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Upper and Lower Sequence of a Cage[1]

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The notation and terminology used in this paper are introduced in the following papers: [21], [7], [15], [8], [2], [19], [4], [17], [3], [14], [13], [6], [1], [5], [11], [22], [12], [18], [20], [16], [9], and [10].

1. Preliminaries

In this paper n is a natural number.

One can prove the following propositions:

(1) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds N-bound $X \leqslant$ N-bound Y.

(2) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds E-bound $X \leqslant$ E-bound Y.

(3) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds S-bound $X \geqslant$ S-bound Y.

(4) For every non empty subset X of $\mathcal{E}^2_{\rm T}$ and for every compact subset Y of $\mathcal{E}^2_{\rm T}$ such that $X \subseteq Y$ holds W-bound $X \geqslant$ W-bound Y.

(5) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of the carrier of $\mathcal{E}^2_{\rm T}$. If $p \in \operatorname{rng} f$, then $f -: p$ is in the area of g.

(6) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is in the area of g. Let p be an element of the carrier of $\mathcal{E}^2_{\rm T}$. If $p \in \operatorname{rng} f$, then $f :- p$ is in the area of g.

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(7) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\downharpoonleft p, f \neq \emptyset$.

(8) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $\mathrm{len} \downharpoonright f, p \geqslant 2$, then $f(1) \in \widetilde{\mathcal{L}}(\downharpoonright f, p)$.

(9) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $f(1) \notin \widetilde{\mathcal{L}}(\mathrm{mid}(f, \mathrm{Index}(p, f) + 1, \mathrm{len}\, f))$.

(10) For all natural numbers i, j, m, n such that $i + j = m + n$ and $i \leqslant m$ and $j \leqslant n$ holds $i = m$.

(11) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a special sequence. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $f(1) \in \widetilde{\mathcal{L}}(\downharpoonleft p, f)$, then $f(1) = p$.

2. About Upper and Lower Sequence of a Cage

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. The functor $\mathrm{UpperSeq}(C, n)$ yielding a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 1) $\mathrm{UpperSeq}(C, n) = ((\mathrm{Cage}(C, n))^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) -: \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

The following proposition is true

(12) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C, n) = (\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))) \hookleftarrow ((\mathrm{Cage}(C, n))^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft})$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. The functor $\mathrm{LowerSeq}(C, n)$ yields a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $\mathrm{LowerSeq}(C, n) = ((\mathrm{Cage}(C, n))^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) :- \text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

Next we state the proposition

(13) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and n be a natural number. Then $\mathrm{len}\,\mathrm{LowerSeq}(C, n) = (\mathrm{len}((\mathrm{Cage}(C, n))^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft}) - (\text{E-max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))) \hookleftarrow ((\mathrm{Cage}(C, n))^{\text{W-min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft})) + 1$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Note that $\mathrm{UpperSeq}(C, n)$ is non empty and $\mathrm{LowerSeq}(C, n)$ is non empty.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\mathrm{UpperSeq}(C, n)$ is one-to-one special unfolded and s.n.c. and $\mathrm{LowerSeq}(C, n)$ is one-to-one special unfolded and s.n.c..

The following propositions are true:

(14) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) + \mathrm{len}\,\mathrm{LowerSeq}(C,n) = \mathrm{len}\,\mathrm{Cage}(C,n) + 1$.

(15) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $(\mathrm{Cage}(C,n))^{\mathrm{W\text{-}min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))}_{\circlearrowleft} = \mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n)$.

(16) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n) \frown \mathrm{LowerSeq}(C,n))$.

(17) For every compact non vertical non horizontal non empty subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cup \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$.

(18) For every simple closed curve P holds $\mathrm{W\text{-}min}\,P \neq \mathrm{E\text{-}min}\,P$.

(19) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\mathrm{len}\,\mathrm{UpperSeq}(C,n) \geqslant 3$ and $\mathrm{len}\,\mathrm{LowerSeq}(C,n) \geqslant 3$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\mathrm{UpperSeq}(C,n)$ is special sequence and $\mathrm{LowerSeq}(C,n)$ is special sequence.

Next we state several propositions:

(20) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ and for every natural number n holds $\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{W\text{-}min}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)), \mathrm{E\text{-}max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))\}$.

(21) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UpperSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(22) For every compact non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{LowerSeq}(C,n)$ is in the area of $\mathrm{Cage}(C,n)$.

(23) For every compact connected non vertical non horizontal subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $((\mathrm{Cage}(C,n))_2)_{\mathbf{2}} = \mathrm{N\text{-}bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(24) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{E\text{-}max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{1}} = \mathrm{E\text{-}bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(25) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \mathrm{len}\,\mathrm{Cage}(C,n)$ and $(\mathrm{Cage}(C,n))_k = \mathrm{S\text{-}max}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$, then $((\mathrm{Cage}(C,n))_{k+1})_{\mathbf{2}} = \mathrm{S\text{-}bound}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$.

(26) Let C be a compact connected non vertical non horizontal subset of

$\mathcal{E}^2_{\rm T}$ and k be a natural number. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} \operatorname{Cage}(C,n)$ and $(\operatorname{Cage}(C,n))_k = \text{W-min}\,\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$, then $((\operatorname{Cage}(C,n))_{k+1})_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

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