FORMALIZED MATHEMATICS
Volume 9, Number 3, 2001
University of Białystok

Gauges and Cages. Part I[1]

Artur Korniłowicz
University of Białystok

Robert Milewski
University of Białystok

Adam Naumowicz
University of Białystok

Andrzej Trybulec
University of Białystok

MML Identifier: `JORDAN1A`.

The notation and terminology used in this paper have been introduced in the following articles: [28], [24], [32], [9], [25], [10], [2], [3], [30], [29], [4], [5], [18], [21], [23], [22], [6], [8], [14], [1], [19], [26], [7], [27], [13], [33], [17], [16], [20], [31], [11], [12], and [15].

1. Preliminaries

For simplicity, we use the following convention: i, i_1, i_2, j, j_1, j_2, k, m, n, t denote natural numbers, D denotes a non empty subset of $\mathcal{E}^2_{\rm T}$, E denotes a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, C denotes a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$, G denotes a Go-board, p, q, x denote points of $\mathcal{E}^2_{\rm T}$, and r, s denote real numbers.

The following propositions are true:

(1) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 \leqslant s_4$ and $0 \leqslant l$ and $l \leqslant 1$ holds $s_1 \leqslant (1-l) \cdot s_3 + l \cdot s_4$.

(2) For all real numbers s_1, s_3, s_4, l such that $s_3 \leqslant s_1$ and $s_4 \leqslant s_1$ and $0 \leqslant l$ and $l \leqslant 1$ holds $(1-l) \cdot s_3 + l \cdot s_4 \leqslant s_1$.

(3) If $n > 0$, then $m^n \bmod m = 0$.

(4) If $j > 0$ and $i \bmod j = 0$, then $i \div j = \frac{i}{j}$.

(5) If $n > 0$, then $i^n \div i = \frac{i^n}{i}$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(6) If $0 < n$ and $1 < r$, then $1 < r^n$.

(7) If $r > 1$ and $m > n$, then $r^m > r^n$.

(8) Let T be a non empty topological space, A be a subset of T, and B, C be subsets of the carrier of T. If A is connected and C is a component of B and $A \cap C \neq \emptyset$ and $A \subseteq B$, then $A \subseteq C$.

Let f be a finite sequence. The functor $\mathrm{Center}\, f$ yields a natural number and is defined as follows:

(Def. 1) $\mathrm{Center}\, f = (\mathrm{len}\, f \div 2) + 1$.

The following two propositions are true:

(9) For every finite sequence f such that $\mathrm{len}\, f$ is odd holds $\mathrm{len}\, f = 2 \cdot \mathrm{Center}\, f - 1$.

(10) For every finite sequence f such that $\mathrm{len}\, f$ is even holds $\mathrm{len}\, f = 2 \cdot \mathrm{Center}\, f - 2$.

2. Some Subsets of the Plane

One can check the following observations:

* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non vertical, non horizontal, and non empty and satisfies conditions of simple closed curve,
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and horizontal, and
* there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is compact, non empty, and vertical.

The following propositions are true:

(11) If $p \in \text{N-most}\, D$, then $p_{\mathbf{2}} = \text{N-bound}\, D$.

(12) If $p \in \text{E-most}\, D$, then $p_{\mathbf{1}} = \text{E-bound}\, D$.

(13) If $p \in \text{S-most}\, D$, then $p_{\mathbf{2}} = \text{S-bound}\, D$.

(14) If $p \in \text{W-most}\, D$, then $p_{\mathbf{1}} = \text{W-bound}\, D$.

(15) $\mathrm{BDD}\, D$ misses D.

(16) For every compact non empty subset S of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve holds $\mathrm{LowerArc}\, S \subseteq S$ and $\mathrm{UpperArc}\, S \subseteq S$.

(17) $p \in \mathrm{VerticalLine}\, p_{\mathbf{1}}$.

(18) $[r, s] \in \mathrm{VerticalLine}\, r$.

(19) For every subset A of $\mathcal{E}^2_{\mathrm{T}}$ such that $A \subseteq \mathrm{VerticalLine}\, s$ holds A is vertical.

(20) $(\mathrm{proj2})([r, s]) = s$ and $(\mathrm{proj1})([r, s]) = r$.

(21) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $r \in [(\mathrm{proj2})(p), (\mathrm{proj2})(q)]$, then $[p_{\mathbf{1}}, r] \in \mathcal{L}(p, q)$.

(22) If $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $r \in [(\mathrm{proj1})(p), (\mathrm{proj1})(q)]$, then $[r, p_{\mathbf{2}}] \in \mathcal{L}(p, q)$.

(23) If $p \in \mathrm{VerticalLine}\, s$ and $q \in \mathrm{VerticalLine}\, s$, then $\mathcal{L}(p,q) \subseteq \mathrm{VerticalLine}\, s$.

Let S be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve. Observe that $\mathrm{LowerArc}\, S$ is non empty and compact and $\mathrm{UpperArc}\, S$ is non empty and compact.

We now state several propositions:

(24) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that A meets B holds $(\mathrm{proj2})^\circ A$ meets $(\mathrm{proj2})^\circ B$.

(25) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that A misses B and $A \subseteq \mathrm{VerticalLine}\, s$ and $B \subseteq \mathrm{VerticalLine}\, s$ holds $(\mathrm{proj2})^\circ A$ misses $(\mathrm{proj2})^\circ B$.

(26) For every closed subset S of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Bounded holds $(\mathrm{proj2})^\circ S$ is closed.

(27) For every subset S of $\mathcal{E}^2_{\mathrm{T}}$ such that S is Bounded holds $(\mathrm{proj2})^\circ S$ is bounded.

(28) For every compact subset S of $\mathcal{E}^2_{\mathrm{T}}$ holds $(\mathrm{proj2})^\circ S$ is compact.

In this article we present several logical schemes. The scheme *TRSubsetEx* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset A of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ such that for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$

for all values of the parameters.

The scheme *TRSubsetUniq* deals with a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let A, B be subsets of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$. Suppose for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in A$ iff $\mathcal{P}[p]$ and for every point p of $\mathcal{E}^{\mathcal{A}}_{\mathrm{T}}$ holds $p \in B$ iff $\mathcal{P}[p]$. Then $A = B$

for all values of the parameters.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{NorthHalfline}\, p$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 2) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in \mathrm{NorthHalfline}\, p$ iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \geqslant p_{\mathbf{2}}$.

The functor $\mathrm{EastHalfline}\, p$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 3) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in \mathrm{EastHalfline}\, p$ iff $x_{\mathbf{1}} \geqslant p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The functor $\mathrm{SouthHalfline}\, p$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 4) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in \mathrm{SouthHalfline}\, p$ iff $x_{\mathbf{1}} = p_{\mathbf{1}}$ and $x_{\mathbf{2}} \leqslant p_{\mathbf{2}}$.

The functor $\mathrm{WestHalfline}\, p$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 5) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ holds $x \in \mathrm{WestHalfline}\, p$ iff $x_{\mathbf{1}} \leqslant p_{\mathbf{1}}$ and $x_{\mathbf{2}} = p_{\mathbf{2}}$.

The following propositions are true:

(29) $\mathrm{NorthHalfline}\, p = \{q;\, q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \ \wedge\ q_{\mathbf{2}} \geqslant p_{\mathbf{2}}\}$.

(30) $\mathrm{NorthHalfline}\, p = \{[p_{\mathbf{1}}, r];\, r$ ranges over elements of $\mathbb{R}$: $r \geqslant p_{\mathbf{2}}\}$.

(31) $\mathrm{EastHalfline}\, p = \{q;\, q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \geqslant p_{\mathbf{1}} \ \wedge\ q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(32) $\mathrm{EastHalfline}\, p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \geqslant p_{\mathbf{1}}\}$.

(33) $\mathrm{SouthHalfline}\, p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} = p_{\mathbf{1}} \ \wedge \ q_{\mathbf{2}} \leqslant p_{\mathbf{2}}\}$.

(34) $\mathrm{SouthHalfline}\, p = \{[p_{\mathbf{1}}, r]; r$ ranges over elements of $\mathbb{R}$: $r \leqslant p_{\mathbf{2}}\}$.

(35) $\mathrm{WestHalfline}\, p = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \leqslant p_{\mathbf{1}} \ \wedge \ q_{\mathbf{2}} = p_{\mathbf{2}}\}$.

(36) $\mathrm{WestHalfline}\, p = \{[r, p_{\mathbf{2}}]; r$ ranges over elements of $\mathbb{R}$: $r \leqslant p_{\mathbf{1}}\}$.

Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. One can check the following observations:

* NorthHalfline p is non empty and convex,
* EastHalfline p is non empty and convex,
* SouthHalfline p is non empty and convex, and
* WestHalfline p is non empty and convex.

3. Goboards

We now state a number of propositions:

(37) If $1 \leqslant i$ and $i \leqslant \mathrm{len}\, G$ and $1 \leqslant j$ and $j \leqslant \mathrm{width}\, G$, then $G_{i,j} \in \mathcal{L}(G_{i,1}, G_{i,\mathrm{width}\, G})$.

(38) If $1 \leqslant i$ and $i \leqslant \mathrm{len}\, G$ and $1 \leqslant j$ and $j \leqslant \mathrm{width}\, G$, then $G_{i,j} \in \mathcal{L}(G_{1,j}, G_{\mathrm{len}\, G,j})$.

(39) If $1 \leqslant j_1$ and $j_1 \leqslant \mathrm{width}\, G$ and $1 \leqslant j_2$ and $j_2 \leqslant \mathrm{width}\, G$ and $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \mathrm{len}\, G$, then $(G_{i_1,j_1})_{\mathbf{1}} \leqslant (G_{i_2,j_2})_{\mathbf{1}}$.

(40) If $1 \leqslant i_1$ and $i_1 \leqslant \mathrm{len}\, G$ and $1 \leqslant i_2$ and $i_2 \leqslant \mathrm{len}\, G$ and $1 \leqslant j_1$ and $j_1 \leqslant j_2$ and $j_2 \leqslant \mathrm{width}\, G$, then $(G_{i_1,j_1})_{\mathbf{2}} \leqslant (G_{i_2,j_2})_{\mathbf{2}}$.

(41) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \mathrm{len}\, G$. Then $(G_{t,\mathrm{width}\, G})_{\mathbf{2}} \geqslant \text{N-bound}\, \widetilde{\mathcal{L}}(f)$.

(42) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \mathrm{width}\, G$. Then $(G_{1,t})_{\mathbf{1}} \leqslant \text{W-bound}\, \widetilde{\mathcal{L}}(f)$.

(43) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \mathrm{len}\, G$. Then $(G_{t,1})_{\mathbf{2}} \leqslant \text{S-bound}\, \widetilde{\mathcal{L}}(f)$.

(44) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G and $1 \leqslant t$ and $t \leqslant \mathrm{width}\, G$. Then $(G_{\mathrm{len}\, G,t})_{\mathbf{1}} \geqslant \text{E-bound}\, \widetilde{\mathcal{L}}(f)$.

(45) If $i \leqslant \mathrm{len}\, G$ and $j \leqslant \mathrm{width}\, G$, then $\mathrm{cell}(G, i, j)$ is non empty.

(46) If $i \leqslant \mathrm{len}\, G$ and $j \leqslant \mathrm{width}\, G$, then $\mathrm{cell}(G, i, j)$ is connected.

(47) If $i \leqslant \mathrm{len}\, G$, then $\mathrm{cell}(G, i, 0)$ is not Bounded.

(48) If $i \leqslant \operatorname{len} G$, then $\operatorname{cell}(G, i, \operatorname{width} G)$ is not Bounded.

4. Gauges

One can prove the following propositions:

(49) $\operatorname{width} \operatorname{Gauge}(D, n) = 2^n + 3$.

(50) If $i < j$, then $\operatorname{len} \operatorname{Gauge}(D, i) < \operatorname{len} \operatorname{Gauge}(D, j)$.

(51) If $i \leqslant j$, then $\operatorname{len} \operatorname{Gauge}(D, i) \leqslant \operatorname{len} \operatorname{Gauge}(D, j)$.

(52) If $m \leqslant n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{len} \operatorname{Gauge}(D, n)$.

(53) If $m \leqslant n$ and $1 < i$ and $i < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 2) + 2$ and $2^{n -' m} \cdot (i - 2) + 2 < \operatorname{width} \operatorname{Gauge}(D, n)$.

(54) Suppose $m \leqslant n$ and $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(D, m)$ and $1 < j$ and $j < \operatorname{width} \operatorname{Gauge}(D, m)$. Let i_1, j_1 be natural numbers. If $i_1 = 2^{n -' m} \cdot (i - 2) + 2$ and $j_1 = 2^{n -' m} \cdot (j - 2) + 2$, then $(\operatorname{Gauge}(D, m))_{i,j} = (\operatorname{Gauge}(D, n))_{i_1,j_1}$.

(55) If $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{len} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$.

(56) If $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{width} \operatorname{Gauge}(D, m)$, then $1 < 2^{n -' m} \cdot (i - 1) + 2$ and $2^{n -' m} \cdot (i - 1) + 2 \leqslant \operatorname{width} \operatorname{Gauge}(D, n)$.

(57) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $((\operatorname{Gauge}(D, n))_{\operatorname{Center} \operatorname{Gauge}(D,n),i})_{\mathbf{1}} = ((\operatorname{Gauge}(D, m))_{\operatorname{Center} \operatorname{Gauge}(D,m),j})_{\mathbf{1}}$.

(58) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(D, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(D, m)$ and $n > 0$ and $m > 0$ or $n = 0$ and $m = 0$, then $((\operatorname{Gauge}(D, n))_{i,\operatorname{Center} \operatorname{Gauge}(D,n)})_{\mathbf{2}} = ((\operatorname{Gauge}(D, m))_{j,\operatorname{Center} \operatorname{Gauge}(D,m)})_{\mathbf{2}}$.

(59) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, 1)$, then $((\operatorname{Gauge}(C, 1))_{\operatorname{Center} \operatorname{Gauge}(C,1),i})_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(60) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, 1)$, then $((\operatorname{Gauge}(C, 1))_{i,\operatorname{Center} \operatorname{Gauge}(C,1)})_{\mathbf{2}} = \frac{\text{S-bound}\, C + \text{N-bound}\, C}{2}$.

(61) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, n))_{i,\operatorname{len} \operatorname{Gauge}(E,n)})_{\mathbf{2}} \leqslant ((\operatorname{Gauge}(E, m))_{j,\operatorname{len} \operatorname{Gauge}(E,m)})_{\mathbf{2}}$.

(62) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, n))_{\operatorname{len} \operatorname{Gauge}(E,n),i})_{\mathbf{1}} \leqslant ((\operatorname{Gauge}(E, m))_{\operatorname{len} \operatorname{Gauge}(E,m),j})_{\mathbf{1}}$.

(63) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, m))_{1,j})_{\mathbf{1}} \leqslant ((\operatorname{Gauge}(E, n))_{1,i})_{\mathbf{1}}$.

(64) If $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(E, n)$ and $1 \leqslant j$ and $j \leqslant \operatorname{len} \operatorname{Gauge}(E, m)$ and $m \leqslant n$, then $((\operatorname{Gauge}(E, m))_{j,1})_{\mathbf{2}} \leqslant ((\operatorname{Gauge}(E, n))_{i,1})_{\mathbf{2}}$.

(65) If $1 \leqslant m$ and $m \leqslant n$, then $\mathcal{L}((\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),1}, (\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),\operatorname{len}\operatorname{Gauge}(E,n)}) \subseteq \mathcal{L}((\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),\operatorname{len}\operatorname{Gauge}(E,m)})$.

(66) If $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant \operatorname{width}\operatorname{Gauge}(E,n)$, then $\mathcal{L}((\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),1}, (\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),j}) \subseteq \mathcal{L}((\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),j})$.

(67) If $1 \leqslant m$ and $m \leqslant n$ and $1 \leqslant j$ and $j \leqslant \operatorname{width}\operatorname{Gauge}(E,n)$, then $\mathcal{L}((\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E,n))_{\operatorname{Center}\operatorname{Gauge}(E,n),j}) \subseteq \mathcal{L}((\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),1}, (\operatorname{Gauge}(E,m))_{\operatorname{Center}\operatorname{Gauge}(E,m),\operatorname{len}\operatorname{Gauge}(E,m)})$.

(68) Suppose $m \leqslant n$ and $1 < i$ and $i + 1 < \operatorname{len}\operatorname{Gauge}(E, m)$ and $1 < j$ and $j + 1 < \operatorname{width}\operatorname{Gauge}(E, m)$. Let i_1, j_1 be natural numbers. Suppose $2^{n-'m} \cdot (i-2) + 2 \leqslant i_1$ and $i_1 < 2^{n-'m} \cdot (i-1) + 2$ and $2^{n-'m} \cdot (j-2) + 2 \leqslant j_1$ and $j_1 < 2^{n-'m} \cdot (j-1) + 2$. Then $\operatorname{cell}(\operatorname{Gauge}(E,n), i_1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E,m), i, j)$.

(69) Suppose $m \leqslant n$ and $3 \leqslant i$ and $i < \operatorname{len}\operatorname{Gauge}(E, m)$ and $1 < j$ and $j + 1 < \operatorname{width}\operatorname{Gauge}(E, m)$. Let i_1, j_1 be natural numbers. If $i_1 = 2^{n-'m} \cdot (i-2)+2$ and $j_1 = 2^{n-'m} \cdot (j-2)+2$, then $\operatorname{cell}(\operatorname{Gauge}(E,n), i_1 -' 1, j_1) \subseteq \operatorname{cell}(\operatorname{Gauge}(E,m), i -' 1, j)$.

(70) If $i \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$, then $\operatorname{cell}(\operatorname{Gauge}(C,n), i, 0) \subseteq \operatorname{UBD} C$.

(71) If $i \leqslant \operatorname{len}\operatorname{Gauge}(E,n)$, then $\operatorname{cell}(\operatorname{Gauge}(E,n), i, \operatorname{width}\operatorname{Gauge}(E,n)) \subseteq \operatorname{UBD} E$.

5. Cages

The following propositions are true:

(72) If $p \in C$, then NorthHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(73) If $p \in C$, then EastHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(74) If $p \in C$, then SouthHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(75) If $p \in C$, then WestHalfline p meets $\widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$.

(76) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{1,t}$.

(77) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{t,1}$.

(78) There exist k, t such that $1 \leqslant k$ and $k < \operatorname{len}\operatorname{Cage}(C,n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C,n)$ and $(\operatorname{Cage}(C,n))_k = (\operatorname{Gauge}(C,n))_{\operatorname{len}\operatorname{Gauge}(C,n),t}$.

(79) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{t, \operatorname{width}\operatorname{Gauge}(C,n)}$, then $(\operatorname{Cage}(C, n))_k \in$ N-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(80) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{1,t}$, then $(\operatorname{Cage}(C, n))_k \in$ W-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(81) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{t,1}$, then $(\operatorname{Cage}(C, n))_k \in$ S-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(82) If $1 \leqslant k$ and $k \leqslant \operatorname{len}\operatorname{Cage}(C, n)$ and $1 \leqslant t$ and $t \leqslant \operatorname{width}\operatorname{Gauge}(C, n)$ and $(\operatorname{Cage}(C, n))_k = (\operatorname{Gauge}(C, n))_{\operatorname{len}\operatorname{Gauge}(C,n),t}$, then $(\operatorname{Cage}(C, n))_k \in$ E-most $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(83) W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) =$ W-bound $C - \frac{\text{E-bound}\,C - \text{W-bound}\,C}{2^n}$.

(84) S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) =$ S-bound $C - \frac{\text{N-bound}\,C - \text{S-bound}\,C}{2^n}$.

(85) E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) =$ E-bound $C + \frac{\text{E-bound}\,C - \text{W-bound}\,C}{2^n}$.

(86) N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) +$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) =$ N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, m)) +$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, m))$.

(87) E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) +$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) =$ E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, m)) +$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, m))$.

(88) If $i < j$, then E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, j)) <$ E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, i))$.

(89) If $i < j$, then W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, i)) <$ W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, j))$.

(90) If $i < j$, then S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, i)) <$ S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, j))$.

(91) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$, then N-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = ((\operatorname{Gauge}(C, n))_{i,\operatorname{len}\operatorname{Gauge}(C,n)})_{\mathbf{2}}$.

(92) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$, then E-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = ((\operatorname{Gauge}(C, n))_{\operatorname{len}\operatorname{Gauge}(C,n),i})_{\mathbf{1}}$.

(93) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$, then S-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = ((\operatorname{Gauge}(C, n))_{i,1})_{\mathbf{2}}$.

(94) If $1 \leqslant i$ and $i \leqslant \operatorname{len}\operatorname{Gauge}(C, n)$, then W-bound $\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = ((\operatorname{Gauge}(C, n))_{1,i})_{\mathbf{1}}$.

(95) If $x \in C$ and $p \in \operatorname{NorthHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p_{\mathbf{2}} > x_{\mathbf{2}}$.

(96) If $x \in C$ and $p \in \operatorname{EastHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p_{\mathbf{1}} > x_{\mathbf{1}}$.

(97) If $x \in C$ and $p \in \operatorname{SouthHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p_{\mathbf{2}} < x_{\mathbf{2}}$.

(98) If $x \in C$ and $p \in \operatorname{WestHalfline} x \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$, then $p_{\mathbf{1}} < x_{\mathbf{1}}$.

(99) If $x \in$ N-most C and $p \in \operatorname{NorthHalfline} x$ and $1 \leqslant i$ and $i < \operatorname{len}\operatorname{Cage}(C, n)$ and $p \in \mathcal{L}(\operatorname{Cage}(C, n), i)$, then $\mathcal{L}(\operatorname{Cage}(C, n), i)$ is horizontal.

(100) If $x \in$ E-most C and $p \in$ EastHalfline x and $1 \leqslant i$ and $i <$ len Cage(C, n) and $p \in \mathcal{L}(\mathrm{Cage}(C, n), i)$, then $\mathcal{L}(\mathrm{Cage}(C, n), i)$ is vertical.

(101) If $x \in$ S-most C and $p \in$ SouthHalfline x and $1 \leqslant i$ and $i <$ len Cage(C, n) and $p \in \mathcal{L}(\mathrm{Cage}(C, n), i)$, then $\mathcal{L}(\mathrm{Cage}(C, n), i)$ is horizontal.

(102) If $x \in$ W-most C and $p \in$ WestHalfline x and $1 \leqslant i$ and $i <$ len Cage(C, n) and $p \in \mathcal{L}(\mathrm{Cage}(C, n), i)$, then $\mathcal{L}(\mathrm{Cage}(C, n), i)$ is vertical.

(103) If $x \in$ N-most C and $p \in$ NorthHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$, then $p_{\mathbf{2}} =$ N-bound $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(104) If $x \in$ E-most C and $p \in$ EastHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$, then $p_{\mathbf{1}} =$ E-bound $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(105) If $x \in$ S-most C and $p \in$ SouthHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$, then $p_{\mathbf{2}} =$ S-bound $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(106) If $x \in$ W-most C and $p \in$ WestHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$, then $p_{\mathbf{1}} =$ W-bound $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$.

(107) If $x \in$ N-most C, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that NorthHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) = \{p\}$.

(108) If $x \in$ E-most C, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that EastHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) = \{p\}$.

(109) If $x \in$ S-most C, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that SouthHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) = \{p\}$.

(110) If $x \in$ W-most C, then there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that WestHalfline $x \cap \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) = \{p\}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Gauges. *Formalized Mathematics*, 8(**1**):25–27, 1999.

[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[8] Czesław Byliński and Mariusz Żynel. Cages - the external approximation of Jordan's curve. *Formalized Mathematics*, 9(**1**):19–24, 2001.

[9] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[15] Andrzej Kondracki. The Chinese Remainder Theorem. *Formalized Mathematics*, 6(**4**):573–577, 1997.
[16] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[17] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[18] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.
[19] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[20] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons. Part I. *Formalized Mathematics*, 5(**1**):97–102, 1996.
[21] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.
[22] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[23] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[24] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[25] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[26] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[27] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[28] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[29] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(**3**):317–322, 1996.
[30] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[31] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[32] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[33] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received September 12, 2000