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Some Properties of Dyadic Numbers and Intervals

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Summary. The article is the second part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we introduce some definitions and theorems concerning properties of intervals; in the second we prove some of properties of dyadic numbers used in proving Urysohn Lemma.

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The terminology and notation used here have been introduced in the following articles: [9], [10], [11], [3], [4], [8], [7], [6], [12], [1], [2], and [5].

The following proposition is true

(1) For every interval A such that $A \neq \emptyset$ holds if $\inf A < \sup A$, then $\mathrm{vol}(A) = \sup A - \inf A$ and if $\sup A = \inf A$, then $\mathrm{vol}(A) = 0_{\overline{\mathbb{R}}}$.

Let A be a subset of $\mathbb{R}$ and let x be a real number. The functor $x \cdot A$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 1) For every real number y holds $y \in x \cdot A$ iff there exists a real number z such that $z \in A$ and $y = x \cdot z$.

Next we state a number of propositions:

(2) For every subset A of $\mathbb{R}$ and for every real number x such that $x \neq 0$ holds $x^{-1} \cdot (x \cdot A) = A$.

(3) For every real number x such that $x \neq 0$ and for every subset A of $\mathbb{R}$ such that $A = \mathbb{R}$ holds $x \cdot A = A$.

(4) For every subset A of $\mathbb{R}$ such that $A \neq \emptyset$ holds $0 \cdot A = \{0\}$.

(5) For every subset A of $\mathbb{R}$ such that $A \neq \emptyset$ holds $0 \cdot A = \{0\}$.

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(6) For every real number x holds $x \cdot \emptyset = \emptyset$.

(7) For every real number y holds $y < 0$ or $y = 0$ or $0 < y$.

(8) Let a, b be extended real numbers. Suppose $a \leqslant b$. Then $a = -\infty$ and $b = -\infty$ or $a = -\infty$ and $b \in \mathbb{R}$ or $a = -\infty$ and $b = +\infty$ or $a \in \mathbb{R}$ and $b \in \mathbb{R}$ or $a \in \mathbb{R}$ and $b = +\infty$ or $a = +\infty$ and $b = +\infty$.

(9) For every extended real number x holds $[x, x]$ is an interval.

(10) For every interval A holds $0 \cdot A$ is an interval.

(11) For all real numbers q, x such that $x \neq 0$ holds $q = x \cdot \frac{q}{x}$.

(12) For all real numbers p, q, x such that $0 < x$ and $x \cdot p < x \cdot q$ holds $p < q$.

(13) For all real numbers p, q, x such that $x < 0$ and $x \cdot p < x \cdot q$ holds $q < p$.

(14) For all real numbers p, q, x such that $0 < x$ and $x \cdot p \leqslant x \cdot q$ holds $p \leqslant q$.

(15) For all real numbers p, q, x such that $x < 0$ and $x \cdot p \leqslant x \cdot q$ holds $q \leqslant p$.

(16) Let A be an interval and x be a real number. If $x \neq 0$, then if A is open interval, then $x \cdot A$ is open interval.

(17) Let A be an interval and x be a real number. If $x \neq 0$, then if A is closed interval, then $x \cdot A$ is closed interval.

(18) Let A be an interval and x be a real number. Suppose $0 < x$. If A is right open interval, then $x \cdot A$ is right open interval.

(19) Let A be an interval and x be a real number. Suppose $x < 0$. If A is right open interval, then $x \cdot A$ is left open interval.

(20) Let A be an interval and x be a real number. Suppose $0 < x$. If A is left open interval, then $x \cdot A$ is left open interval.

(21) Let A be an interval and x be a real number. Suppose $x < 0$. If A is left open interval, then $x \cdot A$ is right open interval.

(22) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A = [\inf A, \sup A]$. Then $B = [\inf B, \sup B]$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(23) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A =]\inf A, \sup A]$. Then $B =]\inf B, \sup B]$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(24) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A =]\inf A, \sup A[$. Then $B =]\inf B, \sup B[$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(25) Let A be an interval. Suppose $A \neq \emptyset$. Let x be a real number. Suppose $0 < x$. Let B be an interval. Suppose $B = x \cdot A$. Suppose $A = [\inf A, \sup A[$.

Then $B = [\inf B, \sup B[$ and for all real numbers s, t such that $s = \inf A$ and $t = \sup A$ holds $\inf B = x \cdot s$ and $\sup B = x \cdot t$.

(26) For every interval A and for every real number x holds $x \cdot A$ is an interval.

Let A be an interval and let x be a real number. Observe that $x \cdot A$ is interval.

The following propositions are true:

(27) Let A be an interval and x be a real number. If $0 \leqslant x$, then for every real number y such that $y = \mathrm{vol}(A)$ holds $x \cdot y = \mathrm{vol}(x \cdot A)$.

(28) For all real numbers x, y, z such that $x < y$ and $y \leqslant z$ or $x \leqslant y$ and $y < z$ holds $x < z$.

(29) For every natural number n holds $n < 2^n$.

(30) For every integer n such that $0 \leqslant n$ holds n is a natural number.

(31) For all natural numbers n, m such that $n < m$ holds $2^n < 2^m$.

(32) For every real number e_1 such that $0 < e_1$ there exists a natural number n such that $1 < 2^n \cdot e_1$.

(33) For all real numbers a, b such that $0 \leqslant a$ and $1 < b - a$ there exists a natural number n such that $a < n$ and $n < b$.

(34) For every integer n such that $0 < n$ holds n is a natural number.

(35) For every rational number n such that $0 \leqslant n$ holds $0 \leqslant \mathrm{num}\, n$.

(36) For every rational number n such that $0 < n$ holds $0 < \mathrm{num}\, n$.

(37) For all real numbers a, b, c, d such that $0 < b$ and $0 < d$ or $b < 0$ and $d < 0$ holds if $\frac{a}{b} < \frac{c}{d}$, then $a \cdot d < c \cdot b$.

(38) For every natural number n holds $\mathrm{dyadic}(n) \subseteq \mathrm{DYADIC}$.

(39) For all real numbers a, b such that $a < b$ and $0 \leqslant a$ and $b \leqslant 1$ there exists a real number c such that $c \in \mathrm{DYADIC}$ and $a < c$ and $c < b$.

(40) For all real numbers a, b such that $a < b$ there exists a real number c such that $c \in \mathrm{DOM}$ and $a < c$ and $c < b$.

(41) For every non empty subset A of $\overline{\mathbb{R}}$ and for all extended real numbers a, b such that $A \subseteq [a, b]$ holds $a \leqslant \inf A$ and $\sup A \leqslant b$.

(42) $0 \in \mathrm{DYADIC}$ and $1 \in \mathrm{DYADIC}$.

(43) For all extended real numbers a, b such that $a = 0$ and $b = 1$ holds $\mathrm{DYADIC} \subseteq [a, b]$.

(44) For all natural numbers n, k such that $n \leqslant k$ holds $\mathrm{dyadic}(n) \subseteq \mathrm{dyadic}(k)$.

(45) For all real numbers a, b, c, d such that $a < c$ and $c < b$ and $a < d$ and $d < b$ holds $|d - c| < b - a$.

(46) Let e_1 be a real number. Suppose $0 < e_1$. Let d be a real number. Suppose $0 < d$ and $d \leqslant 1$. Then there exist real numbers r_1, r_2 such that $r_1 \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $r_2 \in \mathrm{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r_1$ and $r_1 < d$ and $d < r_2$ and $r_2 - r_1 < e_1$.

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