FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

Function Spaces in the Category of Directed Suprema Preserving Maps[1]

Grzegorz Bancerek
University of Białystok

Adam Naumowicz
University of Białystok

Summary. Formalization of [15, pp. 115–117], chapter II, section 2 (2.5 – 2.10).

MML Identifier: `WAYBEL27`.

The notation and terminology used here are introduced in the following papers: [33], [2], [10], [11], [9], [1], [26], [3], [31], [16], [29], [23], [24], [27], [4], [34], [35], [32], [28], [14], [30], [17], [19], [22], [8], [6], [13], [7], [25], [21], [5], [18], [36], [20], and [12].

1. Currying, Uncurrying and Commuting Functions

Let F be a function. We say that F is uncurrying if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function yielding function, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{uncurry} f$.

We say that F is currying if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function and $\pi_1(x)$ is a binary relation, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{curry} f$.

We say that F is commuting if and only if the conditions (Def. 3) are satisfied.

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(Def. 3)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function yielding function, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{commute}(f)$.

Let us note that every function which is empty is also uncurrying, currying, and commuting.

Let us mention that there exists a function which is uncurrying, currying, and commuting.

Let F be an uncurrying function and let X be a set. Observe that $F \restriction X$ is uncurrying.

Let F be a currying function and let X be a set. Note that $F \restriction X$ is currying.

The following propositions are true:

(1) Let X, Y, Z, D be sets. Suppose $D \subseteq (Z^Y)^X$. Then there exists a many sorted set F indexed by D such that F is uncurrying and $\operatorname{rng} F \subseteq Z^{[: X, Y :]}$.

(2) Let X, Y, Z, D be sets. Suppose $D \subseteq Z^{[: X, Y :]}$. Then there exists a many sorted set F indexed by D such that F is currying and if if $Y = \emptyset$, then $X = \emptyset$, then $\operatorname{rng} F \subseteq (Z^Y)^X$.

Let X, Y, Z be sets. Note that there exists a many sorted set indexed by $(Z^Y)^X$ which is uncurrying and there exists a many sorted set indexed by $Z^{[: X, Y :]}$ which is currying.

Next we state several propositions:

(3) Let A, B be non empty sets, C be a set, and f, g be commuting functions. If $\operatorname{dom} f \subseteq (C^B)^A$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \operatorname{id}_{\operatorname{dom} f}$.

(4) Let B be a non empty set, A, C be sets, f be an uncurrying function, and g be a currying function. If $\operatorname{dom} f \subseteq (C^B)^A$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \operatorname{id}_{\operatorname{dom} f}$.

(5) Let A, B, C be sets, f be a currying function, and g be an uncurrying function. If $\operatorname{dom} f \subseteq C^{[: A, B :]}$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \operatorname{id}_{\operatorname{dom} f}$.

(6) For every function yielding function f and for all sets i, A such that $i \in \operatorname{dom} \operatorname{commute}(f)$ holds $(\operatorname{commute}(f))(i)^\circ A \subseteq \pi_i f^\circ A$.

(7) Let f be a function yielding function and i, A be sets. If for every function g such that $g \in f^\circ A$ holds $i \in \operatorname{dom} g$, then $\pi_i f^\circ A \subseteq (\operatorname{commute}(f))(i)^\circ A$.

(8) For all sets X, Y and for every function f such that $\operatorname{rng} f \subseteq Y^X$ and for all sets i, A such that $i \in X$ holds $(\operatorname{commute}(f))(i)^\circ A = \pi_i f^\circ A$.

(9) For every function f and for all sets i, A such that $[: A, \{i\} :] \subseteq \operatorname{dom} f$ holds $\pi_i(\operatorname{curry} f)^\circ A = f^\circ [: A, \{i\} :]$.

Let X be a set and let Y be a non empty functional set. One can verify that every function from X into Y is function yielding.

Let T be a constituted functions 1-sorted structure. Observe that the carrier of T is functional.

Let X be a set and let L be a non empty relational structure. One can check that L^X is constituted functions.

One can verify that there exists a lattice which is constituted functions, complete, and strict and there exists a 1-sorted structure which is constituted functions and non empty.

Let T be a constituted functions non empty relational structure. Note that every non empty relational substructure of T is constituted functions.

Next we state four propositions:

(10) Let S, T be complete lattices, f be an idempotent map from T into T, and h be a map from S into $\operatorname{Im} f$. Then $f \cdot h = h$.

(11) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a relational substructure of T_1. Let f be a map from S into T and f_1 be a map from S into T_1. If f is monotone and $f = f_1$, then f_1 is monotone.

(12) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a full relational substructure of T_1. Let f be a map from S into T and f_1 be a map from S into T_1. If f_1 is monotone and $f = f_1$, then f is monotone.

(13) For every set X and for every subset V of X holds $(\chi_{V,X})^{-1}(\{1\}) = V$ and $(\chi_{V,X})^{-1}(\{0\}) = X \setminus V$.

2. Maps of Power Posets

Let X be a non empty set, let T be a non empty relational structure, let f be an element of T^X, and let x be an element of X. Then $f(x)$ is an element of T.

Next we state several propositions:

(14) Let X be a non empty set, T be a non empty relational structure, and f, g be elements of T^X. Then $f \leqslant g$ if and only if for every element x of X holds $f(x) \leqslant g(x)$.

(15) Let X be a set and L, S be non empty relational structures. Suppose the relational structure of $L =$ the relational structure of S. Then $L^X = S^X$.

(16) Let S_1, S_2, T_1, T_2 be non empty topological spaces. Suppose that
 (i) the topological structure of $S_1 =$ the topological structure of S_2, and
 (ii) the topological structure of $T_1 =$ the topological structure of T_2.
 Then $[S_1 \to T_1] = [S_2 \to T_2]$.

(17) Let X be a set. Then there exists a map f from $2^X_\subseteq$ into $(2^1_\subseteq)^X$ such that f is isomorphic and for every subset Y of X holds $f(Y) = \chi_{Y,X}$.

(18) For every set X holds $2^X_\subseteq$ and $(2^1_\subseteq)^X$ are isomorphic.

(19) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^X)^Y$, S_2 be a full non empty relational substructure of $(T^Y)^X$, and F be a map from S_1 into S_2. If F is commuting, then F is monotone.

(20) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[: X, Y :]}$, and F be a map from S_1 into S_2. If F is uncurrying, then F is monotone.

(21) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[: X, Y :]}$, and F be a map from S_2 into S_1. If F is currying, then F is monotone.

3. Posets of Directed Suprema Preserving Maps

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. The functor $\mathrm{UPS}(S,T)$ yielding a strict relational structure is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{UPS}(S,T)$ is a full relational substructure of $T^{\text{the carrier of } S}$, and

(ii) for every set x holds $x \in$ the carrier of $\mathrm{UPS}(S,T)$ iff x is a directed-sups-preserving map from S into T.

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. Observe that $\mathrm{UPS}(S,T)$ is non empty reflexive antisymmetric and constituted functions.

Let S be a non empty relational structure and let T be a non empty poset. One can verify that $\mathrm{UPS}(S,T)$ is transitive.

We now state the proposition

(22) Let S be a non empty relational structure and T be a non empty reflexive antisymmetric relational structure. Then the carrier of $\mathrm{UPS}(S,T) \subseteq$ (the carrier of $T)^{\text{the carrier of } S}$.

Let S be a non empty relational structure, let T be a non empty reflexive antisymmetric relational structure, let f be an element of $\mathrm{UPS}(S,T)$, and let s be an element of S. Then $f(s)$ is an element of T.

Next we state three propositions:

(23) Let S be a non empty relational structure, T be a non empty reflexive antisymmetric relational structure, and f, g be elements of $\mathrm{UPS}(S,T)$. Then $f \leqslant g$ if and only if for every element s of S holds $f(s) \leqslant g(s)$.

(24) For all complete Scott top-lattices S, T holds $\mathrm{UPS}(S,T) = \mathrm{SCMaps}(S,T)$.

(25) Let S, S' be non empty relational structures and T, T' be non empty reflexive antisymmetric relational structures. Suppose that
  (i) the relational structure of S = the relational structure of S', and
  (ii) the relational structure of T = the relational structure of T'.
  Then $\mathrm{UPS}(S,T) = \mathrm{UPS}(S',T')$.

Let S, T be complete lattices. Note that $\mathrm{UPS}(S,T)$ is complete.

The following propositions are true:

(26) Let S, T be complete lattices. Then $\mathrm{UPS}(S,T)$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

(27) For all complete lattices S, T and for every subset A of $\mathrm{UPS}(S,T)$ holds $\sup A = \bigsqcup_{(T^{\text{the carrier of } S})} A$.

Let S_1, S_2, T_1, T_2 be non empty reflexive antisymmetric relational structures and let f be a map from S_1 into S_2. Let us assume that f is directed-sups-preserving. Let g be a map from T_1 into T_2. Let us assume that g is directed-sups-preserving. The functor $\mathrm{UPS}(f,g)$ yields a map from $\mathrm{UPS}(S_2,T_1)$ into $\mathrm{UPS}(S_1,T_2)$ and is defined by:

(Def. 5) For every directed-sups-preserving map h from S_2 into T_1 holds $(\mathrm{UPS}(f,g))(h) = g \cdot h \cdot f$.

Next we state a number of propositions:

(28) Let S_1, S_2, S_3, T_1, T_2, T_3 be non empty posets, f_1 be a directed-sups-preserving map from S_2 into S_3, f_2 be a directed-sups-preserving map from S_1 into S_2, g_1 be a directed-sups-preserving map from T_1 into T_2, and g_2 be a directed-sups-preserving map from T_2 into T_3. Then $\mathrm{UPS}(f_2,g_2) \cdot \mathrm{UPS}(f_1,g_1) = \mathrm{UPS}(f_1 \cdot f_2, g_2 \cdot g_1)$.

(29) For all non empty reflexive antisymmetric relational structures S, T holds $\mathrm{UPS}(\mathrm{id}_S, \mathrm{id}_T) = \mathrm{id}_{\mathrm{UPS}(S,T)}$.

(30) Let S_1, S_2, T_1, T_2 be complete lattices, f be a directed-sups-preserving map from S_1 into S_2, and g be a directed-sups-preserving map from T_1 into T_2. Then $\mathrm{UPS}(f,g)$ is directed-sups-preserving.

(31) Ω(the Sierpiński space) is Scott.

(32) For every complete Scott top-lattice S holds $[S \to \text{the Sierpiński space}] = \mathrm{UPS}(S, 2^1_{\subseteq})$.

(33) Let S be a complete lattice. Then there exists a map F from $\mathrm{UPS}(S, 2^1_{\subseteq})$ into $\langle \sigma(S), \subseteq \rangle$ such that F is isomorphic and for every directed-sups-preserving map f from S into $2^1_{\subseteq}$ holds $F(f) = f^{-1}(\{1\})$.

(34) For every complete lattice S holds $\mathrm{UPS}(S, 2^1_{\subseteq})$ and $\langle \sigma(S), \subseteq \rangle$ are isomorphic.

(35) Let S_1, S_2, T_1, T_2 be complete lattices, f be a map from S_1 into S_2, and g be a map from T_1 into T_2. If f is isomorphic and g is isomorphic, then $\mathrm{UPS}(f,g)$ is isomorphic.

(36) Let S_1, S_2, T_1, T_2 be complete lattices. Suppose S_1 and S_2 are isomorphic and T_1 and T_2 are isomorphic. Then $\mathrm{UPS}(S_2, T_1)$ and $\mathrm{UPS}(S_1, T_2)$ are isomorphic.

(37) Let S, T be complete lattices and f be a directed-sups-preserving projection map from T into T. Then $\mathrm{Im}\,\mathrm{UPS}(\mathrm{id}_S, f) = \mathrm{UPS}(S, \mathrm{Im}\, f)$.

(38) Let X be a non empty set, S, T be non empty posets, f be a directed-sups-preserving map from S into T^X, and i be an element of X. Then $(\mathrm{commute}(f))(i)$ is a directed-sups-preserving map from S into T.

(39) Let X be a non empty set, S, T be non empty posets, and f be a directed-sups-preserving map from S into T^X. Then $\mathrm{commute}(f)$ is a function from X into the carrier of $\mathrm{UPS}(S, T)$.

(40) Let X be a non empty set, S, T be non empty posets, and f be a function from X into the carrier of $\mathrm{UPS}(S, T)$. Then $\mathrm{commute}(f)$ is a directed-sups-preserving map from S into T^X.

(41) For every non empty set X and for all non empty posets S, T holds there exists a map from $\mathrm{UPS}(S, T^X)$ into $\mathrm{UPS}(S, T)^X$ which is commuting and isomorphic.

(42) For every non empty set X and for all non empty posets S, T holds $\mathrm{UPS}(S, T^X)$ and $(\mathrm{UPS}(S, T))^X$ are isomorphic.

(43) For all continuous complete lattices S, T holds $\mathrm{UPS}(S, T)$ is continuous.

(44) For all algebraic complete lattices S, T holds $\mathrm{UPS}(S, T)$ is algebraic.

(45) Let R, S, T be complete lattices and f be a directed-sups-preserving map from R into $\mathrm{UPS}(S, T)$. Then uncurry f is a directed-sups-preserving map from $[\!: R, S :\!]$ into T.

(46) Let R, S, T be complete lattices and f be a directed-sups-preserving map from $[\!: R, S :\!]$ into T. Then curry f is a directed-sups-preserving map from R into $\mathrm{UPS}(S, T)$.

(47) For all complete lattices R, S, T holds there exists a map from $\mathrm{UPS}(R, \mathrm{UPS}(S, T))$ into $\mathrm{UPS}([\!: R, S :\!], T)$ which is uncurrying and isomorphic.

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Received November 26, 1999