FORMALIZED MATHEMATICS
Volume 9, Number 1, 2001
University of Białystok

A Characterization of Concept Lattices. Dual Concept Lattices

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Summary. In this article we continue the formalization of concept lattices following [4]. We give necessary and sufficient conditions for a complete lattice to be isomorphic to a given formal context. As a by-product we get that a lattice is complete if and only if it is isomorphic to a concept lattice. In addition we introduce dual formal concepts and dual concept lattices and prove that the dual of a concept lattice over a formal context is isomorphic to the concept lattice over the dual formal context.

MML Identifier: CONLAT_2.

The notation and terminology used in this paper have been introduced in the following articles: [8], [10], [2], [3], [11], [1], [5], [9], [15], [7], [14], [6], [13], [12], and [16].

1. Preliminaries

Let C be a FormalContext and let C_1 be a strict FormalConcept of C. The functor ${}^{@}C_1$ yielding an element of ConceptLattice C is defined as follows:

(Def. 1) ${}^{@}C_1 = C_1$.

Next we state four propositions:

(1) For every FormalContext C holds $\bot_{\mathrm{ConceptLattice}\, C} = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Attributes}\, C$ and $\top_{\mathrm{ConceptLattice}\, C} = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Objects}\, C$.

(2) Let C be a FormalContext and D be a non empty subset of $2^{\text{the objects of } C}$. Then $(\mathrm{ObjectDerivation}\, C)(\bigcup D) = \bigcap\{(\mathrm{ObjectDerivation}\, C)(O); O$ ranges over subsets of the objects of C: $O \in D\}$.

ISSN 1426–2630

(3) Let C be a FormalContext and D be a non empty subset of $2^{\text{the Attributes of } C}$. Then $(\text{AttributeDerivation}\, C)(\bigcup D) = \bigcap\{(\text{AttributeDerivation}\, C)(A); A$ ranges over subsets of the Attributes of C: $A \in D\}$.

(4) Let C be a FormalContext and D be a subset of the carrier of ConceptLattice C. Then $\bigsqcap_{\text{ConceptLattice}\, C} D$ is a FormalConcept of C and $\bigsqcup_{\text{ConceptLattice}\, C} D$ is a FormalConcept of C.

Let C be a FormalContext and let D be a subset of the carrier of ConceptLattice C. The functor $\bigsqcap_C D$ yields a FormalConcept of C and is defined as follows:

(Def. 2) $\bigsqcap_C D = \bigsqcap_{\text{ConceptLattice}\, C} D$.

The functor $\bigsqcup_C D$ yields a FormalConcept of C and is defined by:

(Def. 3) $\bigsqcup_C D = \bigsqcup_{\text{ConceptLattice}\, C} D$.

Next we state several propositions:

(5) For every FormalContext C holds $\bigsqcup_C(\emptyset_{\text{ConceptLattice}\, C}) = \text{Concept} - \text{with} - \text{all} - \text{Attributes}\, C$ and $\bigsqcap_C(\emptyset_{\text{ConceptLattice}\, C}) = \text{Concept} - \text{with} - \text{all} - \text{Objects}\, C$.

(6) For every FormalContext C holds $\bigsqcup_C(\Omega_{\text{the carrier of ConceptLattice}\, C}) = \text{Concept} - \text{with} - \text{all} - \text{Objects}\, C$ and $\bigsqcap_C(\Omega_{\text{the carrier of ConceptLattice}\, C}) = \text{Concept} - \text{with} - \text{all} - \text{Attributes}\, C$.

(7) Let C be a FormalContext and D be a non empty subset of ConceptLattice C. Then

(i) the Extent of $\bigsqcup_C D = (\text{AttributeDerivation}\, C)((\text{ObjectDerivation}\, C)(\bigcup\{\text{the Extent of } \langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}))$, and

(ii) the Intent of $\bigsqcup_C D = \bigcap\{\text{the Intent of } \langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}$.

(8) Let C be a FormalContext and D be a non empty subset of ConceptLattice C. Then

(i) the Extent of $\bigsqcap_C D = \bigcap\{\text{the Extent of } \langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}$, and

(ii) the Intent of $\bigsqcap_C D = (\text{ObjectDerivation}\, C)((\text{AttributeDerivation}\, C)(\bigcup\{\text{the Intent of } \langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}))$.

(9) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcup_{\text{ConceptLattice}\, C}\{\langle O, A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the Attributes of C: $\bigvee_{o\,:\,\text{object of}\, C}$ $(o \in$ the Extent of $C_1 \wedge O = (\text{AttributeDerivation}\, C)((\text{ObjectDerivation}\, C)(\{o\})) \wedge A = (\text{ObjectDerivation}\, C)(\{o\}))\} = C_1$.

(10) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcap_{\mathrm{ConceptLattice}\,C}\{\langle O, A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the Attributes of C: $\bigvee_{a\,:\,\mathrm{Attribute\ of}\ C}$ $(a \in$ the Intent of C_1 $\wedge$ $O =$ (AttributeDerivation C)($\{a\}$) $\wedge$ $A =$ (ObjectDerivation C)((AttributeDerivation C)($\{a\}$)))$\} = C_1$.

Let C be a FormalContext. The functor $\gamma(C)$ yields a function from the objects of C into the carrier of ConceptLattice C and is defined by the condition (Def. 4).

(Def. 4) Let o be an element of the objects of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that $(\gamma(C))(o) = \langle O, A\rangle$ and $O =$ (AttributeDerivation C)((ObjectDerivation C)($\{o\}$)) and $A =$ (ObjectDerivation C)($\{o\}$).

Let C be a FormalContext. The functor δ_C yielding a function from the Attributes of C into the carrier of ConceptLattice C is defined by the condition (Def. 5).

(Def. 5) Let a be an element of the Attributes of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that $\delta_C(a) = \langle O, A\rangle$ and $O =$ (AttributeDerivation C)($\{a\}$) and $A =$ (ObjectDerivation C)((AttributeDerivation C)($\{a\}$)).

The following propositions are true:

(11) Let C be a FormalContext, o be an object of C, and a be a Attribute of C. Then $(\gamma(C))(o)$ is a FormalConcept of C and $\delta_C(a)$ is a FormalConcept of C.

(12) For every FormalContext C holds $\mathrm{rng}\,\gamma(C)$ is supremum-dense and $\mathrm{rng}(\delta_C)$ is infimum-dense.

(13) Let C be a FormalContext, o be an object of C, and a be a Attribute of C. Then o is connected with a if and only if $(\gamma(C))(o) \sqsubseteq \delta_C(a)$.

2. The Characterization

We now state the proposition

(14) Let L be a complete lattice and C be a FormalContext. Then ConceptLattice C and L are isomorphic if and only if there exists a function g from the objects of C into the carrier of L and there exists a function d from the Attributes of C into the carrier of L such that $\mathrm{rng}\,g$ is supremum-dense and $\mathrm{rng}\,d$ is infimum-dense and for every object o of C and for every Attribute a of C holds o is connected with a iff $g(o) \sqsubseteq d(a)$.

Let L be a lattice. The functor Context L yields a strict non quasi-empty ContextStr and is defined as follows:

(Def. 6) Context $L = \langle$the carrier of L, the carrier of L, LattRel$(L)\rangle$.

One can prove the following proposition

(15) For every complete lattice L holds ConceptLattice Context L and L are isomorphic.

Let L_1, L_2 be lattices. Let us note that the predicate L_1 and L_2 are isomorphic is symmetric.

Next we state the proposition

(16) For every lattice L holds L is complete iff there exists a FormalContext C such that ConceptLattice C and L are isomorphic.

3. Dual Concept Lattices

Let L be a complete lattice. Observe that L° is complete.

Let C be a FormalContext. The functor C° yielding a strict non quasi-empty ContextStr is defined as follows:

(Def. 7) $C^{\circ} = \langle$the Attributes of C, the objects of C, (the Information of $C)^{\smile}\rangle$.

We now state three propositions:

(17) For every strict FormalContext C holds $(C^{\circ})^{\circ} = C$.

(18) For every FormalContext C and for every subset O of the objects of C holds (ObjectDerivation $C)(O) =$ (AttributeDerivation $C^{\circ})(O)$.

(19) For every FormalContext C and for every subset A of the Attributes of C holds (AttributeDerivation $C)(A) =$ (ObjectDerivation $C^{\circ})(A)$.

Let C be a FormalContext and let C_1 be a ConceptStr over C. The functor ${C_1}^{\circ}$ yields a strict ConceptStr over C° and is defined as follows:

(Def. 8) The Extent of ${C_1}^{\circ} =$ the Intent of C_1 and the Intent of ${C_1}^{\circ} =$ the Extent of C_1.

Let C be a FormalContext and let C_1 be a FormalConcept of C. Then ${C_1}^{\circ}$ is a strict FormalConcept of C°.

We now state the proposition

(20) For every FormalContext C and for every strict FormalConcept C_1 of C holds $({C_1}^{\circ})^{\circ} = C_1$.

Let C be a FormalContext. The functor DualHomomorphism C yielding a homomorphism from (ConceptLattice $C)^{\circ}$ to ConceptLattice C° is defined as follows:

(Def. 9) For every strict FormalConcept C_1 of C holds (DualHomomorphism $C)(C_1) = {C_1}^{\circ}$.

We now state two propositions:

(21) For every FormalContext C holds DualHomomorphism C is isomorphism.

(22) For every FormalContext C holds ConceptLattice C° and (ConceptLattice $C)^{\circ}$ are isomorphic.

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Received August 17, 1999