FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

Darboux's Theorem

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Summary. In this article, we have proved the Darboux's theorem. This theorem is important to prove the Riemann integrability. We can replace an upper bound and a lower bound of a function which is the definition of Riemann integration with convergence of sequence by Darboux's theorem.

MML Identifier: `INTEGRA3`.

The articles [18], [14], [1], [2], [3], [12], [7], [8], [13], [4], [6], [9], [19], [11], [5], [10], [15], [17], and [16] provide the notation and terminology for this paper.

1. Lemmas of Division

We adopt the following convention: x, y are real numbers, i, j, k are natural numbers, and p, q are finite sequences of elements of $\mathbb{R}$.

The following propositions are true:

(1) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of divs A. If $\mathrm{vol}(A) \neq 0$, then there exists i such that $i \in \mathrm{dom}\, D$ and $\mathrm{vol}(\mathrm{divset}(D,i)) > 0$.

(2) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of divs A, and given x. If $x \in A$, then there exists j such that $j \in \mathrm{dom}\, D$ and $x \in \mathrm{divset}(D,j)$.

(3) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of divs A. Then there exists an element D of divs A such that $D_1 \leqslant D$ and $D_2 \leqslant D$ and $\mathrm{rng}\, D = \mathrm{rng}\, D_1 \cup \mathrm{rng}\, D_2$.

ISSN 1426–2630

(4) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of divs A. Suppose $\delta_{(D_1)} < \min \operatorname{rng} \text{upper\_volume}(\chi_{A,A}, D)$. Let given x, y, i. If $i \in \operatorname{dom} D_1$ and $x \in \operatorname{rng} D \cap \operatorname{divset}(D_1, i)$ and $y \in \operatorname{rng} D \cap \operatorname{divset}(D_1, i)$, then $x = y$.

(5) For all p, q such that $\operatorname{rng} p = \operatorname{rng} q$ and p is increasing and q is increasing holds $p = q$.

(6) Let A be a closed-interval subset of $\mathbb{R}$, D, D_1 be elements of divs A, and given i, j. Suppose $D \leqslant D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leqslant j$. Then $\operatorname{indx}(D_1, D, i) \leqslant \operatorname{indx}(D_1, D, j)$ and $\operatorname{indx}(D_1, D, i) \in \operatorname{dom} D_1$ and $\operatorname{indx}(D_1, D, j) \in \operatorname{dom} D_1$.

(7) Let A be a closed-interval subset of $\mathbb{R}$, D, D_1 be elements of divs A, and given i, j. Suppose $D \leqslant D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i < j$. Then $\operatorname{indx}(D_1, D, i) < \operatorname{indx}(D_1, D, j)$ and $\operatorname{indx}(D_1, D, i) \in \operatorname{dom} D_1$ and $\operatorname{indx}(D_1, D, j) \in \operatorname{dom} D_1$.

(8) For every closed-interval subset A of $\mathbb{R}$ and for every element D of divs A holds $\delta_D \geqslant 0$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, g be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geqslant 2$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is total and bounded on A. Then $\sum \text{lower\_volume}(g, D_2) - \sum \text{lower\_volume}(g, D_1) \leqslant (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, g be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geqslant 2$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is total and bounded on A. Then $\sum \text{upper\_volume}(g, D_1) - \sum \text{upper\_volume}(g, D_2) \leqslant (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(11) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of divs A, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leqslant j$ and $r < (\operatorname{mid}(D, i, j))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $r = \inf B$ and $\sup B = (\operatorname{mid}(D, i, j))(\operatorname{len} \operatorname{mid}(D, i, j))$ and $\operatorname{len} \operatorname{mid}(D, i, j) = (j - i) + 1$ and $\operatorname{mid}(D, i, j)$ is a DivisionPoint of B.

(12) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is total and bounded on A and $x > \inf A$. Then $\sum \text{lower\_volume}(f, D_2) - \sum \text{lower\_volume}(f, D_1) \leqslant (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(13) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose

$x \in \mathrm{divset}(D_1, \mathrm{len}\, D_1)$ and $\mathrm{vol}(A) \neq 0$ and $D_1 \leqslant D_2$ and $\mathrm{rng}\, D_2 = \mathrm{rng}\, D_1 \cup \{x\}$ and f is total and bounded on A and $x > \inf A$. Then $\sum \mathrm{upper\_volume}(f, D_1) - \sum \mathrm{upper\_volume}(f, D_2) \leqslant (\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f) \cdot \delta_{(D_1)}$.

(14) Let A be a closed-interval subset of $\mathbb{R}$, D_1, D_2 be elements of $\mathrm{divs}\, A$, r be a real number, and i, j be natural numbers. Suppose $i \in \mathrm{dom}\, D_1$ and $j \in \mathrm{dom}\, D_1$ and $i \leqslant j$ and $D_1 \leqslant D_2$ and $r < (\mathrm{mid}(D_2, \mathrm{indx}(D_2, D_1, i), \mathrm{indx}(D_2, D_1, j)))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ and there exist elements M_1, M_2 of $\mathrm{divs}\, B$ such that $r = \inf B$ and $\sup B = M_2(\mathrm{len}\, M_2)$ and $\sup B = M_1(\mathrm{len}\, M_1)$ and $M_1 \leqslant M_2$ and $M_1 = \mathrm{mid}(D_1, i, j)$ and $M_2 = \mathrm{mid}(D_2, \mathrm{indx}(D_2, D_1, i), \mathrm{indx}(D_2, D_1, j))$.

(15) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of $\mathrm{divs}\, A$, and given x. If $x \in \mathrm{rng}\, D$, then $D(1) \leqslant x$ and $x \leqslant D(\mathrm{len}\, D)$.

(16) Let p be a finite sequence of elements of $\mathbb{R}$ and given i, j, k. Suppose p is increasing and $i \in \mathrm{dom}\, p$ and $j \in \mathrm{dom}\, p$ and $k \in \mathrm{dom}\, p$ and $p(i) \leqslant p(k)$ and $p(k) \leqslant p(j)$. Then $p(k) \in \mathrm{rng}\, \mathrm{mid}(p, i, j)$.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D be an element of $\mathrm{divs}\, A$, and given i. If f is total and bounded on A and $i \in \mathrm{dom}\, D$, then $\inf \mathrm{rng}(f \restriction \mathrm{divset}(D, i)) \leqslant \sup \mathrm{rng}\, f$.

(18) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D be an element of $\mathrm{divs}\, A$, and given i. If f is total and bounded on A and $i \in \mathrm{dom}\, D$, then $\sup \mathrm{rng}(f \restriction \mathrm{divset}(D, i)) \geqslant \inf \mathrm{rng}\, f$.

2. Darboux's Theorem

The following two propositions are true:

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and T be a DivSequence of A. Suppose f is total and bounded on A and δ_T is convergent to 0 and $\mathrm{vol}(A) \neq 0$. Then $\mathrm{lower\_sum}(f, T)$ is convergent and $\lim \mathrm{lower\_sum}(f, T) = \mathrm{lower\_integral}\, f$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and T be a DivSequence of A. Suppose f is total and bounded on A and δ_T is convergent to 0 and $\mathrm{vol}(A) \neq 0$. Then $\mathrm{upper\_sum}(f, T)$ is convergent and $\lim \mathrm{upper\_sum}(f, T) = \mathrm{upper\_integral}\, f$.

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Received December 7, 1999