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Asymptotic Notation. Part II: Examples and Problems[1]

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Summary. The widely used textbook by Brassard and Bratley [2] includes a chapter devoted to asymptotic notation (Chapter 3, pp. 79–97). We have attempted to test how suitable the current version of Mizar is for recording this type of material in its entirety. This article is a follow-up to [11] in which we introduced the basic notions and general theory. This article presents a Mizar formalization of examples and solutions to problems from Chapter 3 of [2] (some of the examples and solved problems are also in [11]). Not all problems have been solved as some required solutions not amenable for formalization.

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The articles [11], [10], [14], [15], [3], [4], [17], [1], [12], [13], [6], [19], [8], [9], [7], [16], [18], and [5] provide the terminology and notation for this paper.

1. Examples from the Text

We adopt the following rules: c, e denote real numbers, k, n, m, N, n_1, M denote natural numbers, and x denotes a set.

One can prove the following two propositions:

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(1) Let t, t_1 be sequences of real numbers. Suppose that

(i) $t(0) = 0$,

(ii) for every n such that $n > 0$ holds $t(n) = (12 \cdot n^3 \cdot \log_2 n - 5 \cdot n^{\mathbf{2}}) + (\log_2 n)^{\mathbf{2}} + 36$,

(iii) $t_1(0) = 0$, and

(iv) for every n such that $n > 0$ holds $t_1(n) = n^3 \cdot \log_2 n$.

Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = t$ and $s_1 = t_1$ and $s \in O(s_1)$.

(2) Let a, b be logbase real numbers and f, g be sequences of real numbers. Suppose $a > 1$ and $b > 1$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_a n$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = \log_b n$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $O(s) = O(s_1)$.

Let a, b, c be real numbers. The functor $\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined by:

(Def. 1) $(\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}})(n) = a^{b \cdot n + c}$.

Let a be a positive real number and let b, c be real numbers. One can verify that $\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}}$ is eventually-positive.

The following proposition is true

(3) For all positive real numbers a, b such that $a < b$ holds $\{b^{1 \cdot n + 0)}\}_{n \in \mathbb{N}} \notin O(\{a^{1 \cdot n + 0)}\}_{n \in \mathbb{N}})$.

The sequence $\{\log_2 n\}_{n \in \mathbb{N}}$ of real numbers is defined as follows:

(Def. 2) $\{\log_2 n\}_{n \in \mathbb{N}}(0) = 0$ and for every n such that $n > 0$ holds $\{\log_2 n\}_{n \in \mathbb{N}}(n) = \log_2 n$.

Let a be a real number. The functor $\{n^a\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def. 3) $\{n^a\}_{n \in \mathbb{N}}(0) = 0$ and for every n such that $n > 0$ holds $\{n^a\}_{n \in \mathbb{N}}(n) = n^a$.

Let us mention that $\{\log_2 n\}_{n \in \mathbb{N}}$ is eventually-positive.

Let a be a real number. Observe that $\{n^a\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state several propositions:

(4) Let f, g be eventually-nonnegative sequences of real numbers. Then $O(f) \subseteq O(g)$ and $O(f) \neq O(g)$ if and only if $f \in O(g)$ and $f \notin \Omega(g)$.

(5) $O(\{\log_2 n\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$ and $O(\{\log_2 n\}_{n \in \mathbb{N}}) \neq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(6) $\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}} \in \Omega(\{\log_2 n\}_{n \in \mathbb{N}})$ and $\{\log_2 n\}_{n \in \mathbb{N}} \notin \Omega(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(7) For every sequence f of real numbers and for every natural number k such that for every n holds $f(n) = \sum_{\kappa=0}^{n} (\{n^k\}_{n \in \mathbb{N}})(\kappa)$ holds $f \in \Theta(\{n^{(k+1)}\}_{n \in \mathbb{N}})$.

(8) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and for every

n such that $n > 0$ holds $f(n) = n^{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and s is not smooth.

Let b be a real number. The functor $\{b\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined as follows:

(Def. 4) $\{b\}_{n \in \mathbb{N}} = \mathbb{N} \longmapsto b$.

Let us note that $\{1\}_{n \in \mathbb{N}}$ is eventually-nonnegative.

One can prove the following proposition

(9) Let f be an eventually-nonnegative sequence of real numbers. Then there exists a non empty set F of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n \in \mathbb{N}}\}$ and $f \in F^{O(\{1\}_{n \in \mathbb{N}})}$ iff there exist N, c, k such that $c > 0$ and for every n such that $n \geqslant N$ holds $1 \leqslant f(n)$ and $f(n) \leqslant c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$.

2. Problem 3.1

One can prove the following proposition

(10) For every sequence f of real numbers such that for every n holds $f(n) = (3 \cdot 10^6 - 18 \cdot 10^3 \cdot n) + 27 \cdot n^{\mathbf{2}}$ holds $f \in O(\{n^2\}_{n \in \mathbb{N}})$.

3. Problem 3.5

We now state three propositions:

(11) $\{n^2\}_{n \in \mathbb{N}} \in O(\{n^3\}_{n \in \mathbb{N}})$.

(12) $\{n^2\}_{n \in \mathbb{N}} \notin \Omega(\{n^3\}_{n \in \mathbb{N}})$.

(13) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{1 \cdot n+1)}\}_{n \in \mathbb{N}}$ and $\{2^{1 \cdot n+0)}\}_{n \in \mathbb{N}} \in \Theta(s)$.

Let a be a natural number. The functor $\{(n+a)!\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined by:

(Def. 5) $\{(n+a)!\}_{n \in \mathbb{N}}(n) = (n+a)!$.

Let a be a natural number. Observe that $\{(n+a)!\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state the proposition

(14) $\{(n+0)!\}_{n \in \mathbb{N}} \notin \Theta(\{(n+1)!\}_{n \in \mathbb{N}})$.

4. Problem 3.6

The following proposition is true

(15) For every sequence f of real numbers such that $f \in O(\{n^1\}_{n \in \mathbb{N}})$ holds $f\,f \in O(\{n^2\}_{n \in \mathbb{N}})$.

5. Problem 3.7

We now state the proposition

(16) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{1 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $2\,\{n^1\}_{n \in \mathbb{N}} \in O(\{n^1\}_{n \in \mathbb{N}})$ and $\{2^{2 \cdot n+0)}\}_{n \in \mathbb{N}} \notin O(s)$.

6. Problem 3.8

One can prove the following proposition

(17) If $\log_2 3 < \frac{159}{100}$, then $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \in O(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \notin \Omega(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \notin \Theta(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$.

7. Problem 3.11

We now state the proposition

(18) Let f, g be sequences of real numbers. Suppose for every n holds $f(n) = n \bmod 2$ and for every n holds $g(n) = (n+1) \bmod 2$. Then there exist eventually-nonnegative sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $s \notin O(s_1)$ and $s_1 \notin O(s)$.

8. Problem 3.19

We now state two propositions:

(19) For all eventually-nonnegative sequences f, g of real numbers holds $O(f) = O(g)$ iff $f \in \Theta(g)$.

(20) For all eventually-nonnegative sequences f, g of real numbers holds $f \in \Theta(g)$ iff $\Theta(f) = \Theta(g)$.

9. Problem 3.21

The following propositions are true:

(21) Let e be a real number and f be a sequence of real numbers. Suppose $0 < e$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n \cdot \log_2 n$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$.

(22) Let e be a real number and g be a sequence of real numbers. Suppose $0 < e$ and $e < 1$ and $g(0) = 0$ and $g(1) = 0$ and for every n such that $n > 1$ holds $g(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \neq O(s)$.

(23) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and $f(1) = 0$ and for every n such that $n > 1$ holds $f(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^8\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^8\}_{n \in \mathbb{N}})$.

(24) Let g be a sequence of real numbers. Suppose that for every n holds $g(n) = ((n^{\mathbf{2}} - n) + 1)^4$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{n^8\}_{n \in \mathbb{N}}) = O(s)$.

(25) Let e be a real number. Suppose $0 < e$ and $e < 1$. Then there exists an eventually-positive sequence s of real numbers such that $s = \{1 + e^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(\{n^8\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^8\}_{n \in \mathbb{N}}) \neq O(s)$.

10. Problem 3.22

One can prove the following propositions:

(26) Let f, g be sequences of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{\log_2 n}$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(27) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(28) There exist eventually-positive sequences s, s_1 of real numbers such that $s = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{1 \cdot n + 1)}\}_{n \in \mathbb{N}}$ and $O(s) = O(s_1)$.

(29) There exist eventually-positive sequences s, s_1 of real numbers such that $s = \{2^{1 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{2 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(30) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{2 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(\{(n+0)!\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{(n+0)!\}_{n \in \mathbb{N}})$.

(31) $O(\{(n+0)!\}_{n \in \mathbb{N}}) \subseteq O(\{(n+1)!\}_{n \in \mathbb{N}})$ and $O(\{(n+0)!\}_{n \in \mathbb{N}}) \neq O(\{(n+1)!\}_{n \in \mathbb{N}})$.

(32) Let g be a sequence of real numbers. Suppose $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \neq O(s)$.

11. Problem 3.23

One can prove the following proposition

(33) Let given n. Suppose $n \geqslant 1$. Let f be a sequence of real numbers and k be a natural number. If for every n holds $f(n) = \sum_{\kappa=0}^{n}(\{n^k\}_{n \in \mathbb{N}})(\kappa)$, then $f(n) \geqslant \frac{n^{k+1}}{k+1}$.

12. Problem 3.24

One can prove the following proposition

(34) Let f, g be sequences of real numbers. Suppose $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n \cdot \log_2 n$ and for every n holds $f(n) = \log_2(n!)$. Then there exists an eventually-nonnegative sequence s of real numbers such that $s = g$ and $f \in \Theta(s)$.

13. Problem 3.26

The following proposition is true

(35) Let f be an eventually-nondecreasing eventually-nonnegative sequence of real numbers and t be a sequence of real numbers. Suppose that for every n holds if $n \bmod 2 = 0$, then $t(n) = 1$ and if $n \bmod 2 = 1$, then $t(n) = n$. Then $t \notin \Theta(f)$.

14. Problem 3.28

Let f be a function from $\mathbb{N}$ into $\mathbb{R}^*$ and let n be a natural number. Then $f(n)$ is a finite sequence of elements of $\mathbb{R}$.

Let n be a natural number and let a, b be positive real numbers. The functor $\mathrm{Prob28}(n, a, b)$ yields a real number and is defined by:

(Def. 6)(i) $\mathrm{Prob28}(n, a, b) = 0$ if $n = 0$,

(ii) there exists a natural number l and there exists a function p_{28} from $\mathbb{N}$ into $\mathbb{R}^*$ such that $l+1 = n$ and $\mathrm{Prob28}(n, a, b) = \pi_n p_{28}(l)$ and $p_{28}(0) = \langle a \rangle$ and for every natural number n there exists a natural number n_1 such that $n_1 = \lceil \frac{n+1+1}{2} \rceil$ and $p_{28}(n + 1) = p_{28}(n) \frown \langle 4 \cdot \pi_{n_1} p_{28}(n) + b \cdot (n + 1 + 1) \rangle$, otherwise.

Let a, b be positive real numbers. The functor $\{\mathrm{Prob28}(n, a, b)\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined by:

(Def. 7) $(\{\mathrm{Prob28}(n, a, b)\}_{n \in \mathbb{N}})(n) = \mathrm{Prob28}(n, a, b)$.

The following proposition is true

(36) For all positive real numbers a, b holds $\{\mathrm{Prob28}(n, a, b)\}_{n \in \mathbb{N}}$ is eventually-nondecreasing.

15. Problem 3.30

The non empty subset $\{2^n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is defined by:

(Def. 8) $\{2^n : n \in \mathbb{N}\} = \{2^n : n$ ranges over natural numbers$\}$.

Next we state three propositions:

(37) Let f be a sequence of real numbers. Suppose that for every n holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $f(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $f(n) = 2^n$. Then $f \in \Theta(\{n^1\}_{n \in \mathbb{N}} | \{2^n : n \in \mathbb{N}\})$ and $f \notin \Theta(\{n^1\}_{n \in \mathbb{N}})$ and $\{n^1\}_{n \in \mathbb{N}}$ is smooth and f is not eventually-nondecreasing.

(38) Let f, g be sequences of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{2^{\lfloor \log_2 n \rfloor}}$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence s of real numbers such that

(i) $s = g$,

(ii) $f \in \Theta(s | \{2^n : n \in \mathbb{N}\})$,

(iii) $f \notin \Theta(s)$,

(iv) f is eventually-nondecreasing,

(v) s is eventually-nondecreasing, and

(vi) s is not smooth w.r.t. 2.

(39) Let g be a sequence of real numbers. Suppose that for every n holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $g(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $g(n) = n^2$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $\{n^1\}_{n\in\mathbb{N}} \in \Theta(s|\{2^n : n \in \mathbb{N}\})$ and $\{n^1\}_{n\in\mathbb{N}} \notin \Theta(s)$ and $s_2 \in O(s)$ and $\{n^1\}_{n\in\mathbb{N}}$ is eventually-nondecreasing and s is not eventually-nondecreasing.

16. Problem 3.31

Let x be a natural number. The functor $x¡$ yielding a natural number is defined as follows:

(Def. 9)(i) There exists n such that $n! \leqslant x$ and $x < (n+1)!$ and $x¡ = n!$ if $x \neq 0$,

(ii) $x¡ = 0$, otherwise.

Next we state the proposition

(40) Let f be a sequence of real numbers. Suppose that for every n holds $f(n) = n¡$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and f is eventually-nondecreasing and for every n holds $f(n) \leqslant \{n^1\}_{n\in\mathbb{N}}(n)$ and s is not smooth.

17. Problem 3.34

Let us mention that $\{n^1\}_{n\in\mathbb{N}} - \{1\}_{n\in\mathbb{N}}$ is eventually-positive.

One can prove the following proposition

(41) $\Theta(\{n^1\}_{n\in\mathbb{N}} - \{1\}_{n\in\mathbb{N}}) + \Theta(\{n^1\}_{n\in\mathbb{N}}) = \Theta(\{n^1\}_{n\in\mathbb{N}})$.

18. Problem 3.35

One can prove the following proposition

(42) There exists a non empty set F of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n\in\mathbb{N}}\}$ and for every n holds $\{n^{(-1)}\}_{n\in\mathbb{N}}(n) \leqslant \{n^1\}_{n\in\mathbb{N}}(n)$ and $\{n^{(-1)}\}_{n\in\mathbb{N}} \notin F^{O(\{1\}_{n\in\mathbb{N}})}$.

19. Addition

The following proposition is true

(43) Let c be a non negative real number and x, f be eventually-nonnegative sequences of real numbers. Given e, N such that $e > 0$ and for every n such that $n \geqslant N$ holds $f(n) \geqslant e$. If $x \in O(c+f)$, then $x \in O(f)$.

20. Potentatially Useful

The following propositions are true:

(44) $2^2 = 4$.

(45) $2^3 = 8$.

(46) $2^4 = 16$.

(47) $2^5 = 32$.

(48) $2^6 = 64$.

(49) $2^{12} = 4096$.

(50) For every n such that $n \geqslant 3$ holds $n^{\mathbf{2}} > 2 \cdot n + 1$.

(51) For every n such that $n \geqslant 10$ holds $2^{n-1} > (2 \cdot n)^{\mathbf{2}}$.

(52) For every n such that $n \geqslant 9$ holds $(n+1)^6 < 2 \cdot n^6$.

(53) For every n such that $n \geqslant 30$ holds $2^n > n^6$.

(54) For every real number x such that $x > 9$ holds $2^x > (2 \cdot x)^{\mathbf{2}}$.

(55) There exists N such that for every n such that $n \geqslant N$ holds $\sqrt{n} - \log_2 n > 1$.

(56) For all real numbers a, b, c such that $a > 0$ and $c > 0$ and $c \neq 1$ holds $a^b = c^{b \cdot \log_c a}$.

(57) $(4+1)! = 120$.

(58) $5^5 = 3125$.

(59) $4^4 = 256$.

(60) For every n holds $(n^{\mathbf{2}} - n) + 1 > 0$.

(61) For every n such that $n \geqslant 2$ holds $n! > 1$.

(62) For all n_1, n such that $n \leqslant n_1$ holds $n! \leqslant n_1!$.

(63) For every k such that $k \geqslant 1$ there exists n such that $n! \leqslant k$ and $k < (n+1)!$ and for every m such that $m! \leqslant k$ and $k < (m+1)!$ holds $m = n$.

(64) For every n such that $n \geqslant 2$ holds $\lceil \frac{n}{2} \rceil < n$.

(65) For every n such that $n \geqslant 3$ holds $n! > n$.

(66) For all natural numbers m, n such that $m > 0$ holds m^n is a natural number.

(67) For every n such that $n \geqslant 2$ holds $2^n > n + 1$.

(68) Let a be a logbase real number and f be a sequence of real numbers. Suppose $a > 1$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_a n$. Then f is eventually-positive.

(69) For all eventually-nonnegative sequences f, g of real numbers holds $f \in O(g)$ and $g \in O(f)$ iff $O(f) = O(g)$.

(70) For all real numbers a, b, c such that $0 < a$ and $a \leqslant b$ and $c \geqslant 0$ holds $a^c \leqslant b^c$.

(71) For every n such that $n \geqslant 4$ holds $2 \cdot n + 3 < 2^n$.

(72) For every n such that $n \geqslant 6$ holds $(n + 1)^{\mathbf{2}} < 2^n$.

(73) For every real number c such that $c > 6$ holds $c^{\mathbf{2}} < 2^c$.

(74) Let e be a positive real number and f be a sequence of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_2(n^e)$. Then $f/\{n^e\}_{n\in\mathbb{N}}$ is convergent and $\lim(f/\{n^e\}_{n\in\mathbb{N}}) = 0$.

(75) For every real number e such that $e > 0$ holds $\{\log_2 n\}_{n\in\mathbb{N}}/\{n^e\}_{n\in\mathbb{N}}$ is convergent and $\lim(\{\log_2 n\}_{n\in\mathbb{N}}/\{n^e\}_{n\in\mathbb{N}}) = 0$.

(76) For every sequence f of real numbers and for every N such that for every n such that $n \leqslant N$ holds $f(n) \geqslant 0$ holds $\sum_{\kappa=0}^{N} f(\kappa) \geqslant 0$.

(77) For all sequences f, g of real numbers and for every N such that for every n such that $n \leqslant N$ holds $f(n) \leqslant g(n)$ holds $\sum_{\kappa=0}^{N} f(\kappa) \leqslant \sum_{\kappa=0}^{N} g(\kappa)$.

(78) Let f be a sequence of real numbers and b be a real number. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = b$. Let N be a natural number. Then $\sum_{\kappa=0}^{N} f(\kappa) = b \cdot N$.

(79) For all sequences f, g of real numbers and for all natural numbers N, M holds $\sum_{\kappa=N+1}^{M} f(\kappa) + f(N + 1) = \sum_{\kappa=N+1+1}^{M} f(\kappa)$.

(80) Let f, g be sequences of real numbers, M be a natural number, and given N. Suppose $N \geqslant M + 1$. If for every n such that $M + 1 \leqslant n$ and $n \leqslant N$ holds $f(n) \leqslant g(n)$, then $\sum_{\kappa=N+1}^{M} f(\kappa) \leqslant \sum_{\kappa=N+1}^{M} g(\kappa)$.

(81) For every n holds $\lceil \frac{n}{2} \rceil \leqslant n$.

(82) Let f be a sequence of real numbers, b be a real number, and N be a natural number. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = b$. Let M be a natural number. Then $\sum_{\kappa=N+1}^{M} f(\kappa) = b \cdot (N - M)$.

(83) Let f, g be sequences of real numbers, N be a natural number, and c be a real number. Suppose f is convergent and $\lim f = c$ and for every n such that $n \geqslant N$ holds $f(n) = g(n)$. Then g is convergent and $\lim g = c$.

(84) For every n such that $n \geqslant 1$ holds $(n^{\mathbf{2}} - n) + 1 \leqslant n^{\mathbf{2}}$.

(85) For every n such that $n \geqslant 1$ holds $n^{\mathbf{2}} \leqslant 2 \cdot ((n^{\mathbf{2}} - n) + 1)$.

(86) For every real number e such that $0 < e$ and $e < 1$ there exists N such that for every n such that $n \geqslant N$ holds $n \cdot \log_2(1+e) - 8 \cdot \log_2 n > 8 \cdot \log_2 n$.

(87) For every n such that $n \geqslant 10$ holds $\frac{2^{2 \cdot n}}{n!} < \frac{1}{2^{n-9}}$.

(88) For every n such that $n \geqslant 3$ holds $2 \cdot (n-2) \geqslant n-1$.

(89) For every real number c such that $c \geqslant 0$ holds $c^{\frac{1}{2}} = \sqrt{c}$.

(90) There exists N such that for every n such that $n \geqslant N$ holds $n - \sqrt{n} \cdot \log_2 n > \frac{n}{2}$.

(91) For every sequence s of real numbers such that for every n holds $s(n) = (1 + \frac{1}{n+1})^{n+1}$ holds s is non-decreasing.

(92) For every n such that $n \geqslant 1$ holds $(\frac{n+1}{n})^n \leqslant (\frac{n+2}{n+1})^{n+1}$.

(93) For all k, n such that $k \leqslant n$ holds $\binom{n}{k} \geqslant \frac{\binom{n+1}{k}}{n+1}$.

(94) For every sequence f of real numbers such that for every n holds $f(n) = \log_2(n!)$ and for every n holds $f(n) = \sum_{\kappa=0}^{n}(\{\log_2 n\}_{n \in \mathbb{N}})(\kappa)$.

(95) For every n such that $n \geqslant 4$ holds $n \cdot \log_2 n \geqslant 2 \cdot n$.

(96) Let a, b be positive real numbers. Then $\mathrm{Prob28}(0, a, b) = 0$ and $\mathrm{Prob28}(1, a, b) = a$ and for every n such that $n \geqslant 2$ there exists n_1 such that $n_1 = \lceil \frac{n}{2} \rceil$ and $\mathrm{Prob28}(n, a, b) = 4 \cdot \mathrm{Prob28}(n_1, a, b) + b \cdot n$.

(97) For every n such that $n \geqslant 2$ holds $n^{\mathbf{2}} > n+1$.

(98) For every n such that $n \geqslant 1$ holds $2^{n+1} - 2^n > 1$.

(99) For every n such that $n \geqslant 2$ holds $2^n - 1 \notin \{2^n : n \in \mathbb{N}\}$.

(100) For all n, k such that $k \geqslant 1$ and $n! \leqslant k$ and $k < (n+1)!$ holds $k¡ = n!$.

(101) For all real numbers a, b, c such that $a > 1$ and $b \geqslant a$ and $c \geqslant 1$ holds $\log_a c \geqslant \log_b c$.

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