FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

Cages - the External Approximation of Jordan's Curve

Czesław Byliński[1]
University of Białystok

Mariusz Żynel[2]
University of Białystok

Summary. On the Euclidean plane Jordan's curve may be approximated with a polygonal path of sides parallel to coordinate axes, either externally, or internally. The paper deals with the external approximation, and the existence of a *Cage* – an external polygonal path – is proved.

MML Identifier: `JORDAN9`.

The papers [17], [25], [8], [18], [9], [2], [3], [23], [4], [22], [14], [16], [21], [6], [5], [11], [1], [19], [7], [13], [12], [15], [24], [20], [10], and [26] provide the terminology and notation for this paper.

1. Generalities

We adopt the following rules: k, n are natural numbers, D is a non empty set, and f, g are finite sequences of elements of D.

One can prove the following propositions:

(1) For all sets A, B such that A meets B holds $A \cap B$ meets B.

(2) For every non empty set A and for all sets B_1, B_2 such that $A \subseteq B_1$ and $A \subseteq B_2$ holds B_1 meets B_2.

[1]The paper was started during the author's visit in the Shinshu University, Nagano, Japan, summer 1998.

[2]The paper was finished during the author's visit in the Shinshu University, Nagano, Japan, summer 1999.

ISSN 1426–2630

(3) Let T be a non empty topological space and B, C_1, C_2, D be subsets of T. Suppose B is connected and C_1 is a component of D and C_2 is a component of D and B meets C_1 and B meets C_2 and $B \subseteq D$. Then $C_1 = C_2$.

(4) If for every n holds $f \restriction n = g \restriction n$, then $f = g$.

(5) If $n \in \operatorname{dom} f$, then there exists k such that $k \in \operatorname{dom} \operatorname{Rev}(f)$ and $n + k = \operatorname{len} f + 1$ and $\pi_n f = \pi_k \operatorname{Rev}(f)$.

(6) If $n \in \operatorname{dom} \operatorname{Rev}(f)$, then there exists k such that $k \in \operatorname{dom} f$ and $n + k = \operatorname{len} f + 1$ and $\pi_n \operatorname{Rev}(f) = \pi_k f$.

2. Go-Board Preliminaries

For simplicity, we adopt the following convention: G denotes a Go-board, f, g denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$, r, s denote real numbers, i, j, k denote natural numbers, and x denotes a set.

Next we state a number of propositions:

(7) f is a sequence which elements belong to G iff $\operatorname{Rev}(f)$ is a sequence which elements belong to G.

(8) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k \leqslant \operatorname{len} f$, then $\pi_k f \in \operatorname{Values} G$.

(9) If $n \leqslant \operatorname{len} f$ and $x \in \widetilde{\mathcal{L}}(f_{\lfloor n})$, then there exists a natural number i such that $n + 1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$ and $x \in \mathcal{L}(f, i)$.

(10) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\pi_k f \in \operatorname{left\_cell}(f, k, G)$ and $\pi_k f \in \operatorname{right\_cell}(f, k, G)$.

(11) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\operatorname{Int} \operatorname{left\_cell}(f, k, G) \neq \emptyset$ and $\operatorname{Int} \operatorname{right\_cell}(f, k, G) \neq \emptyset$.

(12) Suppose f is a sequence which elements belong to G and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f, k, G)$ is connected and $\operatorname{Int} \operatorname{right\_cell}(f, k, G)$ is connected.

(13) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\overline{\operatorname{Int} \operatorname{left\_cell}(f, k, G)} = \operatorname{left\_cell}(f, k, G)$ and $\overline{\operatorname{Int} \operatorname{right\_cell}(f, k, G)} = \operatorname{right\_cell}(f, k, G)$.

(14) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f, k)$ is horizontal. Then there exists j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} G$ and for every p such that $p \in \mathcal{L}(f, k)$ holds $p_{\mathbf{2}} = (G_{1,j})_{\mathbf{2}}$.

(15) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f, k)$ is vertical. Then there exists i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and for every p such that $p \in \mathcal{L}(f, k)$ holds $p_{\mathbf{1}} = (G_{i,1})_{\mathbf{1}}$.

(16) If f is a sequence which elements belong to G and special and $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{Int} \operatorname{cell}(G, i, j)$ misses $\widetilde{\mathcal{L}}(f)$.

(17) Suppose f is a sequence which elements belong to G and special and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f, k, G)$ misses $\widetilde{\mathcal{L}}(f)$ and $\operatorname{Int} \operatorname{right\_cell}(f, k, G)$ misses $\widetilde{\mathcal{L}}(f)$.

(18) Suppose $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$. Then $(G_{i,j})_{\mathbf{1}} = (G_{i,j+1})_{\mathbf{1}}$ and $(G_{i,j})_{\mathbf{2}} = (G_{i+1,j})_{\mathbf{2}}$ and $(G_{i+1,j+1})_{\mathbf{1}} = (G_{i+1,j})_{\mathbf{1}}$ and $(G_{i+1,j+1})_{\mathbf{2}} = (G_{i,j+1})_{\mathbf{2}}$.

(19) Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$. Then $p \in \operatorname{cell}(G, i, j)$ if and only if the following conditions are satisfied:

- (i) $(G_{i,j})_{\mathbf{1}} \leqslant p_{\mathbf{1}}$,
- (ii) $p_{\mathbf{1}} \leqslant (G_{i+1,j})_{\mathbf{1}}$,
- (iii) $(G_{i,j})_{\mathbf{2}} \leqslant p_{\mathbf{2}}$, and
- (iv) $p_{\mathbf{2}} \leqslant (G_{i,j+1})_{\mathbf{2}}$.

(20) If $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j) = \{[r, s] : (G_{i,j})_{\mathbf{1}} \leqslant r \wedge r \leqslant (G_{i+1,j})_{\mathbf{1}} \wedge (G_{i,j})_{\mathbf{2}} \leqslant s \wedge s \leqslant (G_{i,j+1})_{\mathbf{2}}\}$.

(21) Suppose $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G, i, j)$. Then $p = G_{i,j}$ or $p = G_{i,j+1}$ or $p = G_{i+1,j+1}$ or $p = G_{i+1,j}$.

(22) If $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$, then $G_{i,j} \in \operatorname{cell}(G, i, j)$ and $G_{i,j+1} \in \operatorname{cell}(G, i, j)$ and $G_{i+1,j+1} \in \operatorname{cell}(G, i, j)$ and $G_{i+1,j} \in \operatorname{cell}(G, i, j)$.

(23) If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G, i, j)$, then p is extremal in $\operatorname{cell}(G, i, j)$.

(24) Suppose $2 \leqslant \operatorname{len} G$ and $2 \leqslant \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$. Then there exist i, j such that $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j + 1 \leqslant \operatorname{width} G$ and $\mathcal{L}(f, k) \subseteq \operatorname{cell}(G, i, j)$.

(25) Suppose $2 \leqslant \operatorname{len} G$ and $2 \leqslant \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ and $p \in \operatorname{Values} G$ and $p \in \mathcal{L}(f, k)$. Then $p = \pi_k f$ or $p = \pi_{k+1} f$.

(26) If $\langle i, j \rangle \in$ the indices of G and $1 \leqslant k$ and $k \leqslant \operatorname{width} G$, then $(G_{i,j})_{\mathbf{1}} \leqslant (G_{\operatorname{len} G,k})_{\mathbf{1}}$.

(27) If $\langle i, j \rangle \in$ the indices of G and $1 \leqslant k$ and $k \leqslant \operatorname{len} G$, then $(G_{i,j})_{\mathbf{2}} \leqslant (G_{k,\operatorname{width} G})_{\mathbf{2}}$.

(28) Suppose f is a sequence which elements belong to G and special and $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$ and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$. Let A be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A = \operatorname{right\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$ or $A = \operatorname{left\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$, then A is

connected.

(29) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\text{right\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \text{RightComp}(f)$ and $\text{left\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \text{LeftComp}(f)$.

3. Cages

We follow the rules: C is a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, k, n, i_1, i_2 are natural numbers.

Next we state three propositions:

(30) There exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \text{Gauge}(C,n)$ and $\text{N-min}\, C \in \text{cell}(\text{Gauge}(C,n), i, \text{width}\, \text{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\text{Gauge}(C,n))_{i, \text{width}\, \text{Gauge}(C,n) -' 1}$.

(31) Suppose that $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} \text{Gauge}(C,n)$ and $\text{N-min}\, C \in \text{cell}(\text{Gauge}(C,n), i_1, \text{width}\, \text{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\text{Gauge}(C,n))_{i_1, \text{width}\, \text{Gauge}(C,n) -' 1}$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} \text{Gauge}(C,n)$ and $\text{N-min}\, C \in \text{cell}(\text{Gauge}(C,n), i_2, \text{width}\, \text{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\text{Gauge}(C,n))_{i_2, \text{width}\, \text{Gauge}(C,n) -' 1}$. Then $i_1 = i_2$.

(32) Let f be a standard non constant special circular sequence. Suppose that

(i) f is a sequence which elements belong to $\text{Gauge}(C,n)$,

(ii) for every k such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds $\text{left\_cell}(f,k,\text{Gauge}(C,n)) \cap C = \emptyset$ and $\text{right\_cell}(f,k,\text{Gauge}(C,n)) \cap C \neq \emptyset$, and

(iii) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \text{Gauge}(C,n)$ and $\pi_1 f = (\text{Gauge}(C,n))_{i, \text{width}\, \text{Gauge}(C,n)}$ and $\pi_2 f = (\text{Gauge}(C,n))_{i+1, \text{width}\, \text{Gauge}(C,n)}$ and $\text{N-min}\, C \in \text{cell}(\text{Gauge}(C,n), i, \text{width}\, \text{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\text{Gauge}(C,n))_{i, \text{width}\, \text{Gauge}(C,n) -' 1}$.

Then $\text{N-min}\, \widetilde{\mathcal{L}}(f) = \pi_1 f$.

Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Let us assume that C is connected. The functor $\text{Cage}(C,n)$ yields a clockwise oriented standard non constant special circular sequence and is defined by the conditions (Def. 1).

(Def. 1)(i) $\text{Cage}(C,n)$ is a sequence which elements belong to $\text{Gauge}(C,n)$,

(ii) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \text{Gauge}(C,n)$ and $\pi_1 \text{Cage}(C,n) = (\text{Gauge}(C,n))_{i, \text{width}\, \text{Gauge}(C,n)}$ and $\pi_2 \text{Cage}(C,n) = (\text{Gauge}(C,n))_{i+1, \text{width}\, \text{Gauge}(C,n)}$ and $\text{N-min}\, C \in \text{cell}(\text{Gauge}(C,n), i, \text{width}\, \text{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\text{Gauge}(C,n))_{i, \text{width}\, \text{Gauge}(C,n) -' 1}$, and

(iii) for every k such that $1 \leqslant k$ and $k + 2 \leqslant \operatorname{len} \operatorname{Cage}(C, n)$ holds if $\text{front\_left\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C = \emptyset$ and $\text{front\_right\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C = \emptyset$, then $\operatorname{Cage}(C, n)$ turns right k, $\operatorname{Gauge}(C, n)$ and if $\text{front\_left\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C = \emptyset$ and $\text{front\_right\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C \neq \emptyset$, then $\operatorname{Cage}(C, n)$ goes straight k, $\operatorname{Gauge}(C, n)$ and if $\text{front\_left\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C \neq \emptyset$, then $\operatorname{Cage}(C, n)$ turns left k, $\operatorname{Gauge}(C, n)$.

One can prove the following propositions:

(33) If C is connected and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} \operatorname{Cage}(C, n)$, then $\text{left\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C = \emptyset$ and $\text{right\_cell}(\operatorname{Cage}(C, n), k, \operatorname{Gauge}(C, n)) \cap C \neq \emptyset$.

(34) If C is connected, then $\text{N-min}\,\widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = \pi_1 \operatorname{Cage}(C, n)$.

Acknowledgments

Thanks are due to Professor Yatsuka Nakamura and people in the Kiso Laboratory for their gracious support during the work on this paper. Authors would like also to express their gratitude to Andrzej Trybulec for his help in preparing the paper.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.

[5] Czesław Byliński. Gauges. *Formalized Mathematics*, 8(**1**):25–27, 1999.

[6] Czesław Byliński. Some properties of cells on go-board. *Formalized Mathematics*, 8(**1**):139–146, 1999.

[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[11] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[13] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[14] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[15] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons. Part I. *Formalized Mathematics*, 5(**1**):97–102, 1996.

[16] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.

[17] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[18] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[21] Andrzej Trybulec. Left and right component of the complement of a special closed curve. *Formalized Mathematics*, 5(**4**):465–468, 1996.

[22] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(**3**):317–322, 1996.

[23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[25] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[26] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received June 22, 1999