(P, T) reduces $m_1 * p_1 - m_2 * p_2$ to $0_n L$. Then P is a Groebner basis wrt T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let p_1, p_2 be polynomials of n, L. The functor S-Poly(p_1, p_2, T) yielding a polynomial of n, L is defined by:

(Def. 4) $\text{S-Poly}(p_1, p_2, T) = \text{HC}(p_2, T) \cdot (\frac{\text{lcm}(\text{HT}(p_1,T),\text{HT}(p_2,T))}{\text{HT}(p_1,T)} * p_1) - \text{HC}(p_1, T) \cdot (\frac{\text{lcm}(\text{HT}(p_1,T),\text{HT}(p_2,T))}{\text{HT}(p_2,T)} * p_2)$.

One can prove the following propositions:

(22) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like Abelian non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and p_1, p_2 be polynomials of n, L. If $p_1 \in P$ and $p_2 \in P$, then $\text{S-Poly}(p_1, p_2, T) \in P$–ideal.

(23) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. If $p_1 = p_2$, then $\text{S-Poly}(p_1, p_2, T) = 0_n L$.

(24) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and m_1, m_2 be monomials of n, L. Then $\text{S-Poly}(m_1, m_2, T) = 0_n L$.

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p be a polynomial of n, L. Then $\text{S-Poly}(p, 0_n L, T) = 0_n L$ and $\text{S-Poly}(0_n L, p, T) = 0_n L$.

(26) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. Then $\text{S-Poly}(p_1, p_2, T) = 0_n L$ or $\text{HT}(\text{S-Poly}(p_1, p_2, T), T) <_T \text{lcm}(\text{HT}(p_1, T), \text{HT}(p_2, T))$.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p_1, p_2 be non-zero polynomials of n, L. If $\text{HT}(p_2, T) \mid \text{HT}(p_1, T)$, then $\text{HC}(p_2, T) \cdot p_1$ top reduces to $\text{S-Poly}(p_1, p_2, T)$, p_2, T.

(28) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of Polynom-Ring(n, L). Suppose G is a Groebner basis wrt T. Let g_1, g_2, h be polynomials of n, L. If $g_1 \in G$ and $g_2 \in G$ and h is a normal form of $\text{S-Poly}(g_1, g_2, T)$ w.r.t. PolyRedRel(G, T), then $h = 0_n L$.

(29) Let n be a natural number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed com-

mutative associative well unital distributive field-like non degenerated non empty double loop structure, and G be a subset of Polynom-Ring(n, L). Suppose that for all polynomials g_1, g_2, h of n, L such that $g_1 \in G$ and $g_2 \in G$ and h is a normal form of S-Poly(g_1, g_2, T) w.r.t. PolyRedRel(G, T) holds $h = 0_n L$. Let g_1, g_2 be polynomials of n, L. If $g_1 \in G$ and $g_2 \in G$, then PolyRedRel(G, T) reduces S-Poly(g_1, g_2, T) to $0_n L$.

(30) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of Polynom-Ring(n, L). Suppose $0_n L \notin G$. Suppose that for all polynomials g_1, g_2 of n, L such that $g_1 \in G$ and $g_2 \in G$ holds PolyRedRel(G, T) reduces S-Poly(g_1, g_2, T) to $0_n L$. Then G is a Groebner basis wrt T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let P be a subset of Polynom-Ring(n, L). The functor S-Poly(P, T) yielding a subset of Polynom-Ring(n, L) is defined by:

(Def. 5) S-Poly$(P, T) = \{$S-Poly$(p_1, p_2, T); p_1$ ranges over polynomials of n, L, p_2 ranges over polynomials of n, L: $p_1 \in P \ \wedge\ p_2 \in P\}$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let P be a finite subset of Polynom-Ring(n, L). One can check that S-Poly(P, T) is finite.

One can prove the following proposition

(31) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of Polynom-Ring(n, L). Suppose $0_n L \notin G$ and for every polynomial g of n, L such that $g \in G$ holds g is a monomial of n, L. Then G is a Groebner basis wrt T.

3. Standard Representations

The following three propositions are true:

(32) Let L be a non empty multiplicative loop structure, P be a non empty subset of L, A be a left linear combination of P, and i be a natural number. Then $A{\restriction}i$ is a left linear combination of P.

(33) Let L be a non empty multiplicative loop structure, P be a non empty subset of L, A be a left linear combination of P, and i be a natural number. Then $A_{\downharpoonleft i}$ is a left linear combination of P.

(34) Let L be a non empty multiplicative loop structure, P, Q be non empty subsets of the carrier of L, and A be a left linear combination of P. If $P \subseteq Q$, then A is a left linear combination of Q.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let P be a non empty subset of Polynom-Ring(n, L), and let A, B be left linear combinations of P. Then $A \frown B$ is a left linear combination of P.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let f be a polynomial of n, L, let P be a non empty subset of Polynom-Ring(n, L), and let A be a left linear combination of P. We say that A is a monomial representation of f if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) $\sum A = f$, and

(ii) for every natural number i such that $i \in \operatorname{dom} A$ there exists a monomial m of n, L and there exists a polynomial p of n, L such that $p \in P$ and $A_i = m * p$.

Next we state two propositions:

(35) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f be a polynomial of n, L, P be a non empty subset of Polynom-Ring(n, L), and A be a left linear combination of P. Suppose A is a monomial representation of f. Then Support $f \subseteq \bigcup\{\mathrm{Support}(m * p); m$ ranges over monomials of n, L, p ranges over polynomials of n, L: $\bigvee_{i\,:\,\text{natural number}} (i \in \operatorname{dom} A \ \wedge\ A_i = m * p)\}$.

(36) Let n be an ordinal number, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f, g be polynomials of n, L, P be a non empty subset of Polynom-Ring(n, L), and A, B be left linear combinations of P. Suppose A is a monomial representation of f and B is a monomial representation of g. Then $A \frown B$ is a monomial representation of $f + g$.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let f be a polynomial of n, L, let P be a non empty subset of Polynom-Ring(n, L), let A be a left linear combination of P, and let b be a bag of n. We say that A is a standard representation of f, P, b, T if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) $\sum A = f$, and

(ii) for every natural number i such that $i \in \operatorname{dom} A$ there exists a non-zero monomial m of n, L and there exists a non-zero polynomial p of n, L such that $p \in P$ and $A_i = m * p$ and $\mathrm{HT}(m * p, T) \leqslant_T b$.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let f be a polynomial of n, L, let P be a non empty subset of Polynom-Ring(n, L), and let A be a left linear combination of P. We say that A is a standard representation of f, P, T if and only if:

(Def. 8) A is a standard representation of f, P, $\mathrm{HT}(f, T)$, T.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let f be a polynomial of n, L, let P be a non empty subset of Polynom-Ring(n, L), and let b be a bag of n. We say that f has a standard representation of P, b, T if and only if:

(Def. 9) There exists a left linear combination of P which is a standard representation of f, P, b, T.

Let n be an ordinal number, let T be a connected term order of n, let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, let f be a polynomial of n, L, and let P be a non empty subset of Polynom-Ring(n, L). We say that f has a standard representation of P, T if and only if:

(Def. 10) There exists a left linear combination of P which is a standard representation of f, P, T.

One can prove the following propositions:

(37) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f be a polynomial of n, L, P be a non empty subset of Polynom-Ring(n, L), A be a left linear combination of P, and b be a bag of n. Suppose A is a standard representation of f, P, b, T. Then A is a monomial representation of f.

(38) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f, g be polynomials of n, L, P be a non empty subset of Polynom-Ring(n, L), A, B be left linear combinations of P, and b be a bag of n. Suppose A is a standard representation of f, P, b, T and B is a standard representation of g, P, b, T. Then $A \frown B$ is a standard representation of $f + g$, P, b, T.

(39) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f, g be polynomials of n, L, P be a non empty subset of Polynom-Ring(n, L), A, B be left linear combinations of P, b be a bag of n, and i be a natural number. Suppose A is a standard representation of f, P, b, T and $B = A{\restriction}i$ and $g = \sum(A_{\lfloor i})$.

Then B is a standard representation of $f - g$, P, b, T.

(40) Let n be an ordinal number, T be a connected term order of n, L be an Abelian right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f, g be polynomials of n, L, P be a non empty subset of Polynom-Ring(n, L), A, B be left linear combinations of P, b be a bag of n, and i be a natural number. Suppose A is a standard representation of f, P, b, T and $B = A_{\restriction i}$ and $g = \sum(A \restriction i)$ and $i \leqslant \operatorname{len} A$. Then B is a standard representation of $f - g$, P, b, T.

(41) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f be a non-zero polynomial of n, L, P be a non empty subset of Polynom-Ring(n, L), and A be a left linear combination of P. Suppose A is a monomial representation of f. Then there exists a natural number i and there exists a non-zero monomial m of n, L and there exists a non-zero polynomial p of n, L such that $i \in \operatorname{dom} A$ and $p \in P$ and $A(i) = m * p$ and $\mathrm{HT}(f, T) \leqslant_T \mathrm{HT}(m * p, T)$.

(42) Let n be an ordinal number, T be a connected term order of n, L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure, f be a non-zero polynomial of n, L, P be a non empty subset of Polynom-Ring(n, L), and A be a left linear combination of P. Suppose A is a standard representation of f, P, T. Then there exists a natural number i and there exists a non-zero monomial m of n, L and there exists a non-zero polynomial p of n, L such that $p \in P$ and $i \in \operatorname{dom} A$ and $A_i = m * p$ and $\mathrm{HT}(f, T) = \mathrm{HT}(m * p, T)$.

(43) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f be a polynomial of n, L, and P be a non empty subset of Polynom-Ring(n, L) such that PolyRedRel(P, T) reduces f to $0_n L$. Then f has a standard representation of P, T.

(44) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f be a non-zero polynomial of n, L, and P be a non empty subset of Polynom-Ring(n, L). If f has a standard representation of P, T, then f is top reducible wrt P, T.

(45) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a non empty subset of Polynom-Ring(n, L). Then G is a Groebner basis wrt T if and only if for

every non-zero polynomial f of n, L such that $f \in G$–ideal holds f has a standard representation of G, T.

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On the Subcontinua of a Real Line[1]

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Summary. In [10] we showed that the only proper subcontinua of the simple closed curve are arcs and single points. In this article we prove that the only proper subcontinua of the real line are closed intervals. We introduce some auxiliary notions such as $]a,b[_{\mathbb{Q}}$, $]a,b[_{\mathbb{IQ}}$ – intervals consisting of rational and irrational numbers respectively. We show also some basic topological properties of intervals.

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The notation and terminology used in this paper are introduced in the following papers: [24], [27], [22], [23], [18], [25], [28], [3], [4], [26], [19], [6], [21], [13], [16], [17], [1], [8], [5], [9], [14], [7], [20], [15], [12], [11], and [2].

1. Preliminaries

The following three propositions are true:

(1) For all sets A, B, C, D holds $(A \cup B \cup C) \cup D = A \cup (B \cup C \cup D)$.

(2) For all sets A, B, a such that $A \subseteq B$ and $B \subseteq A \cup \{a\}$ holds $A \cup \{a\} = B$ or $A = B$.

(3) For all sets x_1, x_2, x_3, x_4, x_5, x_6 holds $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{x_1, x_3, x_6\} \cup \{x_2, x_4, x_5\}$.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7 are sets.

Let x_1, x_2, x_3, x_4, x_5, x_6 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different if and only if the conditions (Def. 1) are satisfied.

(Def. 1) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_5 \neq x_6$.

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Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different if and only if the conditions (Def. 2) are satisfied.

(Def. 2) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_1 \neq x_7$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_2 \neq x_7$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_3 \neq x_7$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_4 \neq x_7$ and $x_5 \neq x_6$ and $x_5 \neq x_7$ and $x_6 \neq x_7$.

One can prove the following propositions:

(4) For all sets x_1, x_2, x_3, x_4, x_5, x_6 such that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different holds $\mathrm{card}\{x_1, x_2, x_3, x_4, x_5, x_6\} = 6$.

(5) For all sets x_1, x_2, x_3, x_4, x_5, x_6, x_7 such that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different holds $\mathrm{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} = 7$.

(6) If $\{x_1, x_2, x_3\}$ misses $\{x_4, x_5, x_6\}$, then $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$.

(7) Suppose x_1, x_2, x_3 are mutually different and x_4, x_5, x_6 are mutually different and $\{x_1, x_2, x_3\}$ misses $\{x_4, x_5, x_6\}$. Then x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(8) Suppose x_1, x_2, x_3, x_4, x_5, x_6 are mutually different and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ misses $\{x_7\}$. Then x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different.

(9) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_7, x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(10) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_1, x_2, x_5, x_3, x_6, x_7, x_4 are mutually different.

(11) Let T be a non empty topological space and a, b be points of T. Given a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. Then there exists a map g from $\mathbb{I}$ into T such that g is continuous and $g(0) = b$ and $g(1) = a$.

Let us observe that $\mathbb{R}^{\mathbf{1}}$ is arcwise connected.

Let us note that there exists a topological space which is connected and non empty.

2. Intervals

The following two propositions are true:

(12) Every subset of $\mathbb{R}$ is a subset of $\mathbb{R}^{\mathbf{1}}$.

(13) $\Omega_{\mathbb{R}^{\mathbf{1}}} = \mathbb{R}$.

Let a be a real number. We introduce $]-\infty, a]$ as a synonym of $]-\infty, a]$. We introduce $]-\infty, a[$ as a synonym of $]-\infty, a[$. We introduce $[a, +\infty[$ as a synonym of $[a, +\infty[$. We introduce $]a, +\infty[$ as a synonym of $]a, +\infty[$.

Next we state a number of propositions:

(14) For all real numbers a, b holds $a \in]b, +\infty[$ iff $a > b$.

(15) For all real numbers a, b holds $a \in [b, +\infty[$ iff $a \geqslant b$.

(16) For all real numbers a, b holds $a \in]-\infty, b]$ iff $a \leqslant b$.

(17) For all real numbers a, b holds $a \in]-\infty, b[$ iff $a < b$.

(18) For every real number a holds $\mathbb{R} \setminus \{a\} =]-\infty, a[\cup]a, +\infty[$.

(19) For all real numbers a, b, c, d such that $a < b$ and $b \leqslant c$ holds $[a, b]$ misses $]c, d]$.

(20) For all real numbers a, b, c, d such that $a < b$ and $b \leqslant c$ holds $[a, b[$ misses $[c, d]$.

(21) Let A, B be subsets of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $b \leqslant c$ and $c < d$ and $A = [a, b[$ and $B =]c, d]$. Then A and B are separated.

(22) For every real number a holds $\mathbb{R} \setminus]-\infty, a[= [a, +\infty[$.

(23) For every real number a holds $\mathbb{R} \setminus]-\infty, a] =]a, +\infty[$.

(24) For every real number a holds $\mathbb{R} \setminus]a, +\infty[=]-\infty, a]$.

(25) For every real number a holds $\mathbb{R} \setminus [a, +\infty[=]-\infty, a[$.

(26) For every real number a holds $]-\infty, a]$ misses $]a, +\infty[$.

(27) For every real number a holds $]-\infty, a[$ misses $[a, +\infty[$.

(28) For all real numbers a, b, c such that $a \leqslant c$ and $c \leqslant b$ holds $[a, b] \cup [c, +\infty[= [a, +\infty[$.

(29) For all real numbers a, b, c such that $a \leqslant c$ and $c \leqslant b$ holds $]-\infty, c] \cup [a, b] =]-\infty, b]$.

(30) For every 1-sorted structure T and for every subset A of T holds $\{A\}$ is a family of subsets of T.

(31) For every 1-sorted structure T and for all subsets A, B of T holds $\{A, B\}$ is a family of subsets of T.

(32) For every 1-sorted structure T and for all subsets A, B, C of T holds $\{A, B, C\}$ is a family of subsets of T.

Let us observe that every element of $\mathbb{Q}$ is real.

Let us observe that every element of the carrier of the metric space of real numbers is real.

Next we state four propositions:

(33) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and p be a point of the metric space of real numbers. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p, r)$ meets A.

(34) For all elements p, q of the carrier of the metric space of real numbers such that $q \geqslant p$ holds $\rho(p, q) = q - p$.

(35) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(36) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a, b[$ and $a \neq b$ holds $\overline{A} = [a, b]$.

3. Rational and Irrational Numbers

Let us mention that e is irrational.

The subset $\mathbb{IQ}$ of $\mathbb{R}$ is defined by:

(Def. 3) $\mathbb{IQ} = \mathbb{R} \setminus \mathbb{Q}$.

Let a, b be real numbers. The functor $]a, b[_{\mathbb{Q}}$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 4) $]a, b[_{\mathbb{Q}} = \mathbb{Q} \cap]a, b[$.

The functor $]a, b[_{\mathbb{IQ}}$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 5) $]a, b[_{\mathbb{IQ}} = \mathbb{IQ} \cap]a, b[$.

One can prove the following proposition

(37) For every real number x holds x is irrational iff $x \in \mathbb{IQ}$.

Let us observe that there exists a real number which is irrational.

Let us note that $\mathbb{IQ}$ is non empty.

Next we state several propositions:

(38) For every rational number a and for every irrational real number b holds $a + b$ is irrational.

(39) For every irrational real number a holds $-a$ is irrational.

(40) For every rational number a and for every irrational real number b holds $a - b$ is irrational.

(41) For every rational number a and for every irrational real number b holds $b - a$ is irrational.

(42) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $a \cdot b$ is irrational.

(43) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{b}{a}$ is irrational.

One can check that every real number which is irrational is also non zero.

The following propositions are true:

(44) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{a}{b}$ is irrational.

(45) For every irrational real number r holds $\operatorname{frac} r$ is irrational.

Let r be an irrational real number. Note that $\operatorname{frac} r$ is irrational.

Let a be an irrational real number. Note that $-a$ is irrational.

Let a be a rational number and let b be an irrational real number. One can verify the following observations:

* $a + b$ is irrational,
* $b + a$ is irrational,
* $a - b$ is irrational, and
* $b - a$ is irrational.

Let us observe that there exists a rational number which is non zero.

Let a be a non zero rational number and let b be an irrational real number. One can check the following observations:

* $a \cdot b$ is irrational,
* $b \cdot a$ is irrational,
* $\frac{a}{b}$ is irrational, and
* $\frac{b}{a}$ is irrational.

The following propositions are true:

(46) For every irrational real number r holds $0 < \operatorname{frac} r$.

(47) For all real numbers a, b such that $a < b$ there exist rational numbers p_1, p_2 such that $a < p_1$ and $p_1 < p_2$ and $p_2 < b$.

(48) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 < s_4$ and $0 < l$ and $l < 1$ holds $s_1 < (1 - l) \cdot s_3 + l \cdot s_4$.

(49) For all real numbers s_1, s_3, s_4, l such that $s_3 < s_1$ and $s_4 \leqslant s_1$ and $0 < l$ and $l < 1$ holds $(1 - l) \cdot s_3 + l \cdot s_4 < s_1$.

(50) For all real numbers a, b such that $a < b$ there exists an irrational real number p such that $a < p$ and $p < b$.

(51) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(52) For all real numbers a, b, c such that $a < b$ holds $c \in]a, b[_{\mathbb{Q}}$ iff c is rational and $a < c$ and $c < b$.

(53) For all real numbers a, b, c such that $a < b$ holds $c \in]a, b[_{\mathbb{IQ}}$ iff c is irrational and $a < c$ and $c < b$.

(54) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[_{\mathbb{Q}}$ holds $\overline{A} = [a, b]$.

(55) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[_{\mathbb{IQ}}$ holds $\overline{A} = [a, b]$.

(56) For every connected topological space T and for every closed open subset A of T holds $A = \emptyset$ or $A = \Omega_T$.

(57) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that A is closed and open holds $A = \emptyset$ or $A = \mathbb{R}$.

4. Topological Properties of Intervals

We now state a number of propositions:

(58) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A = [a, b[$ and $a \neq b$ holds $\overline{A} = [a, b]$.

(59) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a, b]$ and $a \neq b$ holds $\overline{A} = [a, b]$.

(60) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $A = [a, b[\cup]b, c]$ and $a < b$ and $b < c$, then $\overline{A} = [a, c]$.

(61) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = \{a\}$ holds $\overline{A} = \{a\}$.

(62) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds A is open iff B is open.

(63) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]a, +\infty[$ holds A is open.

(64) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a[$ holds A is open.

(65) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a]$ holds A is closed.

(66) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = [a, +\infty[$ holds A is closed.

(67) For every real number a holds $[a, +\infty[= \{a\} \cup]a, +\infty[$.

(68) For every real number a holds $]-\infty, a] = \{a\} \cup]-\infty, a[$.

(69) For every real number a holds $]a, +\infty[\subseteq [a, +\infty[$.

(70) For every real number a holds $]-\infty, a[\subseteq]-\infty, a]$.

Let a be a real number. One can check the following observations:

* $]a, +\infty[$ is non empty,
* $]-\infty, a]$ is non empty,
* $]-\infty, a[$ is non empty, and
* $[a, +\infty[$ is non empty.

The following propositions are true:

(71) For every real number a holds $]a, +\infty[\neq \mathbb{R}$.

(72) For every real number a holds $[a, +\infty[\neq \mathbb{R}$.

(73) For every real number a holds $]-\infty, a] \neq \mathbb{R}$.

(74) For every real number a holds $]-\infty, a[\neq \mathbb{R}$.

(75) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]a, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(76) For every real number a holds $\overline{]a, +\infty[} = [a, +\infty[$.

(77) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a[$ holds $\overline{A} =]-\infty, a]$.

(78) For every real number a holds $\overline{]-\infty, a[} =]-\infty, a]$.

(79) Let A, B be subsets of the carrier of $\mathbb{R}^{\mathbf{1}}$ and b be a real number. If $A =]-\infty, b[$ and $B =]b, +\infty[$, then A and B are separated.

(80) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A = [a, b[\cup]b, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(81) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[\cup]b, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(82) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A =]a, b[_{\mathbb{Q}} \cup]b, c[\cup]c, +\infty[$, then $\overline{A} = [a, +\infty[$.

(83) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ holds $-A = \mathbb{R} \setminus A$.

(84) For all real numbers a, b such that $a < b$ holds $]a, b[_{\mathbb{IQ}}$ misses $]a, b[_{\mathbb{Q}}$.

(85) For all real numbers a, b such that $a < b$ holds $\mathbb{R} \setminus]a, b[_{\mathbb{Q}} =]-\infty, a] \cup]a, b[_{\mathbb{IQ}} \cup [b, +\infty[$.

(86) For all real numbers a, b, c such that $a \leqslant b$ and $b < c$ holds $a \notin]b, c[\cup]c, +\infty[$.

(87) For all real numbers a, b such that $a < b$ holds $b \notin]a, b[\cup]b, +\infty[$.

(88) For all real numbers a, b such that $a < b$ holds $[a, +\infty[\setminus (]a, b[\cup]b, +\infty[) = \{a\} \cup \{b\}$.

(89) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A =]2, 3[_{\mathbb{Q}} \cup]3, 4[\cup]4, +\infty[$ holds $-A =]-\infty, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(90) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = \{a\}$ holds $-A =]-\infty, a[\cup]a, +\infty[$.

(91) For all real numbers a, b such that $a < b$ holds $]a, +\infty[\cap]-\infty, b] =]a, b]$.

(92) $(]-\infty, 1[\cup]1, +\infty[) \cap (]-\infty, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}) =]-\infty, 1[\cup]1, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(93) For all real numbers a, b such that $a \leqslant b$ holds $]-\infty, b[\setminus \{a\} =]-\infty, a[\cup]a, b[$.

(94) For all real numbers a, b such that $a \leqslant b$ holds $]a, +\infty[\setminus \{b\} =]a, b[\cup]b, +\infty[$.

(95) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b be real numbers. If $a \leqslant b$ and $A = \{a\} \cup [b, +\infty[$, then $-A =]-\infty, a[\cup]a, b[$.

(96) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]-\infty, a[\cup]a, b[$ holds $\overline{A} =]-\infty, b]$.

(97) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]-\infty, a[\cup]a, b]$ holds $\overline{A} =]-\infty, b]$.

(98) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a]$ holds $-A =]a, +\infty[$.

(99) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = [a, +\infty[$ holds $-A =]-\infty, a[$.

(100) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\}$, then $\overline{A} =]-\infty, c]$.

(101) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. If $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\} \cup \{d\}$, then $\overline{A} =]-\infty, c] \cup \{d\}$.

(102) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b be real numbers. If $a \leqslant b$ and $A =]-\infty, a] \cup \{b\}$, then $-A =]a, b[\cup]b, +\infty[$.

(103) For all real numbers a, b holds $[a, +\infty[\cup \{b\} \neq \mathbb{R}$.

(104) For all real numbers a, b holds $]-\infty, a] \cup \{b\} \neq \mathbb{R}$.

(105) For every topological structure T_1 and for all subsets A, B of the carrier of T_1 such that $A \neq B$ holds $-A \neq -B$.

(106) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $\mathbb{R} = -A$ holds $A = \emptyset$.

5. Subcontinua of a Real Line

Let us mention that $\mathbb{I}$ is arcwise connected.

We now state several propositions:

(107) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$ and X' be a subset of $\mathbb{R}$. If $X' = X$, then X' is upper bounded and lower bounded.

(108) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leqslant x$ and $x \leqslant \sup X'$.

(109) Let C be a non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(110) Let A be a connected subset of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a \leqslant b$ and $b \leqslant c$ and $a \in A$ and $c \in A$, then $b \in A$.

(111) For every connected subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a, b] \subseteq A$.

(112) Every non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ is a non empty closed-interval subset of $\mathbb{R}$.

(113) For every non empty compact connected subset A of $\mathbb{R}^{\mathbf{1}}$ there exist real numbers a, b such that $a \leqslant b$ and $A = [a, b]$.

6. Sets with Proper Subsets Only

Let T_1 be a topological structure and let F be a family of subsets of T_1. We say that F has proper subsets if and only if:

(Def. 6) The carrier of $T_1 \notin F$.

One can prove the following proposition

(114) Let T_1 be a topological structure and F, G be families of subsets of T_1 such that F has proper subsets and $G \subseteq F$. Then G has proper subsets.

Let T_1 be a non empty topological structure. Observe that there exists a family of subsets of T_1 which has proper subsets.

We now state the proposition

(115) Let T_1 be a non empty topological structure and A, B be families of subsets of T_1 with proper subsets. Then $A \cup B$ has proper subsets.

Let T be a topological structure and let F be a family of subsets of T. We say that F is open if and only if:

(Def. 7) For every subset P of T such that $P \in F$ holds P is open.

We say that F is closed if and only if:

(Def. 8) For every subset P of T such that $P \in F$ holds P is closed.

Let T be a topological space. Note that there exists a family of subsets of T which is open, closed, and non empty.

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On the Kuratowski Closure-Complement Problem

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Summary. In this article we formalize the Kuratowski closure-complement result: there is at most 14 distinct sets that one can produce from a given subset A of a topological space T by applying closure and complement operators and that all 14 can be obtained from a suitable subset of $\mathbb{R}$, namely KuratExSet $= \{1\} \cup \mathbb{Q}(2,3) \cup (3,4) \cup (4,\infty)$.

The second part of the article deals with the maximal number of distinct sets which may be obtained from a given subset A of T by applying closure and interior operators. The subset KuratExSet of $\mathbb{R}$ is also enough to show that 7 can be achieved.

MML Identifier: `KURATO_1`.

The papers [15], [16], [10], [13], [11], [17], [14], [1], [3], [12], [7], [6], [8], [2], [4], [9], and [5] provide the notation and terminology for this paper.

1. Fourteen Kuratowski Sets

In this paper T is a non empty topological space and A is a subset of T.

The following proposition is true

(1) $\overline{-\overline{-\overline{-\overline{A}}}} = \overline{-\overline{A}}$.

Let us consider T, A. The functor Kurat14Part(A) is defined as follows:

(Def. 1) $\mathrm{Kurat14Part}(A) = \{A, \overline{A}, -\overline{A}, \overline{-\overline{A}}, -\overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, -\overline{-\overline{-\overline{A}}}\}$.

Let us consider T, A. One can check that Kurat14Part(A) is finite.

Let us consider T, A. The functor Kurat14Set(A) yields a family of subsets of T and is defined by:

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(Def. 2) $\mathrm{Kurat14Set}(A) = \{A, \overline{A}, -\overline{A}, \overline{-\overline{A}}, -\overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, -\overline{-\overline{-\overline{A}}}\} \cup$ $\{-A, \overline{-A}, -\overline{-A}, \overline{-\overline{-A}}, -\overline{-\overline{-A}}, \overline{-\overline{-\overline{-A}}}, -\overline{-\overline{-\overline{-A}}}\}$.

We now state three propositions:

(2) $\mathrm{Kurat14Set}(A) = \mathrm{Kurat14Part}(A) \cup \mathrm{Kurat14Part}(-A)$.

(3) $A \in \mathrm{Kurat14Set}(A)$ and $\overline{A} \in \mathrm{Kurat14Set}(A)$ and $-\overline{A} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{A}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{A}} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-\overline{A}}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-\overline{A}}} \in \mathrm{Kurat14Set}(A)$.

(4) $-A \in \mathrm{Kurat14Set}(A)$ and $\overline{-A} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-A} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-A}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-A}} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-\overline{-A}}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-\overline{-A}}} \in \mathrm{Kurat14Set}(A)$.

Let us consider T, A. The functor $\mathrm{Kurat14ClosedPart}(A)$ yielding a family of subsets of T is defined by:

(Def. 3) $\mathrm{Kurat14ClosedPart}(A) = \{\overline{A}, \overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, \overline{-A}, \overline{-\overline{-A}}, \overline{-\overline{-\overline{-A}}}\}$.

The functor $\mathrm{Kurat14OpenPart}(A)$ yields a family of subsets of T and is defined as follows:

(Def. 4) $\mathrm{Kurat14OpenPart}(A) = \{-\overline{A}, -\overline{-\overline{A}}, -\overline{-\overline{-\overline{A}}}, -\overline{-A}, -\overline{-\overline{-A}}, -\overline{-\overline{-\overline{-A}}}\}$.

We now state the proposition

(5) $\mathrm{Kurat14Set}(A) = \{A, -A\} \cup \mathrm{Kurat14ClosedPart}(A) \cup \mathrm{Kurat14OpenPart}(A)$.

Let us consider T, A. One can verify that $\mathrm{Kurat14Set}(A)$ is finite.

Next we state two propositions:

(6) For every subset Q of the carrier of T such that $Q \in \mathrm{Kurat14Set}(A)$ holds $-Q \in \mathrm{Kurat14Set}(A)$ and $\overline{Q} \in \mathrm{Kurat14Set}(A)$.

(7) $\mathrm{card}\,\mathrm{Kurat14Set}(A) \leqslant 14$.

2. Seven Kuratowski Sets

Let us consider T, A. The functor $\mathrm{Kurat7Set}(A)$ yielding a family of subsets of T is defined as follows:

(Def. 5) $\mathrm{Kurat7Set}(A) = \{A, \mathrm{Int}\,A, \overline{A}, \mathrm{Int}\,\overline{A}, \overline{\mathrm{Int}\,A}, \overline{\mathrm{Int}\,\overline{A}}, \mathrm{Int}\,\overline{\mathrm{Int}\,A}\}$.

We now state two propositions:

(8) $A \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,A \in \mathrm{Kurat7Set}(A)$ and $\overline{A} \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,\overline{A} \in \mathrm{Kurat7Set}(A)$ and $\overline{\mathrm{Int}\,A} \in \mathrm{Kurat7Set}(A)$ and $\overline{\mathrm{Int}\,\overline{A}} \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \in \mathrm{Kurat7Set}(A)$.

(9) $\mathrm{Kurat7Set}(A) = \{A\} \cup \{\mathrm{Int}\,A, \mathrm{Int}\,\overline{A}, \mathrm{Int}\,\overline{\mathrm{Int}\,A}\} \cup \{\overline{A}, \overline{\mathrm{Int}\,A}, \overline{\mathrm{Int}\,\overline{A}}\}$.

Let us consider T, A. Note that $\mathrm{Kurat7Set}(A)$ is finite.

We now state two propositions:

(10) For every subset Q of the carrier of T such that $Q \in \mathrm{Kurat7Set}(A)$ holds $\mathrm{Int}\, Q \in \mathrm{Kurat7Set}(A)$ and $\overline{Q} \in \mathrm{Kurat7Set}(A)$.

(11) $\mathrm{card}\, \mathrm{Kurat7Set}(A) \leqslant 7$.

3. The Set Generating Exactly Fourteen Kuratowski Sets

The subset KuratExSet of $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 6) $\mathrm{KuratExSet} = \{1\} \cup]2,3[_{\mathbb{Q}} \cup]3,4[\cup]4,+\infty[$.

Next we state a number of propositions:

(12) $\overline{\mathrm{KuratExSet}} = \{1\} \cup [2,+\infty[$.

(13) $-\overline{\mathrm{KuratExSet}} =]-\infty,1[\cup]1,2[$.

(14) $\overline{-\overline{\mathrm{KuratExSet}}} =]-\infty,2]$.

(15) $-\overline{-\overline{\mathrm{KuratExSet}}} =]2,+\infty[$.

(16) $\overline{-\overline{-\overline{\mathrm{KuratExSet}}}} = [2,+\infty[$.

(17) $-\overline{-\overline{-\overline{\mathrm{KuratExSet}}}} =]-\infty,2[$.

(18) $-\mathrm{KuratExSet} =]-\infty,1[\cup]1,2] \cup]2,3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(19) $\overline{-\mathrm{KuratExSet}} =]-\infty,3] \cup \{4\}$.

(20) $-\overline{-\mathrm{KuratExSet}} =]3,4[\cup]4,+\infty[$.

(21) $\overline{-\overline{-\mathrm{KuratExSet}}} = [3,+\infty[$.

(22) $-\overline{-\overline{-\mathrm{KuratExSet}}} =]-\infty,3[$.

(23) $\overline{-\overline{-\overline{-\mathrm{KuratExSet}}}} =]-\infty,3]$.

(24) $-\overline{-\overline{-\overline{-\mathrm{KuratExSet}}}} =]3,+\infty[$.

4. The Set Generating Exactly Seven Kuratowski Sets

Next we state several propositions:

(25) $\mathrm{Int}\, \mathrm{KuratExSet} =]3,4[\cup]4,+\infty[$.

(26) $\overline{\mathrm{Int}\, \mathrm{KuratExSet}} = [3,+\infty[$.

(27) $\mathrm{Int}\, \overline{\mathrm{Int}\, \mathrm{KuratExSet}} =]3,+\infty[$.

(28) $\mathrm{Int}\, \overline{\mathrm{KuratExSet}} =]2,+\infty[$.

(29) $\overline{\mathrm{Int}\, \overline{\mathrm{KuratExSet}}} = [2,+\infty[$.

5. The Difference Between Chosen Kuratowski Sets

One can prove the following propositions:

(30) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(31) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{KuratExSet}}$.

(32) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(33) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(34) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(35) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(36) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(37) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(38) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(39) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(40) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(41) $\mathrm{Int}\,\mathrm{KuratExSet} \neq \overline{\mathrm{KuratExSet}}$.

(42) $\overline{\mathrm{KuratExSet}} \neq \mathrm{KuratExSet}$.

(43) $\mathrm{KuratExSet} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(44) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(45) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(46) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

6. Final Proofs For Seven Sets

The following propositions are true:

(47) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(48) $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$ are mutually different.

(49) $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(50) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds X is an open non empty subset of $\mathbb{R}^{\mathbf{1}}$.

(51) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds X is a closed subset of $\mathbb{R}^{\mathbf{1}}$.

(52) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds $X \neq \mathbb{R}$.

(53) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds $X \neq \mathbb{R}$.

(54) $\{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ misses $\{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$.

(55) $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

Let us note that KuratExSet is non closed and non open.

Next we state three propositions:

(56) $\{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ misses $\{\mathrm{KuratExSet}\}$.

(57) KuratExSet, $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(58) $\mathrm{card}\,\mathrm{Kurat7Set}(\mathrm{KuratExSet}) = 7$.

7. Final Proofs For Fourteen Sets

One can check that Kurat14ClosedPart(KuratExSet) has proper subsets and Kurat14OpenPart(KuratExSet) has proper subsets.

One can verify that Kurat14Set(KuratExSet) has proper subsets.

Let us note that Kurat14Set(KuratExSet) has non empty elements.

We now state the proposition

(59) For every set A with non empty elements and for every set B such that $B \subseteq A$ holds B has non empty elements.

Let us note that Kurat14ClosedPart(KuratExSet) has non empty elements and Kurat14OpenPart(KuratExSet) has non empty elements.

Let us note that there exists a family of subsets of $\mathbb{R}^{\mathbf{1}}$ which has proper subsets and non empty elements.

We now state the proposition

(60) Let F, G be families of subsets of $\mathbb{R}^{\mathbf{1}}$ with proper subsets and non empty elements. If F is open and G is closed, then F misses G.

Let us mention that Kurat14ClosedPart(KuratExSet) is closed and Kurat14OpenPart(KuratExSet) is open.

One can prove the following proposition

(61) Kurat14ClosedPart(KuratExSet) misses Kurat14OpenPart(KuratExSet).

Let us consider T, A. Observe that Kurat14ClosedPart(A) is finite and Kurat14OpenPart(A) is finite.

We now state three propositions:

(62) $\mathrm{card}\,\mathrm{Kurat14ClosedPart}(\mathrm{KuratExSet}) = 6$.

(63) $\mathrm{card}\,\mathrm{Kurat14OpenPart}(\mathrm{KuratExSet}) = 6$.

(64) $\{\mathrm{KuratExSet}, -\mathrm{KuratExSet}\}$ misses Kurat14ClosedPart(KuratExSet).

Let us observe that KuratExSet is non empty.

The following three propositions are true:

(65) KuratExSet $\neq -$KuratExSet.

(66) $\{$KuratExSet, $-$KuratExSet$\}$ misses Kurat14OpenPart(KuratExSet).

(67) card Kurat14Set(KuratExSet) $= 14$.

8. Properties of Kuratowski Sets

Let T be a topological structure and let A be a family of subsets of T. We say that A is closed for closure operator if and only if:

(Def. 7) For every subset P of the carrier of T such that $P \in A$ holds $\overline{P} \in A$.

We say that A is closed for interior operator if and only if:

(Def. 8) For every subset P of the carrier of T such that $P \in A$ holds $\operatorname{Int} P \in A$.

Let T be a 1-sorted structure and let A be a family of subsets of T. We say that A is closed for complement operator if and only if:

(Def. 9) For every subset P of the carrier of T such that $P \in A$ holds $-P \in A$.

Let us consider T, A. One can verify the following observations:

* Kurat14Set(A) is non empty,
* Kurat14Set(A) is closed for closure operator, and
* Kurat14Set(A) is closed for complement operator.

Let us consider T, A. One can check the following observations:

* Kurat7Set(A) is non empty,
* Kurat7Set(A) is closed for interior operator, and
* Kurat7Set(A) is closed for closure operator.

Let us consider T. One can check that there exists a family of subsets of T which is closed for interior operator, closed for closure operator, and non empty and there exists a family of subsets of T which is closed for complement operator, closed for closure operator, and non empty.

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Convex Hull, Set of Convex Combinations and Convex Cone

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Summary. In this article, there are two themes. One of them is the proof that convex hull of a given subset M consists of all convex combinations of M. Another is definitions of cone and convex cone and some properties of them.

MML Identifier: `CONVEX3`.

The terminology and notation used in this paper are introduced in the following articles: [8], [11], [7], [2], [12], [3], [5], [1], [4], [10], [9], and [6].

1. Equality of Convex Hull and Set of Convex Combinations

Let V be a real linear space. The functor ConvexComb(V) yielding a set is defined by:

(Def. 1) For every set L holds $L \in \mathrm{ConvexComb}(V)$ iff L is a convex combination of V.

Let V be a real linear space and let M be a non empty subset of V. The functor ConvexComb(M) yielding a set is defined as follows:

(Def. 2) For every set L holds $L \in \mathrm{ConvexComb}(M)$ iff L is a convex combination of M.

We now state several propositions:

(1) Let V be a real linear space and v be a vector of V. Then there exists a convex combination L of V such that $\sum L = v$ and for every non empty subset A of V such that $v \in A$ holds L is a convex combination of A.

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(2) Let V be a real linear space and v_1, v_2 be vectors of V. Suppose $v_1 \neq v_2$. Then there exists a convex combination L of V such that for every non empty subset A of V if $\{v_1, v_2\} \subseteq A$, then L is a convex combination of A.

(3) Let V be a real linear space and v_1, v_2, v_3 be vectors of V. Suppose $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$. Then there exists a convex combination L of V such that for every non empty subset A of V if $\{v_1, v_2, v_3\} \subseteq A$, then L is a convex combination of A.

(4) Let V be a real linear space and M be a non empty subset of V. Then M is convex if and only if $\{\sum L; L$ ranges over convex combinations of M: $L \in \mathrm{ConvexComb}(V)\} \subseteq M$.

(5) Let V be a real linear space and M be a non empty subset of V. Then $\mathrm{conv}\, M = \{\sum L; L$ ranges over convex combinations of M: $L \in \mathrm{ConvexComb}(V)\}$.

2. Cone and Convex Cone

Let V be a non empty RLS structure and let M be a subset of V. We say that M is cone if and only if:

(Def. 3) For every real number r and for every vector v of V such that $r > 0$ and $v \in M$ holds $r \cdot v \in M$.

One can prove the following proposition

(6) For every non empty RLS structure V and for every subset M of V such that $M = \emptyset$ holds M is cone.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is cone.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is empty and cone.

Let V be a real linear space. Observe that there exists a subset of V which is non empty and cone.

The following propositions are true:

(7) Let V be a non empty RLS structure and M be a cone subset of V. Suppose V is real linear space-like. Then M is convex if and only if for all vectors u, v of V such that $u \in M$ and $v \in M$ holds $u + v \in M$.

(8) Let V be a real linear space and M be a subset of V. Then M is convex and cone if and only if for every linear combination L of M such that the support of $L \neq \emptyset$ and for every vector v of V such that $v \in$ the support of L holds $L(v) > 0$ holds $\sum L \in M$.

(9) For every non empty RLS structure V and for all subsets M, N of V such that M is cone and N is cone holds $M \cap N$ is cone.

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On the Two Short Axiomatizations of Ortholattices

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Summary. In the paper, two short axiom systems for Boolean algebras are introduced. In the first section we show that the single axiom (DN_1) proposed in [2] in terms of disjunction and negation characterizes Boolean algebras. To prove that (DN_1) is a single axiom for Robbins algebras (that is, Boolean algebras as well), we use the Otter theorem prover. The second section contains proof that the two classical axioms ($\mathrm{Meredith}_1$), ($\mathrm{Meredith}_2$) proposed by Meredith [3] may also serve as a basis for Boolean algebras. The results will be used to characterize ortholattices.

MML Identifier: `ROBBINS2`.

The terminology and notation used in this paper have been introduced in the following articles: [4], [5], and [1].

1. Single Axiom for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (DN_1) if and only if:

(Def. 1) For all elements x, y, z, u of the carrier of L holds $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+(z^{\mathrm{c}}+(z+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = z$.

Let us observe that TrivComplLat satisfies (DN_1) and TrivOrtLat satisfies (DN_1).

Let us observe that there exists a non empty complemented lattice structure which is join-commutative and join-associative and satisfies (DN_1).

Next we state a number of propositions:

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(1) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u, v be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + (((z+u)^{\mathrm{c}} + x)^{\mathrm{c}} + (y^{\mathrm{c}} + (y+v)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(2) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (y^{\mathrm{c}} + (y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(3) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $((x + x^{\mathrm{c}})^{\mathrm{c}} + x)^{\mathrm{c}} = x^{\mathrm{c}}$.

(4) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (((y+y^{\mathrm{c}})^{\mathrm{c}} + y)^{\mathrm{c}} + (y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(5) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(6) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(7) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{\mathrm{c}} + x)^{\mathrm{c}} + (x+y)^{\mathrm{c}})^{\mathrm{c}} = x$.

(8) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + ((x+y)^{\mathrm{c}} + x)^{\mathrm{c}})^{\mathrm{c}} = (x+y)^{\mathrm{c}}$.

(9) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x+y)^{\mathrm{c}} + z)^{\mathrm{c}} + (x+z)^{\mathrm{c}})^{\mathrm{c}} = z$.

(10) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x + ((y+z)^{\mathrm{c}} + (y+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = (y+x)^{\mathrm{c}}$.

(11) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((((x+y)^{\mathrm{c}} + z)^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} + y)^{\mathrm{c}} = (x^{\mathrm{c}} + y)^{\mathrm{c}}$.

(12) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x + ((y+z)^{\mathrm{c}} + (z+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = (z+x)^{\mathrm{c}}$.

(13) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (y^{\mathrm{c}} + (u+y)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(14) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^{\mathrm{c}} = (y+x)^{\mathrm{c}}$.

(15) Let L be a non empty complemented lattice structure satisfying (DN_1)

and x, y, z be elements of the carrier of L. Then $(((x+y)^{c}+(y+z)^{c})^{c}+z)^{c} = (y+z)^{c}$.

(16) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+((x+y)^{c}+z)^{c})^{c}+z)^{c} = ((x+y)^{c}+z)^{c}$.

(17) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{c}+x)^{c}+y)^{c} = (y+y)^{c}$.

(18) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x^{c}+(y+x)^{c})^{c} = x$.

(19) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+y^{c})^{c} = y$.

(20) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+(y+x^{c})^{c})^{c} = x^{c}$.

(21) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $(x+x)^{c} = x^{c}$.

(22) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{c}+x)^{c}+y)^{c} = y^{c}$.

(23) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $(x^{c})^{c} = x$.

(24) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+x)^{c}+y = (y^{c})^{c}$.

(25) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c})^{c} = y+x$.

(26) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+z)^{c}+(y+x)^{c})^{c} = ((y+x)^{c})^{c}$.

(27) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $x+y = y+x$.

One can verify that every non empty complemented lattice structure which satisfies (DN_1) is also join-commutative.

Next we state a number of propositions:

(28) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+x)^{c}+y = y$.

(29) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+y)^{c}+x = x$.

(30) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $x+((y+x)^{c}+y)^{c} = x$.

(31) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y^{c})^{c}+(y^{c}+y)^{c} = (x+y^{c})^{c}$.

(32) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^{\mathrm{c}}+(y+y^{\mathrm{c}})^{\mathrm{c}} = (x+y)^{\mathrm{c}}$.

(33) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^{\mathrm{c}}+(y^{\mathrm{c}}+y)^{\mathrm{c}} = (x+y)^{\mathrm{c}}$.

(34) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}+y)^{\mathrm{c}} = (y^{\mathrm{c}}+y)^{\mathrm{c}}$.

(35) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + y^{\mathrm{c}} + y)^{\mathrm{c}} = (y^{\mathrm{c}} + y)^{\mathrm{c}}$.

(36) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x + y^{\mathrm{c}} + z)^{\mathrm{c}} + y)^{\mathrm{c}} + (y^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(37) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+z)^{\mathrm{c}}+(y+x)^{\mathrm{c}})^{\mathrm{c}} = y + x$.

(38) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+((z+y)^{\mathrm{c}}+x)^{\mathrm{c}})^{\mathrm{c}} = (z + y)^{\mathrm{c}} + x$.

(39) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+x)^{\mathrm{c}}+(y+z)^{\mathrm{c}})^{\mathrm{c}} = y + x$.

(40) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x + y)^{\mathrm{c}} + ((x + y)^{\mathrm{c}} + (x + z)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} + y = y$.

(41) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x+y^{\mathrm{c}}+z)^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(42) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x + (y + x^{\mathrm{c}} + z)^{\mathrm{c}} = x$.

(43) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x^{\mathrm{c}} + (y + x + z)^{\mathrm{c}} = x^{\mathrm{c}}$.

(44) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + y)^{\mathrm{c}} + x = x + y^{\mathrm{c}}$.

(45) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + (x + y^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = (x + y)^{\mathrm{c}}$.

(46) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+y)^{\mathrm{c}}+(x+z))^{\mathrm{c}}+y = y$.

(47) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x + y)^{\mathrm{c}} + z)^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} + y = ((x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}}$.

(48) Let L be a non empty complemented lattice structure satisfying (DN_1)

and x, y, z be elements of the carrier of L. Then $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x^{\mathrm{c}}+y)^{\mathrm{c}})^{\mathrm{c}}+y = x^{\mathrm{c}}+y$.

(49) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^{\mathrm{c}}+(((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z) = ((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(50) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z) = ((y+x)^{\mathrm{c}})^{\mathrm{c}}+(y+z)$.

(51) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^{\mathrm{c}}+(y+x+(y+z))^{\mathrm{c}})^{\mathrm{c}}+(y+z) = (y+x)+(y+z)$.

(52) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^{\mathrm{c}})^{\mathrm{c}}+(y+z) = (y+x)+(y+z)$.

(53) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+(x+z) = y+(x+z)$.

(54) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+(x+z) = z+(x+y)$.

(55) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+z) = z+(y+x)$.

(56) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+z) = y+(z+x)$.

(57) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+z = x+(y+z)$.

Let us observe that every non empty complemented lattice structure which satisfies (DN_1) is also join-associative and every non empty complemented lattice structure which satisfies (DN_1) is also Robbins.

One can prove the following propositions:

(58) Let L be a non empty complemented lattice structure and x, z be elements of the carrier of L. Suppose L is join-commutative, join-associative, and Huntington. Then $(z+x)*(z+x^{\mathrm{c}}) = z$.

(59) Let L be a non empty complemented lattice structure such that L is join-commutative, join-associative, and Robbins. Then L satisfies (DN_1).

Let us mention that every non empty complemented lattice structure which is join-commutative, join-associative, and Robbins satisfies also (DN_1).

Let us observe that there exists a pre-ortholattice which is de Morgan and satisfies (DN_1).

One can verify that every pre-ortholattice which is de Morgan satisfies (DN_1) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (DN_1).

2. Meredith Two Axioms for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (Meredith$_1$) if and only if:

(Def. 2) For all elements x, y of the carrier of L holds $(x^{\mathrm{c}} + y)^{\mathrm{c}} + x = x$.

We say that L satisfies (Meredith$_2$) if and only if:

(Def. 3) For all elements x, y, z of the carrier of L holds $(x^{\mathrm{c}} + y)^{\mathrm{c}} + (z + y) = y + (z + x)$.

Let us note that every non empty complemented lattice structure which satisfies (Meredith$_1$) and (Meredith$_2$) is also join-commutative, join-associative, and Huntington and every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington satisfies also (Meredith$_1$) and (Meredith$_2$).

Let us note that there exists a pre-ortholattice which is de Morgan and satisfies (Meredith$_1$), (Meredith$_2$), and (DN$_1$).

Let us observe that every pre-ortholattice which is de Morgan satisfies (Meredith$_1$) and (Meredith$_2$) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (Meredith$_1$) and (Meredith$_2$).

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Contents

Formaliz. Math. 11 (**3**)

Continued on inside back cover