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Fibonacci Numbers

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Summary. We show that Fibonacci commutes with g.c.d.; we then derive the formula connecting the Fibonacci sequence with the roots of the polynomial $x^2 - x - 1$.

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The terminology and notation used here are introduced in the following articles: [3], [9], [5], [1], [2], [4], [7], [6], and [8].

1. Fibonacci Commutes with gcd

One can prove the following three propositions:

(1) For all natural numbers m, n holds $\gcd(m,n) = \gcd(m,n+m)$.

(2) For all natural numbers k, m, n such that $\gcd(k,m) = 1$ holds $\gcd(k,m \cdot n) = \gcd(k,n)$.

(3) For every real number s such that $s > 0$ there exists a natural number n such that $n > 0$ and $0 < \frac{1}{n}$ and $\frac{1}{n} \leqslant s$.

In this article we present several logical schemes. The scheme *Fib Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every natural number k holds $\mathcal{P}[k]$

provided the following conditions are met:

- $\mathcal{P}[0]$,
- $\mathcal{P}[1]$, and
- For every natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k+2]$.

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The scheme *Bin Ind* concerns a binary predicate $\mathcal{P}$, and states that:

For all natural numbers m, n holds $\mathcal{P}[m,n]$

provided the parameters satisfy the following conditions:

- For all natural numbers m, n such that $\mathcal{P}[m,n]$ holds $\mathcal{P}[n,m]$, and
- Let k be a natural number. Suppose that for all natural numbers m, n such that $m < k$ and $n < k$ holds $\mathcal{P}[m,n]$. Let m be a natural number. If $m \leqslant k$, then $\mathcal{P}[k,m]$.

We now state two propositions:

(4) For all natural numbers m, n holds $\mathrm{Fib}(m+(n+1)) = \mathrm{Fib}(n)\cdot\mathrm{Fib}(m) + \mathrm{Fib}(n+1)\cdot\mathrm{Fib}(m+1)$.

(5) For all natural numbers m, n holds $\gcd(\mathrm{Fib}(m),\mathrm{Fib}(n)) = \mathrm{Fib}(\gcd(m,n))$.

2. Fibonacci Numbers and the Golden Mean

Next we state the proposition

(6) Let x, a, b, c be real numbers. Suppose $a \neq 0$ and $\Delta(a,b,c) \geqslant 0$. Then $a \cdot x^{\mathbf{2}} + b \cdot x + c = 0$ if and only if $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}$.

The real number τ is defined by:

(Def. 1) $\tau = \frac{1+\sqrt{5}}{2}$.

The real number $\overline{\tau}$ is defined as follows:

(Def. 2) $\overline{\tau} = \frac{1-\sqrt{5}}{2}$.

One can prove the following propositions:

(7) For every natural number n holds $\mathrm{Fib}(n) = \frac{\tau^n - \overline{\tau}^n}{\sqrt{5}}$.

(8) For every natural number n holds $|\mathrm{Fib}(n) - \frac{\tau^n}{\sqrt{5}}| < 1$.

(9) For all sequences F, G of real numbers such that for every natural number n holds $F(n) = G(n)$ holds $F = G$.

(10) For all sequences f, g, h of real numbers such that g is non-zero holds $(f/g)\,(g/h) = f/h$.

(11) For all sequences f, g of real numbers and for every natural number n holds $(f/g)(n) = \frac{f(n)}{g(n)}$ and $(f/g)(n) = f(n) \cdot g(n)^{-1}$.

(12) Let F be a sequence of real numbers. Suppose that for every natural number n holds $F(n) = \frac{\mathrm{Fib}(n+1)}{\mathrm{Fib}(n)}$. Then F is convergent and $\lim F = \tau$.

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