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Recursive Euclide Algorithm [1]

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Summary. The earlier SCM computer did not contain recursive function, so Trybulec and Nakamura proved the correctness of the Euclid's algorithm only by way of an iterative program. However, the recursive method is a very important programming method, furthermore, for some algorithms, for example Quicksort, only by employing a recursive method (note push-down stack is essentially also a recursive method) can they be implemented. The main goal of the article is to test the recursive function of the SCMPDS computer by proving the correctness of the Euclid's algorithm by way of a recursive program. In this article, we observed that the memory required by the recursive Euclide algorithm is variable but it is still autonomic. Although the algorithm here is more complicated than the non-recursive algorithm, its focus is that the SCMPDS computer will be able to implement many algorithms like Quicksort which the SCM computer cannot do.

MML Identifier: `SCMP_GCD`.

The articles [12], [14], [1], [3], [5], [4], [16], [15], [11], [2], [10], [18], [9], [8], [6], [7], [17], and [13] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: m, n denote natural numbers, i, j denote instructions of SCMPDS, s denotes a state of SCMPDS, and I, J denote Program-block.

One can prove the following three propositions:

(1) If $m > 0$, then $\gcd(n, m) = \gcd(m, n \bmod m)$.

(2) For all integers i, j such that $i \geqslant 0$ and $j > 0$ holds $i \gcd j = j \gcd i \bmod j$.

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(3) For every natural number m and for every integer j such that inspos $m = j$ holds inspos $m + 2 = 2 \cdot (|j| \div 2) + 4$.

Let k be a natural number. The functor intpos k yields a Int position and is defined as follows:

(Def. 1) intpos $k = \mathbf{d}_k$.

Next we state three propositions:

(4) For all natural numbers n_1, n_2 such that $n_1 \neq n_2$ holds intpos $n_1 \neq$ intpos n_2.

(5) For all natural numbers n_1, n_2 holds DataLoc$(n_1, n_2) =$ intpos $n_1 + n_2$.

(6) For every state s of SCMPDS and for all natural numbers m_1, m_2 such that $\mathbf{IC}_s =$ inspos $m_1 + m_2$ holds ICplusConst$(s, -m_2) =$ inspos m_1.

The Int position GBP is defined by:

(Def. 2) GBP = intpos 0.

The Int position SBP is defined as follows:

(Def. 3) SBP = intpos 1.

The following propositions are true:

(7) GBP $\neq$ SBP .

(8) card$(I;i) =$ card $I + 1$.

(9) card$(i;j) = 2$.

(10) $(I;i)$(inspos card $I) = i$ and inspos card $I \in$ dom$(I;i)$.

(11) $(I;i;J)$(inspos card $I) = i$.

2. The Construction of Recursive Euclide Algorithm

The Program-block GCD − Algorithm is defined by:

(Def. 4) GCD − Algorithm = (GBP :=0);(SBP :=7); saveIC(SBP, RetIC);goto 2; $\mathbf{halt}_{\mathrm{SCMPDS}}$;((SBP, 3) <= 0_*goto*9);((SBP, 6) := (SBP, 3)); Divide(SBP, 2, SBP, 3);((SBP, 7) := (SBP, 3));((SBP, 4 + RetSP) := (GBP, 1)); AddTo(GBP, 1, 4); saveIC(SBP, RetIC);goto (−7);((SBP, 2) := (SBP, 6)); return SBP .

3. The Computation of Recursive Euclide Algorithm

One can prove the following propositions:

(12) card GCD − Algorithm = 15.

(13) $n < 15$ iff inspos $n \in$ dom GCD − Algorithm .

(14) (GCD − Algorithm)(inspos 0) = GBP :=0 and (GCD − Algorithm)(inspos 1) = SBP :=7 and (GCD − Algorithm)(inspos 2) = saveIC(SBP, RetIC) and (GCD − Algorithm)(inspos 3) = goto 2 and (GCD − Algorithm)(inspos 4) = $\mathbf{halt}_{\rm SCMPDS}$ and (GCD − Algorithm)(inspos 5) = (SBP, 3) <= 0_*goto*9 and (GCD − Algorithm)(inspos 6) = (SBP, 6) := (SBP, 3) and (GCD − Algorithm)(inspos 7) = Divide(SBP, 2, SBP, 3) and (GCD − Algorithm)(inspos 8) = (SBP, 7) := (SBP, 3) and (GCD − Algorithm)(inspos 9) = (SBP, 4 + RetSP) := (GBP, 1) and (GCD − Algorithm)(inspos 10) = AddTo(GBP, 1, 4) and (GCD − Algorithm)(inspos 11) = saveIC(SBP, RetIC) and (GCD − Algorithm)(inspos 12) = goto (−7) and (GCD − Algorithm)(inspos 13) = (SBP, 2) := (SBP, 6) and (GCD − Algorithm)(inspos 14) = return SBP .

(15) Let s be a state of SCMPDS. Suppose Initialized(GCD − Algorithm) $\subseteq s$. Then $\mathbf{IC}_{(\mathrm{Computation}(s))(4)}$ = inspos 5 and (Computation(s))(4)(GBP) = 0 and (Computation(s))(4)(SBP) = 7 and (Computation(s))(4)(intpos 7+ RetIC) = inspos 2 and (Computation(s))(4)(intpos 9) = s(intpos 9) and (Computation(s))(4)(intpos 10) = s(intpos 10).

(16) Let s be a state of SCMPDS. Suppose GCD − Algorithm $\subseteq s$ and $\mathbf{IC}_s$ = inspos 5 and s(SBP) > 0 and s(GBP) $= 0$ and s(DataLoc(s(SBP), 3)) $\geqslant 0$ and s(DataLoc(s(SBP), 2)) $\geqslant s$(DataLoc(s(SBP), 3)). Then there exists n such that

(i) CurInstr((Computation(s))(n)) = return SBP,

(ii) s(SBP) = (Computation(s))(n)(SBP),

(iii) (Computation(s))(n)(DataLoc(s(SBP), 2)) = s(DataLoc(s(SBP), 2)) gcd s(DataLoc(s(SBP), 3)), and

(iv) for every natural number j such that $1 < j$ and $j \leqslant s(\mathrm{SBP}) + 1$ holds s(intpos j) = (Computation(s))(n)(intpos j).

(17) Let s be a state of SCMPDS. Suppose GCD − Algorithm $\subseteq s$ and $\mathbf{IC}_s$ = inspos 5 and s(SBP) > 0 and s(GBP) $= 0$ and s(DataLoc(s(SBP), 3)) $\geqslant 0$ and s(DataLoc(s(SBP), 2)) $\geqslant 0$. Then there exists n such that

(i) CurInstr((Computation(s))(n)) = return SBP,

(ii) s(SBP) = (Computation(s))(n)(SBP),

(iii) (Computation(s))(n)(DataLoc(s(SBP), 2)) = s(DataLoc(s(SBP), 2)) gcd s(DataLoc(s(SBP), 3)), and

(iv) for every natural number j such that $1 < j$ and $j \leqslant s(\mathrm{SBP}) + 1$ holds s(intpos j) = (Computation(s))(n)(intpos j).

4. The Correctness of Recursive Euclide Algorithm

The following proposition is true

(18) Let s be a state of SCMPDS. Suppose Initialized(GCD − Algorithm) $\subseteq s$. Let x, y be integers. If $s(\mathrm{intpos}\,9) = x$ and $s(\mathrm{intpos}\,10) = y$ and $x \geqslant 0$ and $y \geqslant 0$, then $(\mathrm{Result}(s))(\mathrm{intpos}\,9) = x \gcd y$.

5. The Autonomy of Recursive Euclide Algorithm

We now state the proposition

(19) Let p be a finite partial state of SCMPDS and x, y be integers. If $y \geqslant 0$ and $x \geqslant y$ and $p = [\mathrm{intpos}\,9 \longmapsto x, \mathrm{intpos}\,10 \longmapsto y]$, then Initialized(GCD − Algorithm)$+\cdot p$ is autonomic.

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Scott-Continuous Functions. Part II[1]

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The terminology and notation used here are introduced in the following articles: [13], [5], [1], [16], [6], [14], [11], [18], [17], [12], [15], [7], [3], [4], [10], [2], [8], [19], and [9].

1. Preliminaries

One can prove the following proposition

(1) Let S, T be up-complete Scott top-lattices and M be a subset of $\mathrm{SCMaps}(S,T)$. Then $\bigsqcup_{\mathrm{SCMaps}(S,T)} M$ is a continuous map from S into T.

Let S be a non empty relational structure and let T be a non empty reflexive relational structure. One can check that every map from S into T which is constant is also monotone.

Let S be a non empty relational structure, let T be a reflexive non empty relational structure, and let a be an element of the carrier of T. One can check that $S \longmapsto a$ is monotone.

One can prove the following propositions:

(2) Let S be a non empty relational structure and T be a lower-bounded antisymmetric reflexive non empty relational structure. Then $\perp_{\mathrm{MonMaps}(S,T)} = S \longmapsto \perp_T$.

(3) Let S be a non empty relational structure and T be an upper-bounded antisymmetric reflexive non empty relational structure. Then $\top_{\mathrm{MonMaps}(S,T)} = S \longmapsto \top_T$.

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(4) Let S, T be complete lattices, f be a monotone map from S into T, and x be an element of S. Then $f(x) = \sup(f^{\circ}{\downarrow}x)$.

(5) Let S, T be complete lower-bounded lattices, f be a monotone map from S into T, and x be an element of S. Then $f(x) = \bigsqcup_T \{f(w); w$ ranges over elements of S: $w \leqslant x\}$.

(6) Let S be a relational structure, T be a non empty relational structure, and F be a subset of $T^{\text{the carrier of } S}$. Then $\sup F$ is a map from S into T.

2. On the Scott Continuity of Maps

Let X_1, X_2, Y be non empty relational structures, let f be a map from $[\!: X_1, X_2 :\!]$ into Y, and let x be an element of the carrier of X_1. The functor $\mathrm{Proj}(f, x)$ yields a map from X_2 into Y and is defined as follows:

(Def. 1) $\mathrm{Proj}(f, x) = (\mathrm{curry}\, f)(x)$.

For simplicity, we use the following convention: X_1, X_2, Y denote non empty relational structures, f denotes a map from $[\!: X_1, X_2 :\!]$ into Y, x denotes an element of the carrier of X_1, and y denotes an element of the carrier of X_2.

We now state the proposition

(7) For every element y of the carrier of X_2 holds $(\mathrm{Proj}(f, x))(y) = f(\langle x, y\rangle)$.

Let X_1, X_2, Y be non empty relational structures, let f be a map from $[\!: X_1, X_2 :\!]$ into Y, and let y be an element of the carrier of X_2. The functor $\mathrm{Proj}(f, y)$ yielding a map from X_1 into Y is defined by:

(Def. 2) $\mathrm{Proj}(f, y) = (\mathrm{curry}'\, f)(y)$.

The following propositions are true:

(8) For every element x of the carrier of X_1 holds $(\mathrm{Proj}(f, y))(x) = f(\langle x, y\rangle)$.

(9) Let R, S, T be non empty relational structures, f be a map from $[\!: R, S :\!]$ into T, a be an element of R, and b be an element of S. Then $(\mathrm{Proj}(f, a))(b) = (\mathrm{Proj}(f, b))(a)$.

Let S be a non empty relational structure and let T be a non empty reflexive relational structure. Observe that there exists a map from S into T which is antitone.

The following two propositions are true:

(10) Let R, S, T be non empty reflexive relational structures, f be a map from $[\!: R, S :\!]$ into T, a be an element of the carrier of R, and b be an element of the carrier of S. If f is monotone, then $\mathrm{Proj}(f, a)$ is monotone and $\mathrm{Proj}(f, b)$ is monotone.

(11) Let R, S, T be non empty reflexive relational structures, f be a map from $[: R, S :]$ into T, a be an element of the carrier of R, and b be an element of the carrier of S. If f is antitone, then $\mathrm{Proj}(f, a)$ is antitone and $\mathrm{Proj}(f, b)$ is antitone.

Let R, S, T be non empty reflexive relational structures, let f be a monotone map from $[: R, S :]$ into T, and let a be an element of the carrier of R. Note that $\mathrm{Proj}(f, a)$ is monotone.

Let R, S, T be non empty reflexive relational structures, let f be a monotone map from $[: R, S :]$ into T, and let b be an element of the carrier of S. Note that $\mathrm{Proj}(f, b)$ is monotone.

Let R, S, T be non empty reflexive relational structures, let f be an antitone map from $[: R, S :]$ into T, and let a be an element of the carrier of R. Observe that $\mathrm{Proj}(f, a)$ is antitone.

Let R, S, T be non empty reflexive relational structures, let f be an antitone map from $[: R, S :]$ into T, and let b be an element of the carrier of S. Note that $\mathrm{Proj}(f, b)$ is antitone.

We now state several propositions:

(12) Let R, S, T be lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is monotone and $\mathrm{Proj}(f, b)$ is monotone. Then f is monotone.

(13) Let R, S, T be lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is antitone and $\mathrm{Proj}(f, b)$ is antitone. Then f is antitone.

(14) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, b be an element of S, and X be a subset of R. Then $(\mathrm{Proj}(f, b))^\circ X = f^\circ [: X, \{b\} :]$.

(15) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, b be an element of R, and X be a subset of S. Then $(\mathrm{Proj}(f, b))^\circ X = f^\circ [: \{b\}, X :]$.

(16) Let R, S, T be lattices, f be a map from $[: R, S :]$ into T, a be an element of R, and b be an element of S. Suppose f is directed-sups-preserving. Then $\mathrm{Proj}(f, a)$ is directed-sups-preserving and $\mathrm{Proj}(f, b)$ is directed-sups-preserving.

(17) Let R, S, T be lattices, f be a monotone map from $[: R, S :]$ into T, a be an element of R, b be an element of S, and X be a directed subset of $[: R, S :]$. If $\sup f^\circ X$ exists in T and $a \in \pi_1(X)$ and $b \in \pi_2(X)$, then $f(\langle a, b \rangle) \leqslant \sup(f^\circ X)$.

(18) Let R, S, T be complete lattices and f be a map from $[: R, S :]$ into T. Suppose that for every element a of R and for every element b of S holds $\mathrm{Proj}(f, a)$ is directed-sups-preserving and $\mathrm{Proj}(f, b)$ is directed-sups-preserving. Then f is directed-sups-preserving.

(19) Let S be a non empty 1-sorted structure, T be a non empty relational

structure, and f be a set. Then f is an element of $T^{\text{the carrier of } S}$ if and only if f is a map from S into T.

3. The Poset of Continuous Maps

Let S be a topological structure and let T be a non empty FR-structure. The functor $[S \to T]$ yielding a strict relational structure is defined by the conditions (Def. 3).

(Def. 3)(i) $[S \to T]$ is a full relational substructure of $T^{\text{the carrier of } S}$, and

(ii) for every set x holds $x \in$ the carrier of $([S \to T])$ iff there exists a map f from S into T such that $x = f$ and f is continuous.

Let S be a non empty topological space and let T be a non empty topological space-like FR-structure. Observe that $[S \to T]$ is non empty.

Let S be a non empty topological space and let T be a non empty topological space-like FR-structure. Note that $[S \to T]$ is constituted functions.

One can prove the following propositions:

(20) Let S be a non empty topological space, T be a non empty reflexive topological space-like FR-structure, and x, y be elements of $[S \to T]$. Then $x \leqslant y$ if and only if for every element i of S holds $\langle x(i), y(i) \rangle \in$ the internal relation of T.

(21) Let S be a non empty topological space, T be a non empty reflexive topological space-like FR-structure, and x be a set. Then x is a continuous map from S into T if and only if x is an element of $[S \to T]$.

Let S be a non empty topological space and let T be a non empty reflexive topological space-like FR-structure. Note that $[S \to T]$ is reflexive.

Let S be a non empty topological space and let T be a non empty transitive topological space-like FR-structure. Note that $[S \to T]$ is transitive.

Let S be a non empty topological space and let T be a non empty antisymmetric topological space-like FR-structure. One can check that $[S \to T]$ is antisymmetric.

Let S be a non empty 1-sorted structure and let T be a non empty topological space-like FR-structure. One can verify that $T^{\text{the carrier of } S}$ is constituted functions.

One can prove the following three propositions:

(22) Let S be a non empty 1-sorted structure, T be a complete lattice, f, g, h be maps from S into T, and i be an element of S. If $h = \bigsqcup_{(T^{\text{the carrier of } S})}\{f, g\}$, then $h(i) = \sup\{f(i), g(i)\}$.

(23) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for

every element i of I holds $J(i)$ is a complete lattice. Let X be a subset of $\prod J$ and i be an element of I. Then $(\inf X)(i) = \inf \pi_i X$.

(24) Let S be a non empty 1-sorted structure, T be a complete lattice, f, g, h be maps from S into T, and i be an element of S. If $h = \bigsqcap_{(T^{\text{the carrier of } S})}\{f, g\}$, then $h(i) = \inf\{f(i), g(i)\}$.

Let S be a non empty 1-sorted structure and let T be a lattice. Observe that every element of $T^{\text{the carrier of } S}$ is function-like and relation-like.

Let S, T be top-lattices. One can check that every element of $[S \to T]$ is function-like and relation-like.

One can prove the following propositions:

(25) Let S be a non empty relational structure, T be a complete lattice, F be a non empty subset of $T^{\text{the carrier of } S}$, and i be an element of the carrier of S. Then $(\sup F)(i) = \bigsqcup_T\{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}$.

(26) Let S, T be complete top-lattices, F be a non empty subset of $[S \to T]$, and i be an element of the carrier of S. Then $(\bigsqcup_{(T^{\text{the carrier of } S})} F)(i) = \bigsqcup_T\{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}$.

In the sequel S denotes a non empty relational structure, T denotes a complete lattice, and i denotes an element of S.

Next we state two propositions:

(27) Let F be a non empty subset of $T^{\text{the carrier of } S}$ and D be a non empty subset of S. Then $(\sup F)^{\circ}D = \{\bigsqcup_T\{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}; i$ ranges over elements of S: $i \in D\}$.

(28) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a non empty subset of S. Then $(\bigsqcup_{(T^{\text{the carrier of } S})} F)^{\circ}D = \{\bigsqcup_T\{f(i); f$ ranges over elements of $T^{\text{the carrier of } S}$: $f \in F\}; i$ ranges over elements of S: $i \in D\}$.

The scheme *FraenkelF'RSS* deals with a non empty relational structure $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and and states that:

$\{\mathcal{F}(v_1); v_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2); v_2$ ranges over elements of $\mathcal{A} : \mathcal{P}[v_2]\}$

provided the following condition is met:

- For every element v of $\mathcal{A}$ such that $\mathcal{P}[v]$ holds $\mathcal{F}(v) = \mathcal{G}(v)$.

The following propositions are true:

(29) Let S, T be complete Scott top-lattices and F be a non empty subset of $[S \to T]$. Then $\bigsqcup_{(T^{\text{the carrier of } S})} F$ is a monotone map from S into T.

(30) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a directed non empty subset of S. Then $\bigsqcup_T\{\bigsqcup_T\{g(i); i$ ranges over elements of S: $i \in D\}; g$ ranges over elements of $T^{\text{the carrier of } S}$:

$g \in F\} = \bigsqcup_T \{\bigsqcup_T \{g'(i'); g'$ ranges over elements of $T^{\text{the carrier of } S}$: $g' \in F\}; i'$ ranges over elements of S: $i' \in D\}$.

(31) Let S, T be complete Scott top-lattices, F be a non empty subset of $[S \to T]$, and D be a directed non empty subset of S. Then $\bigsqcup_T((\bigsqcup_{(T^{\text{the carrier of } S})} F)^\circ D) = (\bigsqcup_{(T^{\text{the carrier of } S})} F)(\sup D)$.

(32) Let S, T be complete Scott top-lattices and F be a non empty subset of $[S \to T]$. Then $\bigsqcup_{(T^{\text{the carrier of } S})} F \in$ the carrier of $([S \to T])$.

(33) Let S be a non empty relational structure and T be a lower-bounded antisymmetric non empty relational structure. Then $\bot_{T^{\text{the carrier of } S}} = S \longmapsto \bot_T$.

(34) Let S be a non empty relational structure and T be an upper-bounded antisymmetric non empty relational structure. Then $\top_{T^{\text{the carrier of } S}} = S \longmapsto \top_T$.

Let S be a non empty reflexive relational structure, let T be a complete lattice, and let a be an element of T. Note that $S \longmapsto a$ is directed-sups-preserving.

One can prove the following proposition

(35) Let S, T be complete Scott top-lattices. Then $[S \to T]$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

Let S, T be complete Scott top-lattices. Observe that $[S \to T]$ is complete.

We now state three propositions:

(36) For all non empty Scott complete top-lattices S, T holds $\bot_{[S \to T]} = S \longmapsto \bot_T$.

(37) For all non empty Scott complete top-lattices S, T holds $\top_{[S \to T]} = S \longmapsto \top_T$.

(38) For all Scott complete top-lattices S, T holds $\mathrm{SCMaps}(S, T) = [S \to T]$.

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Some Properties of Isomorphism between Relational Structures. On the Product of Topological Spaces[1]

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The articles [1], [12], [7], [8], [9], [10], [19], [2], [26], [14], [24], [20], [21], [28], [29], [22], [27], [23], [17], [13], [31], [6], [16], [15], [4], [11], [5], [18], [3], [30], and [25] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) $2^1 = \{0, 1\}$.

(2) For every set X and for every subset Y of X holds $\mathrm{rng}(\mathrm{id}_X \restriction Y) = Y$.

(3) For every function f and for all sets a, b holds $(f + \cdot (a \,\dot{\longmapsto}\, b))(a) = b$.

Let us observe that there exists a relational structure which is strict and empty.

Next we state four propositions:

(4) Let S be an empty 1-sorted structure, T be a 1-sorted structure, and f be a map from S into T. If $\mathrm{rng}\, f = \Omega_T$, then T is empty.

(5) Let S be a 1-sorted structure, T be an empty 1-sorted structure, and f be a map from S into T. If $\mathrm{dom}\, f = \Omega_S$, then S is empty.

(6) Let S be a non empty 1-sorted structure, T be a 1-sorted structure, and f be a map from S into T. If $\mathrm{dom}\, f = \Omega_S$, then T is non empty.

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(7) Let S be a 1-sorted structure, T be a non empty 1-sorted structure, and f be a map from S into T. If $\operatorname{rng} f = \Omega_T$, then S is non empty.

Let S be a non empty reflexive relational structure, let T be a non empty relational structure, and let f be a map from S into T. Let us observe that f is directed-sups-preserving if and only if:

(Def. 1) For every non empty directed subset X of S holds f preserves sup of X.

Let R be a 1-sorted structure and let N be a net structure over R. We say that N is function yielding if and only if:

(Def. 2) The mapping of N is function yielding.

Let us note that there exists a 1-sorted structure which is strict, non empty, and constituted functions.

One can verify that there exists a relational structure which is strict, non empty, and constituted functions.

Let R be a constituted functions 1-sorted structure. One can verify that every net structure over R is function yielding.

Let R be a constituted functions 1-sorted structure. Note that there exists a net structure over R which is strict and function yielding.

Let R be a non empty constituted functions 1-sorted structure. Note that there exists a net structure over R which is strict, non empty, and function yielding.

Let R be a constituted functions 1-sorted structure and let N be a function yielding net structure over R. Observe that the mapping of N is function yielding.

Let R be a non empty constituted functions 1-sorted structure. Note that there exists a net in R which is strict and function yielding.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. Note that rng (the mapping of N) is non empty.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. Observe that $\operatorname{rng} \operatorname{netmap}(N, S)$ is non empty.

One can prove the following two propositions:

(8) Let A, B, C be non empty relational structures, f be a map from B into C, and g, h be maps from A into B. If $g \leqslant h$ and f is monotone, then $f \cdot g \leqslant f \cdot h$.

(9) Let S be a non empty topological space, T be a non empty topological space-like FR-structure, f, g be maps from S into T, and x, y be elements of $[S \to T]$. If $x = f$ and $y = g$, then $x \leqslant y$ iff $f \leqslant g$.

Let I be a set and let R be a non empty relational structure. Note that every element of the carrier of R^I is function-like and relation-like.

Let I be a non empty set, let R be a non empty relational structure, let f be an element of the carrier of R^I, and let i be an element of I. Then $f(i)$ is an element of R.

2. Some Properties of Isomorphism between Relational Structures

One can prove the following proposition

(10) For all relational structures S, T and for every map f from S into T such that f is isomorphic holds f is onto.

Let S, T be relational structures. Note that every map from S into T which is isomorphic is also onto.

We now state four propositions:

(11) Let S, T be non empty relational structures and f be a map from S into T. If f is isomorphic, then f^{-1} is isomorphic.

(12) For all non empty relational structures S, T such that S and T are isomorphic and S has g.l.b.'s holds T has g.l.b.'s.

(13) For all non empty relational structures S, T such that S and T are isomorphic and S has l.u.b.'s holds T has l.u.b.'s.

(14) For every relational structure L such that L is empty holds L is bounded.

Let us note that every relational structure which is empty is also bounded.

The following propositions are true:

(15) Let S, T be relational structures. Suppose S and T are isomorphic and S is lower-bounded. Then T is lower-bounded.

(16) Let S, T be relational structures. Suppose S and T are isomorphic and S is upper-bounded. Then T is upper-bounded.

(17) Let S, T be non empty relational structures, A be a subset of S, and f be a map from S into T. Suppose f is isomorphic and $\sup A$ exists in S. Then $\sup f^\circ A$ exists in T.

(18) Let S, T be non empty relational structures, A be a subset of S, and f be a map from S into T. Suppose f is isomorphic and $\inf A$ exists in S. Then $\inf f^\circ A$ exists in T.

3. On the Product of Topological Spaces

Next we state two propositions:

(19) Let S, T be topological structures. Suppose S and T are homeomorphic or there exists a map f from S into T such that $\operatorname{dom} f = \Omega_S$ and $\operatorname{rng} f = \Omega_T$. Then S is empty if and only if T is empty.

(20) For every non empty topological space T holds T and the topological structure of T are homeomorphic.

Let T be a Scott reflexive non empty FR-structure. One can verify that every subset of T which is open is also inaccessible and upper and every subset of T which is inaccessible and upper is also open.

Next we state several propositions:

(21) Let T be a topological structure, x, y be points of T, and X, Y be subsets of T. If $X = \{x\}$ and $\overline{X} \subseteq \overline{Y}$, then $x \in \overline{Y}$.

(22) Let T be a topological structure, x, y be points of T, and Y, V be subsets of T. If $Y = \{y\}$ and $x \in \overline{Y}$ and V is open and $x \in V$, then $y \in V$.

(23) Let T be a topological structure, x, y be points of T, and X, Y be subsets of T. Suppose $X = \{x\}$ and $Y = \{y\}$. Suppose that for every subset V of T such that V is open holds if $x \in V$, then $y \in V$. Then $\overline{X} \subseteq \overline{Y}$.

(24) Let S, T be non empty topological spaces, A be an irreducible subset of S, and B be a subset of T. Suppose $A = B$ and the topological structure of $S =$ the topological structure of T. Then B is irreducible.

(25) Let S, T be non empty topological spaces, a be a point of S, b be a point of T, A be a subset of the carrier of S, and B be a subset of the carrier of T. Suppose $a = b$ and $A = B$ and the topological structure of $S =$ the topological structure of T and a is dense point of A. Then b is dense point of B.

(26) Let S, T be topological structures, A be a subset of S, and B be a subset of T. Suppose $A = B$ and the topological structure of $S =$ the topological structure of T and A is compact. Then B is compact.

(27) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is sober. Then T is sober.

(28) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is locally-compact. Then T is locally-compact.

(29) Let S, T be topological structures. Suppose the topological structure of $S =$ the topological structure of T and S is compact. Then T is compact.

Let I be a non empty set, let T be a non empty topological space, let x be a point of $\prod(I \longmapsto T)$, and let i be an element of I. Then $x(i)$ is an element of T.

The following propositions are true:

(30) Let M be a non empty set, J be a topological space yielding nonempty many sorted set indexed by M, and x, y be points of $\prod J$. Then $x \in \overline{\{y\}}$ if and only if for every element i of M holds $x(i) \in \overline{\{y(i)\}}$.

(31) Let M be a non empty set, T be a non empty topological space, and x, y be points of $\prod(M \longmapsto T)$. Then $x \in \overline{\{y\}}$ if and only if for every element i of M holds $x(i) \in \overline{\{y(i)\}}$.

(32) Let M be a non empty set, i be an element of M, J be a topological

space yielding nonempty many sorted set indexed by M, and x be a point of $\prod J$. Then $\pi_i\overline{\{x\}} = \overline{\{x(i)\}}$.

(33) Let M be a non empty set, i be an element of M, T be a non empty topological space, and x be a point of $\prod(M \longmapsto T)$. Then $\pi_i\overline{\{x\}} = \overline{\{x(i)\}}$.

(34) Let X, Y be non empty topological structures, f be a map from X into Y, and g be a map from Y into X. Suppose $f = \mathrm{id}_X$ and $g = \mathrm{id}_X$ and f is continuous and g is continuous. Then the topological structure of $X =$ the topological structure of Y.

(35) Let X, Y be non empty topological spaces and f be a map from X into Y. If f° is continuous, then f is continuous.

Let X, Y be non empty topological spaces. Observe that every continuous map from X into Y is continuous.

Let X be a non empty topological space and let Y be a non empty subspace of X. Note that $\overset{Y}{\hookrightarrow}$ is continuous.

The following propositions are true:

(36) For every non empty topological space T and for every map f from T into T such that $f \cdot f = f$ holds $f^\circ \cdot (\overset{\mathrm{Im}\,f}{\hookrightarrow}) = \mathrm{id}_{\mathrm{Im}\,f}$.

(37) For every non empty topological space Y and for every non empty subspace W of Y holds $(\overset{W}{\hookrightarrow})^\circ$ is a homeomorphism.

(38) Let M be a non empty set and J be a topological space yielding nonempty many sorted set indexed by M. Suppose that for every element i of M holds $J(i)$ is a T_0 topological space. Then $\prod J$ is T_0.

Let I be a non empty set and let T be a non empty T_0 topological space. One can check that $\prod(I \longmapsto T)$ is T_0.

The following proposition is true

(39) Let M be a non empty set and J be a topological space yielding nonempty many sorted set indexed by M. Suppose that for every element i of M holds $J(i)$ is T_1 and topological space-like. Then $\prod J$ is a T_1 space.

Let I be a non empty set and let T be a non empty T_1 topological space. Observe that $\prod(I \longmapsto T)$ is T_1.

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Cages - the External Approximation of Jordan's Curve

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Summary. On the Euclidean plane Jordan's curve may be approximated with a polygonal path of sides parallel to coordinate axes, either externally, or internally. The paper deals with the external approximation, and the existence of a *Cage* – an external polygonal path – is proved.

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The papers [17], [25], [8], [18], [9], [2], [3], [23], [4], [22], [14], [16], [21], [6], [5], [11], [1], [19], [7], [13], [12], [15], [24], [20], [10], and [26] provide the terminology and notation for this paper.

1. Generalities

We adopt the following rules: k, n are natural numbers, D is a non empty set, and f, g are finite sequences of elements of D.

One can prove the following propositions:

(1) For all sets A, B such that A meets B holds $A \cap B$ meets B.

(2) For every non empty set A and for all sets B_1, B_2 such that $A \subseteq B_1$ and $A \subseteq B_2$ holds B_1 meets B_2.

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(3) Let T be a non empty topological space and B, C_1, C_2, D be subsets of T. Suppose B is connected and C_1 is a component of D and C_2 is a component of D and B meets C_1 and B meets C_2 and $B \subseteq D$. Then $C_1 = C_2$.

(4) If for every n holds $f{\restriction}n = g{\restriction}n$, then $f = g$.

(5) If $n \in \operatorname{dom} f$, then there exists k such that $k \in \operatorname{dom} \operatorname{Rev}(f)$ and $n + k = \operatorname{len} f + 1$ and $\pi_n f = \pi_k \operatorname{Rev}(f)$.

(6) If $n \in \operatorname{dom} \operatorname{Rev}(f)$, then there exists k such that $k \in \operatorname{dom} f$ and $n + k = \operatorname{len} f + 1$ and $\pi_n \operatorname{Rev}(f) = \pi_k f$.

2. Go-Board Preliminaries

For simplicity, we adopt the following convention: G denotes a Go-board, f, g denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$, r, s denote real numbers, i, j, k denote natural numbers, and x denotes a set.

Next we state a number of propositions:

(7) f is a sequence which elements belong to G iff $\operatorname{Rev}(f)$ is a sequence which elements belong to G.

(8) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k \leqslant \operatorname{len} f$, then $\pi_k f \in \operatorname{Values} G$.

(9) If $n \leqslant \operatorname{len} f$ and $x \in \widetilde{\mathcal{L}}(f_{\lfloor n})$, then there exists a natural number i such that $n + 1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$ and $x \in \mathcal{L}(f, i)$.

(10) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\pi_k f \in \text{left\_cell}(f, k, G)$ and $\pi_k f \in \text{right\_cell}(f, k, G)$.

(11) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\operatorname{Int} \text{left\_cell}(f, k, G) \neq \emptyset$ and $\operatorname{Int} \text{right\_cell}(f, k, G) \neq \emptyset$.

(12) Suppose f is a sequence which elements belong to G and $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$. Then $\operatorname{Int} \text{left\_cell}(f, k, G)$ is connected and $\operatorname{Int} \text{right\_cell}(f, k, G)$ is connected.

(13) If f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\overline{\operatorname{Int} \text{left\_cell}(f, k, G)} = \text{left\_cell}(f, k, G)$ and $\overline{\operatorname{Int} \text{right\_cell}(f, k, G)} = \text{right\_cell}(f, k, G)$.

(14) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f, k)$ is horizontal. Then there exists j such that $1 \leqslant j$ and $j \leqslant \operatorname{width} G$ and for every p such that $p \in \mathcal{L}(f, k)$ holds $p_{\mathbf{2}} = (G_{1,j})_{\mathbf{2}}$.

(15) Suppose f is a sequence which elements belong to G and $\mathcal{L}(f, k)$ is vertical. Then there exists i such that $1 \leqslant i$ and $i \leqslant \operatorname{len} G$ and for every p such that $p \in \mathcal{L}(f, k)$ holds $p_{\mathbf{1}} = (G_{i,1})_{\mathbf{1}}$.

(16) If f is a sequence which elements belong to G and special and $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{Int} \operatorname{cell}(G, i, j)$ misses $\widetilde{\mathcal{L}}(f)$.

(17) Suppose f is a sequence which elements belong to G and special and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$. Then $\operatorname{Int} \operatorname{left\_cell}(f, k, G)$ misses $\widetilde{\mathcal{L}}(f)$ and $\operatorname{Int} \operatorname{right\_cell}(f, k, G)$ misses $\widetilde{\mathcal{L}}(f)$.

(18) Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$. Then $(G_{i,j})_{\mathbf{1}} = (G_{i,j+1})_{\mathbf{1}}$ and $(G_{i,j})_{\mathbf{2}} = (G_{i+1,j})_{\mathbf{2}}$ and $(G_{i+1,j+1})_{\mathbf{1}} = (G_{i+1,j})_{\mathbf{1}}$ and $(G_{i+1,j+1})_{\mathbf{2}} = (G_{i,j+1})_{\mathbf{2}}$.

(19) Let i, j be natural numbers. Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$. Then $p \in \operatorname{cell}(G, i, j)$ if and only if the following conditions are satisfied:

(i) $(G_{i,j})_{\mathbf{1}} \leqslant p_{\mathbf{1}}$,

(ii) $p_{\mathbf{1}} \leqslant (G_{i+1,j})_{\mathbf{1}}$,

(iii) $(G_{i,j})_{\mathbf{2}} \leqslant p_{\mathbf{2}}$, and

(iv) $p_{\mathbf{2}} \leqslant (G_{i,j+1})_{\mathbf{2}}$.

(20) If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j) = \{[r, s] : (G_{i,j})_{\mathbf{1}} \leqslant r \ \wedge \ r \leqslant (G_{i+1,j})_{\mathbf{1}} \ \wedge \ (G_{i,j})_{\mathbf{2}} \leqslant s \ \wedge \ s \leqslant (G_{i,j+1})_{\mathbf{2}}\}$.

(21) Suppose $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G, i, j)$. Then $p = G_{i,j}$ or $p = G_{i,j+1}$ or $p = G_{i+1,j+1}$ or $p = G_{i+1,j}$.

(22) If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$, then $G_{i,j} \in \operatorname{cell}(G, i, j)$ and $G_{i,j+1} \in \operatorname{cell}(G, i, j)$ and $G_{i+1,j+1} \in \operatorname{cell}(G, i, j)$ and $G_{i+1,j} \in \operatorname{cell}(G, i, j)$.

(23) If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$ and $p \in \operatorname{Values} G$ and $p \in \operatorname{cell}(G, i, j)$, then p is extremal in $\operatorname{cell}(G, i, j)$.

(24) Suppose $2 \leqslant \operatorname{len} G$ and $2 \leqslant \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$. Then there exist i, j such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} G$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{width} G$ and $\mathcal{L}(f, k) \subseteq \operatorname{cell}(G, i, j)$.

(25) Suppose $2 \leqslant \operatorname{len} G$ and $2 \leqslant \operatorname{width} G$ and f is a sequence which elements belong to G and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ and $p \in \operatorname{Values} G$ and $p \in \mathcal{L}(f, k)$. Then $p = \pi_k f$ or $p = \pi_{k+1} f$.

(26) If $\langle i, j \rangle \in$ the indices of G and $1 \leqslant k$ and $k \leqslant \operatorname{width} G$, then $(G_{i,j})_{\mathbf{1}} \leqslant (G_{\operatorname{len} G, k})_{\mathbf{1}}$.

(27) If $\langle i, j \rangle \in$ the indices of G and $1 \leqslant k$ and $k \leqslant \operatorname{len} G$, then $(G_{i,j})_{\mathbf{2}} \leqslant (G_{k, \operatorname{width} G})_{\mathbf{2}}$.

(28) Suppose f is a sequence which elements belong to G and special and $\widetilde{\mathcal{L}}(g) \subseteq \widetilde{\mathcal{L}}(f)$ and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$. Let A be a subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A = \operatorname{right\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$ or $A = \operatorname{left\_cell}(f, k, G) \setminus \widetilde{\mathcal{L}}(g)$, then A is

connected.

(29) Let f be a non constant standard special circular sequence. Suppose f is a sequence which elements belong to G. Let given k. If $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$, then $\mathrm{right\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \mathrm{RightComp}(f)$ and $\mathrm{left\_cell}(f,k,G) \setminus \widetilde{\mathcal{L}}(f) \subseteq \mathrm{LeftComp}(f)$.

3. Cages

We follow the rules: C is a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, k, n, i_1, i_2 are natural numbers.

Next we state three propositions:

(30) There exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \mathrm{Gauge}(C,n)$ and $\text{N-min}\, C \in \mathrm{cell}(\mathrm{Gauge}(C,n), i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\mathrm{Gauge}(C,n))_{i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1}$.

(31) Suppose that $1 \leqslant i_1$ and $i_1 + 1 \leqslant \operatorname{len} \mathrm{Gauge}(C,n)$ and $\text{N-min}\, C \in \mathrm{cell}(\mathrm{Gauge}(C,n), i_1, \operatorname{width} \mathrm{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\mathrm{Gauge}(C,n))_{i_1, \operatorname{width} \mathrm{Gauge}(C,n) -' 1}$ and $1 \leqslant i_2$ and $i_2 + 1 \leqslant \operatorname{len} \mathrm{Gauge}(C,n)$ and $\text{N-min}\, C \in \mathrm{cell}(\mathrm{Gauge}(C,n), i_2, \operatorname{width} \mathrm{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\mathrm{Gauge}(C,n))_{i_2, \operatorname{width} \mathrm{Gauge}(C,n) -' 1}$. Then $i_1 = i_2$.

(32) Let f be a standard non constant special circular sequence. Suppose that

(i) f is a sequence which elements belong to $\mathrm{Gauge}(C,n)$,

(ii) for every k such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} f$ holds $\mathrm{left\_cell}(f,k,\mathrm{Gauge}(C,n)) \cap C = \emptyset$ and $\mathrm{right\_cell}(f,k,\mathrm{Gauge}(C,n)) \cap C \neq \emptyset$, and

(iii) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \mathrm{Gauge}(C,n)$ and $\pi_1 f = (\mathrm{Gauge}(C,n))_{i, \operatorname{width} \mathrm{Gauge}(C,n)}$ and $\pi_2 f = (\mathrm{Gauge}(C,n))_{i+1, \operatorname{width} \mathrm{Gauge}(C,n)}$ and $\text{N-min}\, C \in \mathrm{cell}(\mathrm{Gauge}(C,n), i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\mathrm{Gauge}(C,n))_{i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1}$.

Then $\text{N-min}\, \widetilde{\mathcal{L}}(f) = \pi_1 f$.

Let C be a compact non vertical non horizontal non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Let us assume that C is connected. The functor $\mathrm{Cage}(C,n)$ yields a clockwise oriented standard non constant special circular sequence and is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{Cage}(C,n)$ is a sequence which elements belong to $\mathrm{Gauge}(C,n)$,

(ii) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} \mathrm{Gauge}(C,n)$ and $\pi_1 \mathrm{Cage}(C,n) = (\mathrm{Gauge}(C,n))_{i, \operatorname{width} \mathrm{Gauge}(C,n)}$ and $\pi_2 \mathrm{Cage}(C,n) = (\mathrm{Gauge}(C,n))_{i+1, \operatorname{width} \mathrm{Gauge}(C,n)}$ and $\text{N-min}\, C \in \mathrm{cell}(\mathrm{Gauge}(C,n), i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1)$ and $\text{N-min}\, C \neq (\mathrm{Gauge}(C,n))_{i, \operatorname{width} \mathrm{Gauge}(C,n) -' 1}$, and

(iii) for every k such that $1 \leqslant k$ and $k+2 \leqslant \operatorname{len}\mathrm{Cage}(C,n)$ holds if front_left_cell$(\mathrm{Cage}(C,n),k,\mathrm{Gauge}(C,n)) \cap C = \emptyset$ and front_right_cell$(\mathrm{Cage}(C,n),k,\mathrm{Gauge}(C,n)) \cap C = \emptyset$, then $\mathrm{Cage}(C,n)$ turns right k, $\mathrm{Gauge}(C,n)$ and if front_left_cell$(\mathrm{Cage}(C,n),k,\mathrm{Gauge}(C,n)) \cap C = \emptyset$ and front_right_cell$(\mathrm{Cage}(C,n),k,\mathrm{Gauge}(C,n)) \cap C \neq \emptyset$, then $\mathrm{Cage}(C,n)$ goes straight k, $\mathrm{Gauge}(C,n)$ and if front_left_cell$(\mathrm{Cage}(C,n),k,$ $\mathrm{Gauge}(C,n)) \cap C \neq \emptyset$, then $\mathrm{Cage}(C,n)$ turns left k, $\mathrm{Gauge}(C,n)$.

One can prove the following propositions:

(33) If C is connected and $1 \leqslant k$ and $k+1 \leqslant \operatorname{len}\mathrm{Cage}(C,n)$, then left_cell$(\mathrm{Cage}(C,n),k,\mathrm{Gauge}(C,n)) \cap C = \emptyset$ and right_cell$(\mathrm{Cage}(C,n),k,$ $\mathrm{Gauge}(C,n)) \cap C \neq \emptyset$.

(34) If C is connected, then N-min $\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \pi_1\,\mathrm{Cage}(C,n)$.

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Components and Basis of Topological Spaces[1]

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Summary. This article contains many facts about components and basis of topological spaces.

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The notation and terminology used here are introduced in the following papers: [21], [15], [1], [14], [6], [7], [19], [9], [8], [17], [2], [22], [18], [13], [12], [20], [16], [23], [11], [4], [5], [10], and [3].

1. Preliminaries

The scheme *SeqLambda1C* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and and states that:

> There exists a finite sequence p of elements of $\mathcal{B}$ such that $\operatorname{len} p = \mathcal{A}$ and for every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $p(i) = \mathcal{F}(i)$ and if not $\mathcal{P}[i]$, then $p(i) = \mathcal{G}(i)$

provided the following requirement is met:

- For every natural number i such that $i \in \operatorname{Seg} \mathcal{A}$ holds if $\mathcal{P}[i]$, then $\mathcal{F}(i) \in \mathcal{B}$ and if not $\mathcal{P}[i]$, then $\mathcal{G}(i) \in \mathcal{B}$.

Let X be a set and let p be a finite sequence of elements of 2^X. Then $\operatorname{rng} p$ is a family of subsets of X.

Let us observe that *Boolean* is finite.

We now state two propositions:

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(2)[2] For every natural number i and for every finite set D holds D^i is finite.

(3) For every finite set T holds every family of subsets of T is finite.

Let T be a finite set. One can check that every family of subsets of T is finite.

Let T be a finite 1-sorted structure. One can verify that every family of subsets of T is finite.

One can prove the following proposition

(4) For every infinite set X there exist sets x, y such that $x \in X$ and $y \in X$ and $x \neq y$.

2. Components

Let X be a set, let p be a finite sequence of elements of 2^X, and let q be a finite sequence of elements of *Boolean*. The functor MergeSequence(p,q) yielding a finite sequence of elements of 2^X is defined as follows:

(Def. 1) $\operatorname{len} \operatorname{MergeSequence}(p,q) = \operatorname{len} p$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $(\operatorname{MergeSequence}(p,q))(i) = (q(i) = \mathit{true} \rightarrow p(i), X \setminus p(i))$.

One can prove the following propositions:

(5) Let X be a set, p be a finite sequence of elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then $\operatorname{dom} \operatorname{MergeSequence}(p,q) = \operatorname{dom} p$.

(6) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $q(i) = \mathit{true}$, then $(\operatorname{MergeSequence}(p,q))(i) = p(i)$.

(7) Let X be a set, p be a finite sequence of elements of 2^X, q be a finite sequence of elements of *Boolean*, and i be a natural number. If $i \in \operatorname{dom} p$ and $q(i) = \mathit{false}$, then $(\operatorname{MergeSequence}(p,q))(i) = X \setminus p(i)$.

(8) For every set X and for every finite sequence q of elements of *Boolean* holds $\operatorname{len} \operatorname{MergeSequence}(\varepsilon_{2^X}, q) = 0$.

(9) For every set X and for every finite sequence q of elements of *Boolean* holds $\operatorname{MergeSequence}(\varepsilon_{2^X}, q) = \varepsilon_{2^X}$.

(10) For every set X and for every element x of 2^X and for every finite sequence q of elements of *Boolean* holds $\operatorname{len} \operatorname{MergeSequence}(\langle x \rangle, q) = 1$.

(11) Let X be a set, x be an element of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1) = \mathit{true}$, then $(\operatorname{MergeSequence}(\langle x \rangle, q))(1) = x$, and

(ii) if $q(1) = \mathit{false}$, then $(\operatorname{MergeSequence}(\langle x \rangle, q))(1) = X \setminus x$.

[2]The proposition (1) has been removed.

(12) For every set X and for all elements x, y of 2^X and for every finite sequence q of elements of *Boolean* holds $\operatorname{len}\operatorname{MergeSequence}(\langle x,y\rangle,q)=2$.

(13) Let X be a set, x, y be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1)=\mathit{true}$, then $(\operatorname{MergeSequence}(\langle x,y\rangle,q))(1)=x$,
(ii) if $q(1)=\mathit{false}$, then $(\operatorname{MergeSequence}(\langle x,y\rangle,q))(1)=X\setminus x$,
(iii) if $q(2)=\mathit{true}$, then $(\operatorname{MergeSequence}(\langle x,y\rangle,q))(2)=y$, and
(iv) if $q(2)=\mathit{false}$, then $(\operatorname{MergeSequence}(\langle x,y\rangle,q))(2)=X\setminus y$.

(14) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then $\operatorname{len}\operatorname{MergeSequence}(\langle x,y,z\rangle,q)=3$.

(15) Let X be a set, x, y, z be elements of 2^X, and q be a finite sequence of elements of *Boolean*. Then

(i) if $q(1)=\mathit{true}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(1)=x$,
(ii) if $q(1)=\mathit{false}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(1)=X\setminus x$,
(iii) if $q(2)=\mathit{true}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(2)=y$,
(iv) if $q(2)=\mathit{false}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(2)=X\setminus y$,
(v) if $q(3)=\mathit{true}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(3)=z$, and
(vi) if $q(3)=\mathit{false}$, then $(\operatorname{MergeSequence}(\langle x,y,z\rangle,q))(3)=X\setminus z$.

(16) Let X be a set and p be a finite sequence of elements of 2^X. Then $\{\operatorname{Intersect}(\operatorname{rng}\operatorname{MergeSequence}(p,q)); q$ ranges over finite sequences of elements of *Boolean*: $\operatorname{len} q=\operatorname{len} p\}$ is a family of subsets of X.

Let X be a set and let Y be a finite family of subsets of X. The functor $\operatorname{Components} Y$ yields a family of subsets of X and is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence p of elements of 2^X such that $\operatorname{len} p=\operatorname{card} Y$ and $\operatorname{rng} p=Y$ and $\operatorname{Components} Y=\{\operatorname{Intersect}(\operatorname{rng}\operatorname{MergeSequence}(p,q)); q$ ranges over finite sequences of elements of *Boolean*: $\operatorname{len} q=\operatorname{len} p\}$.

Let X be a set and let Y be a finite family of subsets of X. Note that $\operatorname{Components} Y$ is finite.

One can prove the following four propositions:

(17) For every set X and for every empty family Y of subsets of X holds $\operatorname{Components} Y=\{X\}$.

(18) For every set X and for all finite families Y, Z of subsets of X such that $Z\subseteq Y$ holds $\operatorname{Components} Y$ is finer than $\operatorname{Components} Z$.

(19) For every set X and for every finite family Y of subsets of X holds $\bigcup\operatorname{Components} Y=X$.

(20) Let X be a set, Y be a finite family of subsets of X, and A, B be sets. If $A\in\operatorname{Components} Y$ and $B\in\operatorname{Components} Y$ and $A\neq B$, then $A\cap B=\emptyset$.

Let X be a set and let Y be a finite family of subsets of X. We say that Y is in general position if and only if:

(Def. 3) $\emptyset \notin \operatorname{Components} Y$.

We now state three propositions:

(21) Let X be a set and Y, Z be finite families of subsets of X. If Z is in general position and $Y \subseteq Z$, then Y is in general position.

(22) For every non empty set X holds every empty family of subsets of X is in general position.

(23) Let X be a non empty set and Y be a finite family of subsets of X. If Y is in general position, then $\operatorname{Components} Y$ is a partition of X.

3. About Basis of Topological Spaces

We now state two propositions:

(24) For every non empty relational structure L holds Ω_L is infs-closed and sups-closed.

(25) For every non empty relational structure L holds Ω_L has bottom and top.

Let L be a non empty relational structure. Observe that Ω_L is infs-closed and sups-closed and has bottom and top.

The following propositions are true:

(26) For every continuous sup-semilattice L holds Ω_L is a CLbasis of L.

(27) For every up-complete non empty poset L such that L is finite holds the carrier of $L =$ the carrier of $\operatorname{CompactSublatt}(L)$.

(28) For every lower-bounded sup-semilattice L and for every subset B of L such that B is infinite holds $\overline{\overline{B}} = \overline{\overline{\operatorname{finsups}(B)}}$.

(29) For every T_0 non empty topological space T holds $\overline{\overline{\text{the carrier of } T}} \subseteq \overline{\overline{\text{the topology of } T}}$.

(30) Let T be a topological structure and X be a subset of T. Suppose X is open. Let B be a finite family of subsets of T. Suppose B is a basis of T. Let Y be a set. If $Y \in \operatorname{Components} B$, then $X \cap Y = \emptyset$ or $Y \subseteq X$.

(31) For every T_0 topological space T such that T is infinite holds every basis of T is infinite.

(32) Let T be a non empty topological space. Suppose T is finite. Let B be a basis of T and x be an element of T. Then $\bigcap\{A; A$ ranges over elements of the topology of T: $x \in A\} \in B$.

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Properties of the External Approximation of Jordan's Curve

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The articles [20], [6], [14], [7], [2], [18], [17], [13], [3], [5], [10], [1], [11], [15], [4], [9], [12], [19], [16], and [8] provide the terminology and notation for this paper.

One can verify that there exists a subset of $\mathcal{E}^2_{\rm T}$ which is connected, compact, non vertical, and non horizontal.

We adopt the following rules: i, j, k, n are natural numbers, P is a subset of $\mathcal{E}^2_{\rm T}$, and C is a connected compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(1) Suppose that
- (i) $1 \leqslant k$,
- (ii) $k+1 \leqslant \operatorname{len} \operatorname{Cage}(C,n)$,
- (iii) $\langle i,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C,n)$,
- (iv) $\langle i,\, j+1\rangle \in$ the indices of $\operatorname{Gauge}(C,n)$,
- (v) $\pi_k \operatorname{Cage}(C,n) = (\operatorname{Gauge}(C,n))_{i,j}$, and
- (vi) $\pi_{k+1} \operatorname{Cage}(C,n) = (\operatorname{Gauge}(C,n))_{i,j+1}$.

Then $i < \operatorname{len} \operatorname{Gauge}(C,n)$.

(2) Suppose that
- (i) $1 \leqslant k$,
- (ii) $k+1 \leqslant \operatorname{len} \operatorname{Cage}(C,n)$,
- (iii) $\langle i,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C,n)$,
- (iv) $\langle i,\, j+1\rangle \in$ the indices of $\operatorname{Gauge}(C,n)$,
- (v) $\pi_k \operatorname{Cage}(C,n) = (\operatorname{Gauge}(C,n))_{i,j+1}$, and
- (vi) $\pi_{k+1} \operatorname{Cage}(C,n) = (\operatorname{Gauge}(C,n))_{i,j}$.

Then $i > 1$.

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(3) Suppose that
(i) $1 \leqslant k$,
(ii) $k+1 \leqslant \operatorname{len} \operatorname{Cage}(C, n)$,
(iii) $\langle i,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C, n)$,
(iv) $\langle i+1,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C, n)$,
(v) $\pi_k \operatorname{Cage}(C, n) = (\operatorname{Gauge}(C, n))_{i,j}$, and
(vi) $\pi_{k+1} \operatorname{Cage}(C, n) = (\operatorname{Gauge}(C, n))_{i+1,j}$.
Then $j > 1$.

(4) Suppose that
(i) $1 \leqslant k$,
(ii) $k+1 \leqslant \operatorname{len} \operatorname{Cage}(C, n)$,
(iii) $\langle i,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C, n)$,
(iv) $\langle i+1,\, j\rangle \in$ the indices of $\operatorname{Gauge}(C, n)$,
(v) $\pi_k \operatorname{Cage}(C, n) = (\operatorname{Gauge}(C, n))_{i+1,j}$, and
(vi) $\pi_{k+1} \operatorname{Cage}(C, n) = (\operatorname{Gauge}(C, n))_{i,j}$.
Then $j < \operatorname{width} \operatorname{Gauge}(C, n)$.

(5) $C \cap \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = \emptyset$.

(6) $\text{N-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) = \text{N-bound}\, C + \frac{\text{N-bound}\, C - \text{S-bound}\, C}{2^n}$.

(7) If $i < j$, then $\text{N-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, j)) < \text{N-bound}\, \widetilde{\mathcal{L}}(\operatorname{Cage}(C, i))$.

Let C be a connected compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Note that $\overline{\operatorname{RightComp}(\operatorname{Cage}(C, n))}$ is compact.

The following propositions are true:

(8) $\text{N-min}\, C \in \operatorname{RightComp}(\operatorname{Cage}(C, n))$.

(9) $C \cap \operatorname{RightComp}(\operatorname{Cage}(C, n)) \neq \emptyset$.

(10) $C \cap \operatorname{LeftComp}(\operatorname{Cage}(C, n)) = \emptyset$.

(11) $C \subseteq \operatorname{RightComp}(\operatorname{Cage}(C, n))$.

(12) $C \subseteq \operatorname{BDD} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$.

(13) $\operatorname{UBD} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) \subseteq \operatorname{UBD} C$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor UBD-Family C is defined as follows:

(Def. 1) UBD-Family $C = \{\operatorname{UBD} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) : n$ ranges over natural numbers$\}$.

The functor BDD-Family C is defined by:

(Def. 2) BDD-Family $C = \{\operatorname{BDD} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n)) : n$ ranges over natural numbers$\}$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$. Then UBD-Family C is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$. Then BDD-Family C is a family of subsets of $\mathcal{E}^2_{\mathrm{T}}$.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$. Note that UBD-Family C is non empty and BDD-Family C is non empty.

One can prove the following propositions:

(14) $\bigcup$ UBD-Family C = UBD C.

(15) $C \subseteq \bigcap$ BDD-Family C.

(16) BDD $C \cap$ LeftComp(Cage(C, n)) $= \emptyset$.

(17) BDD $C \subseteq$ RightComp(Cage(C, n)).

(18) If P is inside component of C, then $P \cap \widetilde{\mathcal{L}}$(Cage($C, n$)) $= \emptyset$.

(19) BDD $C \cap \widetilde{\mathcal{L}}$(Cage($C, n$)) $= \emptyset$.

(20) $\bigcap$ BDD-Family $C = C \cup$ BDD C.

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Irrationality of e

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Summary. We prove the irrationality of square roots of prime numbers and of the number e. In order to be able to prove the last, a proof is given that `number_e = exp(1)` as defined in the Mizar library, that is that

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{k=0}^{\infty}\frac{1}{k!}$$

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The articles [2], [3], [4], [18], [14], [1], [6], [13], [15], [8], [7], [20], [12], [5], [10], [11], [9], [16], [21], [17], and [19] provide the notation and terminology for this paper.

1. Square Roots of Primes are Irrational

For simplicity, we follow the rules: k, n, p, K, N are natural numbers, x, y, e_1 are real numbers, s_1, s_2, s_3 are sequences of real numbers, and s_4 is a finite sequence of elements of $\mathbb{R}$.

Let us consider x. We introduce x is irrational as an antonym of x is rational.

Let us consider x, y. We introduce x^y as a synonym of x^y.

One can prove the following two propositions:

(1) If p is prime, then $\sqrt{p}$ is irrational.

(2) There exist x, y such that x is irrational and y is irrational and x^y is rational.

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2. A proof that $e = e$

The scheme *LambdaRealSeq* deals with a unary functor $\mathcal{F}$ yielding a real number, and states that:

> There exists s_1 such that for every n holds $s_1(n) = \mathcal{F}(n)$ and for all s_2, s_3 such that for every n holds $s_2(n) = \mathcal{F}(n)$ and for every n holds $s_3(n) = \mathcal{F}(n)$ holds $s_2 = s_3$

for all values of the parameter.

Let us consider k. The functor $\mathbf{a}_k$ is a sequence of real numbers and is defined by:

(Def. 1) For every n holds $\mathbf{a}_k(n) = \frac{n-k}{n}$.

Let us consider k. The functor $\mathbf{b}_k$ is a sequence of real numbers and is defined by:

(Def. 2) For every n holds $\mathbf{b}_k(n) = \binom{n}{k} \cdot n^{-k}$.

Let us consider n. The functor $\mathbf{c}_n$ is a sequence of real numbers and is defined as follows:

(Def. 3) For every k holds $\mathbf{c}_n(k) = \binom{n}{k} \cdot n^{-k}$.

Next we state the proposition

(3) $\mathbf{c}_n(k) = \mathbf{b}_k(n)$.

The sequence $\mathbf{d}$ of real numbers is defined as follows:

(Def. 4) For every n holds $\mathbf{d}(n) = (1 + \frac{1}{n})^n$.

The sequence $\mathbf{e}$ of real numbers is defined as follows:

(Def. 5) For every k holds $\mathbf{e}(k) = \frac{1}{k!}$.

We now state a number of propositions:

(4) If $n > 0$, then $n^{-(k+1)} = \frac{n^{-k}}{n}$.

(5) For all real numbers x, y, z, v, w such that $y \neq 0$ and $z \neq 0$ and $v \neq 0$ and $w \neq 0$ holds $\frac{x}{y \cdot z \cdot \frac{v}{w}} = \frac{w}{z} \cdot \frac{x}{y \cdot v}$.

(6) $\binom{n}{k+1} = \frac{n-k}{k+1} \cdot \binom{n}{k}$.

(7) If $n > 0$, then $\mathbf{b}_{k+1}(n) = \frac{1}{k+1} \cdot \mathbf{b}_k(n) \cdot \mathbf{a}_k(n)$.

(8) If $n > 0$, then $\mathbf{a}_k(n) = 1 - \frac{k}{n}$.

(9) $\mathbf{a}_k$ is convergent and $\lim(\mathbf{a}_k) = 1$.

(10) For every s_1 such that for every n holds $s_1(n) = x$ holds s_1 is convergent and $\lim s_1 = x$.

(11) For every n such that $n > 0$ holds $\mathbf{b}_0(n) = 1$.

(12) $\frac{1}{k+1} \cdot \frac{1}{k!} = \frac{1}{(k+1)!}$.

(13) $\mathbf{b}_k$ is convergent and $\lim(\mathbf{b}_k) = \frac{1}{k!}$ and $\lim(\mathbf{b}_k) = \mathbf{e}(k)$.

(14) If $k < n$, then $0 < \mathbf{a}_k(n)$ and $\mathbf{a}_k(n) \leqslant 1$.

(15) If $n > 0$, then $0 \leqslant \mathbf{b}_k(n)$ and $\mathbf{b}_k(n) \leqslant \frac{1}{k!}$ and $\mathbf{b}_k(n) \leqslant \mathbf{e}(k)$ and $0 \leqslant \mathbf{c}_n(k)$ and $\mathbf{c}_n(k) \leqslant \frac{1}{k!}$ and $\mathbf{c}_n(k) \leqslant \mathbf{e}(k)$.

(16) For every s_1 such that $s_1 \uparrow 1$ is summable holds s_1 is summable and $\sum s_1 = s_1(0) + \sum(s_1 \uparrow 1)$.

(17) For every s_4 such that $\operatorname{len} s_4 = n$ and $1 \leqslant k$ and $k < n$ holds $(s_4)_{\lfloor 1}(k) = s_4(k+1)$.

(18) For every s_4 such that $\operatorname{len} s_4 > 0$ holds $\sum s_4 = s_4(1) + \sum((s_4)_{\lfloor 1})$.

(19) Let given n and given s_1, s_4. Suppose $\operatorname{len} s_4 = n$ and for every k such that $k < n$ holds $s_1(k) = s_4(k+1)$ and for every k such that $k \geqslant n$ holds $s_1(k) = 0$. Then s_1 is summable and $\sum s_1 = \sum s_4$.

(20) If $x \neq 0$ and $y \neq 0$ and $k \leqslant n$, then $\langle \binom{n}{0} x^0 y^n, \ldots, \binom{n}{n} x^n y^0 \rangle (k+1) = \binom{n}{k} \cdot x^{n-k} \cdot y^k$.

(21) If $n > 0$ and $k \leqslant n$, then $\mathbf{c}_n(k) = \langle \binom{n}{0} 1^0 (\frac{1}{n})^n, \ldots, \binom{n}{n} 1^n (\frac{1}{n})^0 \rangle (k+1)$.

(22) If $n > 0$, then $\mathbf{c}_n$ is summable and $\sum(\mathbf{c}_n) = (1 + \frac{1}{n})^n$ and $\sum(\mathbf{c}_n) = \mathbf{d}(n)$.

(23) $\mathbf{d}$ is convergent and $\lim \mathbf{d} = e$.

(24) $\mathbf{e}$ is summable and $\sum \mathbf{e} = \exp 1$.

(25) Let given K and d_1 be a sequence of real numbers. If for every n holds $d_1(n) = (\sum_{\alpha=0}^{\kappa} (\mathbf{c}_n)(\alpha))_{\kappa \in \mathbb{N}}(K)$, then d_1 is convergent and $\lim d_1 = (\sum_{\alpha=0}^{\kappa} \mathbf{e}(\alpha))_{\kappa \in \mathbb{N}}(K)$.

(26) If s_1 is convergent and $\lim s_1 = x$, then for every e_1 such that $e_1 > 0$ there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) > x - e_1$.

(27) Suppose that

(i) for every e_1 such that $e_1 > 0$ there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) > x - e_1$, and

(ii) there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) \leqslant x$.

Then s_1 is convergent and $\lim s_1 = x$.

(28) If s_1 is summable, then for every e_1 such that $e_1 > 0$ there exists K such that $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(K) > \sum s_1 - e_1$.

(29) If $n \geqslant 1$, then $\mathbf{d}(n) \leqslant \sum \mathbf{e}$.

(30) If s_1 is summable and for every k holds $s_1(k) \geqslant 0$, then $\sum s_1 \geqslant (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(K)$.

(31) $\mathbf{d}$ is convergent and $\lim \mathbf{d} = \sum \mathbf{e}$.

e can be characterized by the condition:

(Def. 6) $e = \sum \mathbf{e}$.

e can be characterized by the condition:

(Def. 7) $e = \exp 1$.

3. The Number e is Irrational

We now state a number of propositions:

(32) If x is rational, then there exists n such that $n \geqslant 2$ and $n! \cdot x$ is integer.

(33) $n! \cdot \mathbf{e}(k) = \frac{n!}{k!}$.

(34) $\frac{n!}{k!} > 0$.

(35) If s_1 is summable and for every n holds $s_1(n) > 0$, then $\sum s_1 > 0$.

(36) $n! \cdot \sum(\mathbf{e} \uparrow (n+1)) > 0$.

(37) If $k \leqslant n$, then $\frac{n!}{k!}$ is integer.

(38) $n! \cdot (\sum_{\alpha=0}^{\kappa} \mathbf{e}(\alpha))_{\kappa \in \mathbb{N}}(n)$ is integer.

(39) If $x = \frac{1}{n+1}$, then $\frac{n!}{(n+k+1)!} \leqslant x^{k+1}$.

(40) If $n > 0$ and $x = \frac{1}{n+1}$, then $n! \cdot \sum(\mathbf{e} \uparrow (n+1)) \leqslant \frac{x}{1-x}$.

(41) If $n \geqslant 2$ and $x = \frac{1}{n+1}$, then $\frac{x}{1-x} < 1$.

(42) e is irrational.

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Injective Spaces. Part II[1]

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The notation and terminology used in this paper are introduced in the following articles: [11], [8], [6], [1], [19], [23], [10], [17], [18], [24], [9], [26], [22], [14], [12], [3], [7], [15], [4], [16], [2], [13], [25], [21], [20], and [5].

1. Injective Spaces

The following propositions are true:

(1) For every point p of the Sierpiński space such that $p = 0$ holds $\{p\}$ is closed.

(2) For every point p of the Sierpiński space such that $p = 1$ holds $\{p\}$ is non closed.

Let us note that the Sierpiński space is non T_1.

One can check that every top-lattice which is complete and Scott is also discernible.

Let us observe that there exists a T_0-space which is injective and strict.

Let us observe that there exists a top-lattice which is complete, Scott, and strict.

Next we state several propositions:

(3) Let I be a non empty set and T be a Scott topological augmentation of $\prod(I \longmapsto 2^1_\subseteq)$. Then the carrier of T = the carrier of $\prod(I \longmapsto$ the Sierpiński space).

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(4) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, T_2 be a Scott topological augmentation of L_2, h be a map from L_1 into L_2, and H be a map from T_1 into T_2. If $h = H$ and h is isomorphic, then H is a homeomorphism.

(5) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic, then T_1 and T_2 are homeomorphic.

(6) Let S, T be non empty topological spaces. If S is injective and S and T are homeomorphic, then T is injective.

(7) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic and T_1 is injective, then T_2 is injective.

Let X, Y be non empty topological spaces. Let us observe that X is a topological retract of Y if and only if:

(Def. 1) There exists a continuous map c from X into Y and there exists a continuous map r from Y into X such that $r \cdot c = \mathrm{id}_X$.

One can prove the following propositions:

(8) Let S, T, X, Y be non empty topological spaces. Suppose that
(i) the topological structure of $S =$ the topological structure of T,
(ii) the topological structure of $X =$ the topological structure of Y, and
(iii) S is a topological retract of X.
Then T is a topological retract of Y.

(9) Let T, S_1, S_2 be non empty topological structures. Suppose S_1 and S_2 are homeomorphic and S_1 is a topological retract of T. Then S_2 is a topological retract of T.

(10) Let S, T be non empty topological spaces. Suppose T is injective and S is a topological retract of T. Then S is injective.

(11) Let J be an injective non empty topological space and Y be a non empty topological space. If J is a subspace of Y, then J is a topological retract of Y.

(12) For every complete continuous lattice L holds every Scott topological augmentation of L is injective.

Let L be a complete continuous lattice. Observe that every topological augmentation of L which is Scott is also injective.

Let T be an injective non empty topological space. Note that the topological structure of T is injective.

2. Specialization Order

Let T be a topological structure. The functor ΩT yielding a strict FR-structure is defined by the conditions (Def. 2).

(Def. 2)(i) The topological structure of $\Omega T =$ the topological structure of T, and

(ii) for all elements x, y of ΩT holds $x \leqslant y$ iff there exists a subset Y of T such that $Y = \{y\}$ and $x \in \overline{Y}$.

Let T be an empty topological structure. Observe that ΩT is empty.

Let T be a non empty topological structure. Note that ΩT is non empty.

Let T be a topological space. Note that ΩT is topological space-like.

Let T be a topological structure. One can verify that ΩT is reflexive.

Let T be a topological structure. One can verify that ΩT is transitive.

Let T be a T_0-space. One can verify that ΩT is antisymmetric.

One can prove the following propositions:

(13) Let S, T be topological spaces. Suppose the topological structure of $S =$ the topological structure of T. Then $\Omega S = \Omega T$.

(14) Let M be a non empty set and T be a non empty topological space. Then the relational structure of $\Omega \prod(M \longmapsto T) =$ the relational structure of $\prod(M \longmapsto \Omega T)$.

(15) For every Scott complete top-lattice S holds $\Omega S =$ the FR-structure of S.

(16) Let L be a complete lattice and S be a Scott topological augmentation of L. Then the relational structure of $\Omega S =$ the relational structure of L.

Let S be a Scott complete top-lattice. Note that ΩS is complete.

We now state four propositions:

(17) Let T be a non empty topological space and S be a non empty subspace of T. Then ΩS is a full relational substructure of ΩT.

(18) Let S, T be topological spaces, h be a map from S into T, and g be a map from ΩS into ΩT. If $h = g$ and h is a homeomorphism, then g is isomorphic.

(19) For all topological spaces S, T such that S and T are homeomorphic holds ΩS and ΩT are isomorphic.

(20) For every injective T_0-space T holds ΩT is a complete continuous lattice.

Let T be an injective T_0-space. One can verify that ΩT is complete and continuous.

We now state the proposition

(21) For all non empty topological spaces X, Y holds every continuous map from ΩX into ΩY is monotone.

Let X, Y be non empty topological spaces. Note that every map from ΩX into ΩY which is continuous is also monotone.

Next we state the proposition

(22) For every non empty topological space T and for every element x of the carrier of ΩT holds $\overline{\{x\}} = {\downarrow}x$.

Let T be a non empty topological space and let x be an element of the carrier of ΩT. One can verify that $\overline{\{x\}}$ is non empty lower and directed and ${\downarrow}x$ is closed.

Next we state the proposition

(23) For every topological space X holds every open subset of ΩX is upper.

Let T be a topological space. One can verify that every subset of ΩT which is open is also upper.

Let I be a non empty set, let S, T be non empty relational structures, let N be a net in T, and let i be an element of I. Let us assume that the carrier of $T \subseteq$ the carrier of S^I. The functor commute(N, i, S) yielding a strict net in S is defined by the conditions (Def. 3).

(Def. 3)(i) The relational structure of commute(N, i, S) = the relational structure of N, and

(ii) the mapping of commute(N, i, S) = (commute(the mapping of N))(i).

Next we state two propositions:

(24) Let X, Y be non empty topological spaces, N be a net in $[X \to \Omega Y]$, i be an element of the carrier of N, and x be a point of X. Then (the mapping of commute$(N, x, \Omega Y)$)(i) = (the mapping of N)$(i)(x)$.

(25) Let X, Y be non empty topological spaces, N be an eventually-directed net in $[X \to \Omega Y]$, and x be a point of X. Then commute$(N, x, \Omega Y)$ is eventually-directed.

Let X, Y be non empty topological spaces, let N be an eventually-directed net in $[X \to \Omega Y]$, and let x be a point of X. One can verify that commute$(N, x, \Omega Y)$ is eventually-directed.

Let X, Y be non empty topological spaces. Observe that every net in $[X \to \Omega Y]$ is function yielding.

Next we state the proposition

(26) Let X be a non empty topological space, Y be a T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds sup commute$(N, x, \Omega Y)$ exists. Then sup rng (the mapping of N) exists in $(\Omega Y)^{\text{the carrier of } X}$.

3. Monotone Convergence Topological Spaces

Let T be a non empty topological space. We say that T is monotone convergence if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let D be a non empty directed subset of ΩT. Then $\sup D$ exists in ΩT and for every open subset V of T such that $\sup D \in V$ holds D meets V.

One can prove the following proposition

(27) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is monotone convergence. Then T is monotone convergence.

Let us observe that every T_0-space which is trivial is also monotone convergence.

Let us observe that there exists a topological space which is strict, trivial, and non empty.

One can prove the following two propositions:

(28) Let S be a monotone convergence T_0-space and T be a T_0-space. If S and T are homeomorphic, then T is monotone convergence.

(29) Every Scott complete top-lattice is monotone convergence.

Let L be a complete lattice. One can check that every Scott topological augmentation of L is monotone convergence.

Let L be a complete lattice and let S be a Scott topological augmentation of L. One can check that the topological structure of S is monotone convergence.

We now state the proposition

(30) For every monotone convergence T_0-space X holds ΩX is up-complete.

Let X be a monotone convergence T_0-space. Observe that ΩX is up-complete.

One can prove the following three propositions:

(31) Let X be a monotone convergence non empty topological space and N be an eventually-directed prenet over ΩX. Then $\sup N$ exists.

(32) Let X be a monotone convergence non empty topological space and N be an eventually-directed net in ΩX. Then $\sup N \in \operatorname{Lim} N$.

(33) For every monotone convergence non empty topological space X holds every eventually-directed net in ΩX is convergent.

Let X be a monotone convergence non empty topological space. Observe that every eventually-directed net in ΩX is convergent.

We now state two propositions:

(34) Let X be a non empty topological space. Suppose that for every eventually-directed net N in ΩX holds $\sup N$ exists and $\sup N \in \operatorname{Lim} N$. Then X is monotone convergence.

(35) Let X be a monotone convergence non empty topological space and Y be a T_0-space. Then every continuous map from ΩX into ΩY is directed-sups-preserving.

Let X be a monotone convergence non empty topological space and let Y be a T_0-space. One can check that every map from ΩX into ΩY which is continuous is also directed-sups-preserving.

Next we state four propositions:

(36) Let T be a monotone convergence T_0-space and R be a T_0-space. If R is a topological retract of T, then R is monotone convergence.

(37) Let T be an injective T_0-space and S be a Scott topological augmentation of ΩT. Then the topological structure of $S =$ the topological structure of T.

(38) Every injective T_0-space is compact, locally-compact, and sober.

(39) Every injective T_0-space is monotone convergence.

One can verify that every T_0-space which is injective is also monotone convergence.

One can prove the following propositions:

(40) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, and f, g be maps from X into ΩY. Suppose that

(i) $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \operatorname{rng}(\text{the mapping of } N)$,

(ii) $\sup \operatorname{rng}(\text{the mapping of } N)$ exists in $(\Omega Y)^{\text{the carrier of } X}$, and

(iii) $g \in \operatorname{rng}(\text{the mapping of } N)$.

Then $g \leqslant f$.

(41) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, x be a point of X, and f be a map from X into ΩY. Suppose for every point a of X holds $\operatorname{commute}(N, a, \Omega Y)$ is eventually-directed and $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \operatorname{rng}(\text{the mapping of } N)$. Then $f(x) = \sup \operatorname{commute}(N, x, \Omega Y)$.

(42) Let X be a non empty topological space, Y be a monotone convergence T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds $\operatorname{commute}(N, x, \Omega Y)$ is eventually-directed. Then $\bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \operatorname{rng}(\text{the mapping of } N)$ is a continuous map from X into Y.

(43) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \to \Omega Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

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Propositional Calculus for Boolean Valued Functions. Part VI

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Summary. In this paper, we proved some elementary propositional calculus formulae for Boolean valued functions.

MML Identifier: `BVFUNC10`.

The articles [1] and [2] provide the notation and terminology for this paper.

In this paper Y is a non empty set.

The following propositions are true:

(1) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $a \wedge b \vee b \wedge c \vee c \wedge a = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

(2) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $a \wedge \neg b \vee b \wedge \neg c \vee c \wedge \neg a = b \wedge \neg a \vee c \wedge \neg b \vee a \wedge \neg c$.

(3) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a) = (b \vee \neg a) \wedge (c \vee \neg b) \wedge (a \vee \neg c)$.

(4) For all elements a, b, c of $\mathrm{BVF}(Y)$ such that $c \Rightarrow a = \mathit{true}(Y)$ and $c \Rightarrow b = \mathit{true}(Y)$ holds $c \Rightarrow a \vee b = \mathit{true}(Y)$.

(5) For all elements a, b, c of $\mathrm{BVF}(Y)$ such that $a \Rightarrow c = \mathit{true}(Y)$ and $b \Rightarrow c = \mathit{true}(Y)$ holds $a \wedge b \Rightarrow c = \mathit{true}(Y)$.

(6) For all elements a_1, a_2, b_1, b_2, c_1, c_2 of $\mathrm{BVF}(Y)$ holds $(a_1 \Rightarrow a_2) \wedge (b_1 \Rightarrow b_2) \wedge (c_1 \Rightarrow c_2) \wedge (a_1 \vee b_1 \vee c_1) \Subset a_2 \vee b_2 \vee c_2$.

(7) For all elements a_1, a_2, b_1, b_2 of $\mathrm{BVF}(Y)$ holds $(a_1 \Rightarrow b_1) \wedge (a_2 \Rightarrow b_2) \wedge (a_1 \vee a_2) \wedge \neg(b_1 \wedge b_2) = (b_1 \Rightarrow a_1) \wedge (b_2 \Rightarrow a_2) \wedge (b_1 \vee b_2) \wedge \neg(a_1 \wedge a_2)$.

(8) For all elements a, b, c, d of $\mathrm{BVF}(Y)$ holds $(a \vee b) \wedge (c \vee d) = a \wedge c \vee a \wedge d \vee b \wedge c \vee b \wedge d$.

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(9) For all elements a_1, a_2, b_1, b_2, b_3 of $\mathrm{BVF}(Y)$ holds $a_1 \wedge a_2 \vee b_1 \wedge b_2 \wedge b_3 = (a_1 \vee b_1) \wedge (a_1 \vee b_2) \wedge (a_1 \vee b_3) \wedge (a_2 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_2 \vee b_3)$.

(10) For all elements a, b, c, d of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \wedge (c \Rightarrow d) = (a \Rightarrow b \wedge c \wedge d) \wedge (b \Rightarrow c \wedge d) \wedge (c \Rightarrow d)$.

(11) For all elements a, b, c, d of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow c) \wedge (b \Rightarrow d) \wedge (a \vee b) \Subset c \vee d$.

(12) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \wedge b \Rightarrow \neg c) \wedge a \wedge c \Subset \neg b$.

(13) For all elements a_1, a_2, a_3, b_1, b_2, b_3 of $\mathrm{BVF}(Y)$ holds $a_1 \wedge a_2 \wedge a_3 \Rightarrow b_1 \vee b_2 \vee b_3 = \neg b_1 \wedge \neg b_2 \wedge a_3 \Rightarrow \neg a_1 \vee \neg a_2 \vee b_3$.

(14) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \wedge (c \Rightarrow a) = a \wedge b \wedge c \vee \neg a \wedge \neg b \wedge \neg c$.

(15) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \wedge (c \Rightarrow a) \wedge (a \vee b \vee c) = a \wedge b \wedge c$.

(16) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \vee b) \wedge (b \vee c) \wedge (c \vee a) \wedge \neg(a \wedge b \wedge c) = \neg a \wedge b \wedge c \vee a \wedge \neg b \wedge c \vee a \wedge b \wedge \neg c$.

(17) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset a \Rightarrow b \wedge c$.

(18) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset a \vee b \Rightarrow c$.

(19) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset a \Rightarrow b \vee c$.

(20) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset a \Rightarrow b \vee \neg c$.

(21) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset b \Rightarrow c \vee a$.

(22) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset b \Rightarrow c \vee \neg a$.

(23) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset (a \Rightarrow b) \wedge (b \Rightarrow c \vee a)$.

(24) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset (a \Rightarrow b \vee \neg c) \wedge (b \Rightarrow c)$.

(25) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset (a \Rightarrow b \vee c) \wedge (b \Rightarrow c \vee a)$.

(26) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset (a \Rightarrow b \vee \neg c) \wedge (b \Rightarrow c \vee a)$.

(27) For all elements a, b, c of $\mathrm{BVF}(Y)$ holds $(a \Rightarrow b) \wedge (b \Rightarrow c) \Subset (a \Rightarrow b \vee \neg c) \wedge (b \Rightarrow c \vee \neg a)$.

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Predicate Calculus for Boolean Valued Functions. Part III

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: BVFUNC11.

The papers [8], [1], [3], [5], [2], [4], [7], and [6] provide the notation and terminology for this paper.

1. Preliminaries

In this paper Y is a non empty set.

We now state several propositions:

(1) For every element z of Y and for all partitions P_1, P_2 of Y such that $P_1 \Subset P_2$ holds $\mathrm{EqClass}(z, P_1) \subseteq \mathrm{EqClass}(z, P_2)$.

(2) For every element z of Y and for all partitions P_1, P_2 of Y holds $\mathrm{EqClass}(z, P_1) \subseteq \mathrm{EqClass}(z, P_1 \vee P_2)$.

(3) For every element z of Y and for all partitions P_1, P_2 of Y holds $\mathrm{EqClass}(z, P_1 \wedge P_2) \subseteq \mathrm{EqClass}(z, P_1)$.

(4) For every element z of Y and for every partition P_1 of Y holds $\mathrm{EqClass}(z, P_1) \subseteq \mathrm{EqClass}(z, \mathcal{O}(Y))$ and $\mathrm{EqClass}(z, \mathcal{I}(Y)) \subseteq \mathrm{EqClass}(z, P_1)$.

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(5) Let G be a subset of PARTITIONS(Y) and A, B be partitions of Y. Suppose G is an independent family of partitions and $G = \{A, B\}$ and $A \neq B$. Let a, b be sets. If $a \in A$ and $b \in B$, then $a \cap b \neq \emptyset$.

(6) Let G be a subset of PARTITIONS(Y) and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\bigwedge G = A \wedge B$.

(7) Let G be a subset of PARTITIONS(Y) and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\mathrm{CompF}(A, G) = B$ and $\mathrm{CompF}(B, G) = A$.

2. Predicate Calculus

One can prove the following propositions:

(8) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(9) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$, then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(10) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$, then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(11) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(12) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(13) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(14) For every element a of BVF(Y) and for every subset G of PARTITIONS(Y) and for all partitions A, B of Y holds $\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(15) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\neg\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(16) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(17) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(18) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(19) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(20) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,A}G,B}G$.

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A Characterization of Concept Lattices. Dual Concept Lattices

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Summary. In this article we continue the formalization of concept lattices following [4]. We give necessary and sufficient conditions for a complete lattice to be isomorphic to a given formal context. As a by-product we get that a lattice is complete if and only if it is isomorphic to a concept lattice. In addition we introduce dual formal concepts and dual concept lattices and prove that the dual of a concept lattice over a formal context is isomorphic to the concept lattice over the dual formal context.

MML Identifier: CONLAT_2.

The notation and terminology used in this paper have been introduced in the following articles: [8], [10], [2], [3], [11], [1], [5], [9], [15], [7], [14], [6], [13], [12], and [16].

1. Preliminaries

Let C be a FormalContext and let C_1 be a strict FormalConcept of C. The functor ${}^{@}C_1$ yielding an element of ConceptLattice C is defined as follows:

(Def. 1) ${}^{@}C_1 = C_1$.

Next we state four propositions:

(1) For every FormalContext C holds $\perp_{\mathrm{ConceptLattice}\,C} = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Attributes}\,C$ and $\top_{\mathrm{ConceptLattice}\,C} = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Objects}\,C$.

(2) Let C be a FormalContext and D be a non empty subset of $2^{\text{the objects of } C}$. Then $(\mathrm{ObjectDerivation}\,C)(\bigcup D) = \bigcap\{(\mathrm{ObjectDerivation}\,C)(O); O$ ranges over subsets of the objects of C: $O \in D\}$.

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(3) Let C be a FormalContext and D be a non empty subset of $2^{\text{the Attributes of } C}$. Then $(\mathrm{AttributeDerivation}\, C)(\bigcup D) = \bigcap\{(\mathrm{AttributeDerivation}\, C)(A); A$ ranges over subsets of the Attributes of C: $A \in D\}$.

(4) Let C be a FormalContext and D be a subset of the carrier of ConceptLattice C. Then $\bigsqcap_{\mathrm{ConceptLattice}\, C} D$ is a FormalConcept of C and $\bigsqcup_{\mathrm{ConceptLattice}\, C} D$ is a FormalConcept of C.

Let C be a FormalContext and let D be a subset of the carrier of ConceptLattice C. The functor $\bigsqcap_C D$ yields a FormalConcept of C and is defined as follows:

(Def. 2) $\bigsqcap_C D = \bigsqcap_{\mathrm{ConceptLattice}\, C} D$.

The functor $\bigsqcup_C D$ yields a FormalConcept of C and is defined by:

(Def. 3) $\bigsqcup_C D = \bigsqcup_{\mathrm{ConceptLattice}\, C} D$.

Next we state several propositions:

(5) For every FormalContext C holds $\bigsqcup_C(\emptyset_{\mathrm{ConceptLattice}\, C}) = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Attributes}\, C$ and $\bigsqcap_C(\emptyset_{\mathrm{ConceptLattice}\, C}) = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Objects}\, C$.

(6) For every FormalContext C holds $\bigsqcup_C(\Omega_{\text{the carrier of ConceptLattice}\, C}) = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Objects}\, C$ and $\bigsqcap_C(\Omega_{\text{the carrier of ConceptLattice}\, C}) = \mathrm{Concept} - \mathrm{with} - \mathrm{all} - \mathrm{Attributes}\, C$.

(7) Let C be a FormalContext and D be a non empty subset of ConceptLattice C. Then

(i) the Extent of $\bigsqcup_C D = (\mathrm{AttributeDerivation}\, C)((\mathrm{ObjectDerivation}\, C)$ $(\bigcup\{$the Extent of $\langle E, I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}))$, and

(ii) the Intent of $\bigsqcup_C D = \bigcap\{$the Intent of $\langle E, I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}$.

(8) Let C be a FormalContext and D be a non empty subset of ConceptLattice C. Then

(i) the Extent of $\bigsqcap_C D = \bigcap\{$the Extent of $\langle E, I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}$, and

(ii) the Intent of $\bigsqcap_C D = (\mathrm{ObjectDerivation}\, C)((\mathrm{AttributeDerivation}\, C)$ $(\bigcup\{$the Intent of $\langle E, I\rangle$; E ranges over subsets of the objects of C, I ranges over subsets of the Attributes of C: $\langle E, I\rangle \in D\}))$.

(9) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcup_{\mathrm{ConceptLattice}\, C}\{\langle O, A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the Attributes of C: $\bigvee_{o\,:\,\text{object of } C}$ $(o \in$ the Extent of $C_1 \wedge O = (\mathrm{AttributeDerivation}\, C)((\mathrm{ObjectDerivation}\, C)(\{o\})) \wedge A = (\mathrm{ObjectDerivation}\, C)(\{o\}))\} = C_1$.

(10) Let C be a FormalContext and C_1 be a strict FormalConcept of C. Then $\bigsqcap_{\mathrm{ConceptLattice}\,C}\{\langle O, A\rangle; O$ ranges over subsets of the objects of C, A ranges over subsets of the Attributes of C: $\bigvee_{a\,:\,\text{Attribute of } C}$ $(a \in$ the Intent of $C_1 \ \wedge \ O = (\mathrm{AttributeDerivation}\,C)(\{a\}) \ \wedge \ A = (\mathrm{ObjectDerivation}\,C)((\mathrm{AttributeDerivation}\,C)(\{a\})))\} = C_1$.

Let C be a FormalContext. The functor $\gamma(C)$ yields a function from the objects of C into the carrier of ConceptLattice C and is defined by the condition (Def. 4).

(Def. 4) Let o be an element of the objects of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that $(\gamma(C))(o) = \langle O, A\rangle$ and $O = (\mathrm{AttributeDerivation}\,C)((\mathrm{ObjectDerivation}\,C)(\{o\}))$ and $A = (\mathrm{ObjectDerivation}\,C)(\{o\})$.

Let C be a FormalContext. The functor δ_C yielding a function from the Attributes of C into the carrier of ConceptLattice C is defined by the condition (Def. 5).

(Def. 5) Let a be an element of the Attributes of C. Then there exists a subset O of the objects of C and there exists a subset A of the Attributes of C such that $\delta_C(a) = \langle O, A\rangle$ and $O = (\mathrm{AttributeDerivation}\,C)(\{a\})$ and $A = (\mathrm{ObjectDerivation}\,C)((\mathrm{AttributeDerivation}\,C)(\{a\}))$.

The following propositions are true:

(11) Let C be a FormalContext, o be an object of C, and a be a Attribute of C. Then $(\gamma(C))(o)$ is a FormalConcept of C and $\delta_C(a)$ is a FormalConcept of C.

(12) For every FormalContext C holds $\mathrm{rng}\,\gamma(C)$ is supremum-dense and $\mathrm{rng}(\delta_C)$ is infimum-dense.

(13) Let C be a FormalContext, o be an object of C, and a be a Attribute of C. Then o is connected with a if and only if $(\gamma(C))(o) \sqsubseteq \delta_C(a)$.

2. The Characterization

We now state the proposition

(14) Let L be a complete lattice and C be a FormalContext. Then ConceptLattice C and L are isomorphic if and only if there exists a function g from the objects of C into the carrier of L and there exists a function d from the Attributes of C into the carrier of L such that $\mathrm{rng}\,g$ is supremum-dense and $\mathrm{rng}\,d$ is infimum-dense and for every object o of C and for every Attribute a of C holds o is connected with a iff $g(o) \sqsubseteq d(a)$.

Let L be a lattice. The functor Context L yields a strict non quasi-empty ContextStr and is defined as follows:

(Def. 6) Context $L = \langle$the carrier of L, the carrier of L, LattRel$(L)\rangle$.

One can prove the following proposition

(15) For every complete lattice L holds ConceptLattice Context L and L are isomorphic.

Let L_1, L_2 be lattices. Let us note that the predicate L_1 and L_2 are isomorphic is symmetric.

Next we state the proposition

(16) For every lattice L holds L is complete iff there exists a FormalContext C such that ConceptLattice C and L are isomorphic.

3. Dual Concept Lattices

Let L be a complete lattice. Observe that L° is complete.

Let C be a FormalContext. The functor C° yielding a strict non quasi-empty ContextStr is defined as follows:

(Def. 7) $C^\circ = \langle$the Attributes of C, the objects of C, (the Information of $C)^\smile\rangle$.

We now state three propositions:

(17) For every strict FormalContext C holds $(C^\circ)^\circ = C$.

(18) For every FormalContext C and for every subset O of the objects of C holds (ObjectDerivation $C)(O) =$ (AttributeDerivation $C^\circ)(O)$.

(19) For every FormalContext C and for every subset A of the Attributes of C holds (AttributeDerivation $C)(A) =$ (ObjectDerivation $C^\circ)(A)$.

Let C be a FormalContext and let C_1 be a ConceptStr over C. The functor ${C_1}^\circ$ yields a strict ConceptStr over C° and is defined as follows:

(Def. 8) The Extent of ${C_1}^\circ =$ the Intent of C_1 and the Intent of ${C_1}^\circ =$ the Extent of C_1.

Let C be a FormalContext and let C_1 be a FormalConcept of C. Then ${C_1}^\circ$ is a strict FormalConcept of C°.

We now state the proposition

(20) For every FormalContext C and for every strict FormalConcept C_1 of C holds $({C_1}^\circ)^\circ = C_1$.

Let C be a FormalContext. The functor DualHomomorphism C yielding a homomorphism from (ConceptLattice $C)^\circ$ to ConceptLattice C° is defined as follows:

(Def. 9) For every strict FormalConcept C_1 of C holds (DualHomomorphism $C)(C_1) = {C_1}^\circ$.

We now state two propositions:

(21) For every FormalContext C holds DualHomomorphism C is isomorphism.

(22) For every FormalContext C holds ConceptLattice C° and (ConceptLattice $C)^{\circ}$ are isomorphic.

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Predicate Calculus for Boolean Valued Functions. Part IV

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC12`.

The terminology and notation used in this paper are introduced in the following papers: [1], [2], [3], [5], and [4].

In this paper Y is a non empty set.

The following propositions are true:

(1) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\forall_{\forall_{a,A}G,B}G = \exists_{\neg\forall_{a,A}G,B}G$.

(2) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\exists_{\forall_{a,A}G,B}G = \forall_{\neg\forall_{a,A}G,B}G$.

(3) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\neg\forall_{a,A}G,B}G = \forall_{\exists_{\neg a,A}G,B}G$.

(4) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\neg\exists_{a,A}G,B}G = \forall_{\forall_{\neg a,A}G,B}G$.

(5) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\forall_{\exists_{a,A}G,B}G = \exists_{\forall_{\neg a,A}G,B}G$.

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(6) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\exists_{\forall_{a,A}G,B}G = \forall_{\exists_{\neg a,A}G,B}G$.
(7) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\forall_{\forall_{a,A}G,B}G = \exists_{\exists_{\neg a,A}G,B}G$.
(8) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\exists_{\neg\forall_{a,A}G,B}G = \exists_{\exists_{\neg a,A}G,B}G$.
(9) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\exists_{\neg\exists_{a,A}G,B}G = \exists_{\forall_{\neg a,A}G,B}G$.
(10) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\exists_{\exists_{a,A}G,B}G = \forall_{\neg\exists_{a,A}G,B}G$.
(11) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.
(12) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,A}G,B}G$.
(13) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,A}G,B}G$.
(14) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,A}G,B}G$.
(15) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{a,A}G,B}G$.
(16) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,A}G,B}G$.

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Predicate Calculus for Boolean Valued Functions. Part V

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC13`.

The papers [1], [2], [3], [5], and [4] provide the terminology and notation for this paper.

In this paper Y denotes a non empty set.

One can prove the following propositions:

(1) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(2) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(3) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(4) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(5) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

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(6) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(7) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

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(53) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(54) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(55) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(56) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(57) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(58) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\exists_{a,B}G,A}G$.

(59) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and

A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.

(61)[1] Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(62) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

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[1]The proposition (60) has been removed.

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Definitions of Radix-2^k Signed-Digit Number and its Adder Algorithm

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Summary. In this article, a radix-2^k signed-digit number (Radix-2^k SD number) is defined and based on it a high-speed adder algorithm is discussed.

The processes of coding and encoding for public-key cryptograms require a great deal of addition operations of natural number of many figures. This results in a long time for the encoding and decoding processes. It is possible to reduce the processing time using the high-speed adder algorithm.

In the first section of this article, we prepared some useful theorems for natural numbers and integers. In the second section, we defined the concept of radix-2^k, a set named k-SD and proved some properties about them. In the third section, we provide some important functions for generating Radix-2^k SD numbers from natural numbers and natural numbers from Radix-2^k SD numbers. In the fourth section, we defined the carry and data components of addition with Radix-2^k SD numbers and some properties about them. In the fifth section, we defined a theorem for checking whether or not a natural number can be expressed as n digits Radix-2^k SD number.

In the last section, a high-speed adder algorithm on Radix-2^k SD numbers is proposed and we provided some properties. In this algorithm, the carry of each digit has an effect on only the next digit. Properties of the relationships of the results of this algorithm to the operations of natural numbers are also given.

MML Identifier: RADIX_1.

The notation and terminology used here are introduced in the following papers: [9], [6], [2], [3], [12], [4], [11], [1], [5], [7], [13], [10], and [8].

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1. Some Useful Theorems

We adopt the following convention: i, k, m, n, x, y are natural numbers, i_1, i_2, i_3 are integers, and e is a set.

The following propositions are true:

(1) If $n \neq 0$, then $m \div n = (m$ **qua** integer$) \div n$ **qua** integer and $m \bmod n = (m$ **qua** integer$) \bmod n$ **qua** integer.

(2) If $k \neq 0$ and $n \bmod k = k - 1$, then $(n + 1) \bmod k = 0$.

(3) If $k \neq 0$ and $n \bmod k < k - 1$, then $(n + 1) \bmod k = (n \bmod k) + 1$.

(4) If $m \neq 0$ and $n \neq 0$, then $k \bmod m \cdot n \bmod n = k \bmod n$.

(5) If $k \neq 0$, then $(n + 1) \bmod k = 0$ or $(n + 1) \bmod k = (n \bmod k) + 1$.

(6) If $i \neq 0$ and $k \neq 0$, then $(n \bmod i_{\mathbb{N}}^{k}) \div i_{\mathbb{N}}^{k-'1} < i$.

(7) If $k \leqslant n$, then $m_{\mathbb{N}}^{k} \mid m_{\mathbb{N}}^{n}$.

(8) If $i_3 > 0$, then $i_1 \bmod i_2 \cdot i_3 \bmod i_3 = i_1 \bmod i_3$.

2. Definition for Radix-2^k, k-SD

Let us consider n. The functor $\mathrm{Radix}\, n$ yields a natural number and is defined by:

(Def. 1) $\mathrm{Radix}\, n = 2^n$.

Let us consider k. The functor $k\,{-}\mathrm{SD}$ yields a set and is defined by:

(Def. 2) $k\,{-}\mathrm{SD} = \{e; e$ ranges over integers: $e \leqslant \mathrm{Radix}\, k - 1 \wedge e \geqslant -\mathrm{Radix}\, k + 1\}$.

The following propositions are true:

(9) $\mathrm{Radix}\, n \neq 0$.

(10) For every e holds $e \in 0\,{-}\mathrm{SD}$ iff $e = 0$.

(11) $0\,{-}\mathrm{SD} = \{0\}$.

(12) $k\,{-}\mathrm{SD} \subseteq k + 1\,{-}\mathrm{SD}$.

(13) If $e \in k\,{-}\mathrm{SD}$, then e is an integer.

(14) $k\,{-}\mathrm{SD} \subseteq \mathbb{Z}$.

(15) If $i_1 \in k\,{-}\mathrm{SD}$, then $i_1 \leqslant \mathrm{Radix}\, k - 1$ and $i_1 \geqslant -\mathrm{Radix}\, k + 1$.

(16) $0 \in k\,{-}\mathrm{SD}$.

Let us consider k. Note that $k\,{-}\mathrm{SD}$ is non empty.

Let us consider k. Then $k\,{-}\mathrm{SD}$ is a non empty subset of $\mathbb{Z}$.

3. Functions for Generating Radix-2^k SD Numbers from Natural Numbers and Natural Numbers from Radix-2^k SD Numbers

In the sequel a denotes a tuple of n and k –SD.

We now state the proposition

(18)[1] If $i \in \operatorname{Seg} n$, then $a(i)$ is an element of k –SD.

Let i, k, n be natural numbers and let x be a tuple of n and k –SD. The functor $\operatorname{DigA}(x, i)$ yields an integer and is defined by:

(Def. 3)(i) $\operatorname{DigA}(x, i) = x(i)$ if $i \in \operatorname{Seg} n$,

(ii) $\operatorname{DigA}(x, i) = 0$ if $i = 0$.

Let i, k, n be natural numbers and let x be a tuple of n and k –SD. The functor $\operatorname{DigB}(x, i)$ yielding an element of $\mathbb{Z}$ is defined as follows:

(Def. 4) $\operatorname{DigB}(x, i) = \operatorname{DigA}(x, i)$.

One can prove the following propositions:

(19) If $i \in \operatorname{Seg} n$, then $\operatorname{DigA}(a, i)$ is an element of k –SD.

(20) For every tuple x of 1 and $\mathbb{Z}$ such that $\pi_1 x = m$ holds $x = \langle m \rangle$.

Let i, k, n be natural numbers and let x be a tuple of n and k –SD. The functor $\operatorname{SubDigit}(x, i, k)$ yielding an element of $\mathbb{Z}$ is defined by:

(Def. 5) $\operatorname{SubDigit}(x, i, k) = ((\operatorname{Radix} k)_{\mathbb{N}}^{i -' 1}) \cdot \operatorname{DigB}(x, i)$.

Let n, k be natural numbers and let x be a tuple of n and k –SD. The functor $\operatorname{DigitSD} x$ yielding a tuple of n and $\mathbb{Z}$ is defined as follows:

(Def. 6) For every natural number i such that $i \in \operatorname{Seg} n$ holds $\pi_i \operatorname{DigitSD} x = \operatorname{SubDigit}(x, i, k)$.

Let n, k be natural numbers and let x be a tuple of n and k –SD. The functor $\operatorname{SDDec} x$ yields an integer and is defined as follows:

(Def. 7) $\operatorname{SDDec} x = \sum \operatorname{DigitSD} x$.

Let i, k, x be natural numbers. The functor $\operatorname{DigitDC}(x, i, k)$ yielding an element of k –SD is defined as follows:

(Def. 8) $\operatorname{DigitDC}(x, i, k) = (x \operatorname{mod} (\operatorname{Radix} k)_{\mathbb{N}}^{i}) \div (\operatorname{Radix} k)_{\mathbb{N}}^{i -' 1}$.

Let k, n, x be natural numbers. The functor $\operatorname{DecSD}(x, n, k)$ yields a tuple of n and k –SD and is defined as follows:

(Def. 9) For every natural number i such that $i \in \operatorname{Seg} n$ holds $\operatorname{DigA}(\operatorname{DecSD}(x, n, k), i) = \operatorname{DigitDC}(x, i, k)$.

[1] The proposition (17) has been removed.

4. Definition for Carry and Data Components of Addition

Let x be an integer. The functor SD_Add_Carry x yielding an integer is defined as follows:

(Def. 10) $$\mathrm{SD\_Add\_Carry}\, x = \begin{cases} 1, & \text{if } x > 2, \\ -1, & \text{if } x < -2, \\ 0, & \text{otherwise.} \end{cases}$$

One can prove the following proposition

(21) $\mathrm{SD\_Add\_Carry}\, 0 = 0$.

Let x be an integer and let k be a natural number.

The functor $\mathrm{SD\_Add\_Data}(x, k)$ yields an integer and is defined by:

(Def. 11) $\mathrm{SD\_Add\_Data}(x, k) = x - \mathrm{SD\_Add\_Carry}\, x \cdot \mathrm{Radix}\, k$.

Next we state two propositions:

(22) $\mathrm{SD\_Add\_Data}(0, k) = 0$.

(23) If $k \geqslant 2$ and $i_1 \in k-\mathrm{SD}$ and $i_2 \in k-\mathrm{SD}$, then $-\mathrm{Radix}\, k + 2 \leqslant \mathrm{SD\_Add\_Data}(i_1 + i_2, k)$ and $\mathrm{SD\_Add\_Data}(i_1 + i_2, k) \leqslant \mathrm{Radix}\, k - 2$.

5. Definition for Checking whether or not a Natural Number can be Expressed as n Digits Radix-2^k SD Number

Let n, x, k be natural numbers. We say that x is represented by n, k if and only if:

(Def. 12) $x < (\mathrm{Radix}\, k)^n_{\mathbb{N}}$.

Next we state four propositions:

(24) If m is represented by 1, k, then $\mathrm{DigA}(\mathrm{DecSD}(m, 1, k), 1) = m$.

(25) For every n such that $n \geqslant 1$ and for every m such that m is represented by n, k holds $m = \mathrm{SDDec}\, \mathrm{DecSD}(m, n, k)$.

(26) If $k \geqslant 2$ and m is represented by 1, k and n is represented by 1, k, then $\mathrm{SD\_Add\_Carry}\, \mathrm{DigA}(\mathrm{DecSD}(m, 1, k), 1) + \mathrm{DigA}(\mathrm{DecSD}(n, 1, k), 1) = \mathrm{SD\_Add\_Carry}\, m + n$.

(27) If m is represented by $n+1$, k, then $\mathrm{DigA}(\mathrm{DecSD}(m, n+1, k), n+1) = m \div (\mathrm{Radix}\, k)^n_{\mathbb{N}}$.

6. Definition for Addition Operation for a High-Speed Adder Algorithm on Radix-2^k SD Number

Let k, i, n be natural numbers and let x, y be tuples of n and k −SD. Let us assume that $i \in \operatorname{Seg} n$ and $k \geqslant 2$. The functor $\operatorname{Add}(x, y, i, k)$ yields an element of k −SD and is defined as follows:

(Def. 13) $\operatorname{Add}(x, y, i, k) = \text{SD\_Add\_Data}(\operatorname{DigA}(x, i) + \operatorname{DigA}(y, i), k) + \text{SD\_Add\_Carry}\ \operatorname{DigA}(x, i -' 1) + \operatorname{DigA}(y, i -' 1)$.

Let n, k be natural numbers and let x, y be tuples of n and k −SD. The functor $x' +' y$ yielding a tuple of n and k −SD is defined by:

(Def. 14) For every i such that $i \in \operatorname{Seg} n$ holds $\operatorname{DigA}(x' +' y, i) = \operatorname{Add}(x, y, i, k)$.

One can prove the following two propositions:

(28) If $k \geqslant 2$ and m is represented by 1, k and n is represented by 1, k, then $\operatorname{SDDec} \operatorname{DecSD}(m, 1, k)' +' \operatorname{DecSD}(n, 1, k) = \text{SD\_Add\_Data}(m + n, k)$.

(29) Let given n. Suppose $n \geqslant 1$. Let given k, x, y. Suppose $k \geqslant 2$ and x is represented by n, k and y is represented by n, k. Then $x + y = \operatorname{SDDec} \operatorname{DecSD}(x, n, k)' +' \operatorname{DecSD}(y, n, k) + ((\operatorname{Radix} k)^n_{\mathbb{N}}) \cdot \text{SD\_Add\_Carry}\ \operatorname{DigA}(\operatorname{DecSD}(x, n, k), n) + \operatorname{DigA}(\operatorname{DecSD}(y, n, k), n)$.

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Retracts and Inheritance[1]

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The notation and terminology used in this paper are introduced in the following papers: [20], [10], [8], [9], [7], [17], [1], [22], [13], [21], [18], [2], [24], [25], [23], [19], [12], [27], [15], [4], [11], [5], [3], [14], [26], [6], and [16].

1. Poset Retracts

The following three propositions are true:

(1) For all binary relations a, b holds $a \cdot b = a\,b$.

(2) Let X be a set, L be a non empty relational structure, S be a non empty relational substructure of L, f, g be functions from X into the carrier of S, and f', g' be functions from X into the carrier of L. If $f' = f$ and $g' = g$ and $f \leqslant g$, then $f' \leqslant g'$.

(3) Let X be a set, L be a non empty relational structure, S be a full non empty relational substructure of L, f, g be functions from X into the carrier of S, and f', g' be functions from X into the carrier of L. If $f' = f$ and $g' = g$ and $f' \leqslant g'$, then $f \leqslant g$.

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is directed-sups-preserving and monotone.

The following proposition is true

(4) For all functions f, g such that f is idempotent and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and $\operatorname{rng} g \subseteq \operatorname{dom} f$ holds $f \cdot g = g$.

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Let S be a 1-sorted structure. Note that there exists a map from S into S which is idempotent.

One can prove the following propositions:

(5) For every up-complete non empty poset L holds every directed-sups-inheriting full non empty relational substructure of L is up-complete.

(6) Let L be an up-complete non empty poset and f be a map from L into L. Suppose f is idempotent and directed-sups-preserving. Then $\operatorname{Im} f$ is directed-sups-inheriting.

(7) Let T be an up-complete non empty poset and S be a directed-sups-inheriting full non empty relational substructure of T. Then $\operatorname{incl}(S, T)$ is directed-sups-preserving.

(8) Let S, T be non empty relational structures, f be a map from T into S, and g be a map from S into T. If $f \cdot g = \mathrm{id}_S$, then $\operatorname{rng} f =$ the carrier of S.

(9) Let T be a non empty relational structure, S be a non empty relational substructure of T, and f be a map from T into S. If $f \cdot \operatorname{incl}(S, T) = \mathrm{id}_S$, then f is an idempotent map from T into T.

Let S, T be non empty posets and let f be a function. We say that f is a retraction of T into S if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) f is a directed-sups-preserving map from T into S,

(ii) $f \restriction$the carrier of $S = \mathrm{id}_S$, and

(iii) S is a directed-sups-inheriting full relational substructure of T.

We say that f is a UPS retraction of T into S if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) f is a directed-sups-preserving map from T into S, and

(ii) there exists a directed-sups-preserving map g from S into T such that $f \cdot g = \mathrm{id}_S$.

Let S, T be non empty posets. We say that S is a retract of T if and only if:

(Def. 3) There exists a map f from T into S such that f is a retraction of T into S.

We say that S is a UPS retract of T if and only if:

(Def. 4) There exists a map f from T into S such that f is a UPS retraction of T into S.

The following propositions are true:

(10) For all non empty posets S, T and for every function f such that f is a retraction of T into S holds $f \cdot \operatorname{incl}(S, T) = \mathrm{id}_S$.

(11) Let S be a non empty poset, T be an up-complete non empty poset, and f be a function. Suppose f is a retraction of T into S. Then f is a UPS retraction of T into S.

(12) Let S, T be non empty posets and f be a function. If f is a retraction of T into S, then $\operatorname{rng} f =$ the carrier of S.

(13) Let S, T be non empty posets and f be a function. If f is a UPS retraction of T into S, then $\operatorname{rng} f =$ the carrier of S.

(14) Let S, T be non empty posets and f be a function. Suppose f is a retraction of T into S. Then f is an idempotent map from T into T.

(15) Let T, S be non empty posets and f be a map from T into T. Suppose f is a retraction of T into S. Then $\operatorname{Im} f =$ the relational structure of S.

(16) Let T be an up-complete non empty poset, S be a non empty poset, and f be a map from T into T. Suppose f is a retraction of T into S. Then f is directed-sups-preserving and projection.

(17) Let S, T be non empty reflexive transitive relational structures and f be a map from S into T. Then f is isomorphic if and only if the following conditions are satisfied:
  (i) f is monotone, and
  (ii) there exists a monotone map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(18) Let S, T be non empty posets. Then S and T are isomorphic if and only if there exists a monotone map f from S into T and there exists a monotone map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(19) Let S, T be up-complete non empty posets. Suppose S and T are isomorphic. Then S is a UPS retract of T and T is a UPS retract of S.

(20) Let T, S be non empty posets, f be a monotone map from T into S, and g be a monotone map from S into T. Suppose $f \cdot g = \mathrm{id}_S$. Then there exists a projection map h from T into T such that $h = g \cdot f$ and $h {\restriction}$the carrier of $\operatorname{Im} h = \mathrm{id}_{\operatorname{Im} h}$ and S and $\operatorname{Im} h$ are isomorphic.

(21) Let T be an up-complete non empty poset, S be a non empty poset, and f be a function. Suppose f is a UPS retraction of T into S. Then there exists a directed-sups-preserving projection map h from T into T such that h is a retraction of T into $\operatorname{Im} h$ and S and $\operatorname{Im} h$ are isomorphic.

(22) For every up-complete non empty poset L and for every non empty poset S such that S is a retract of L holds S is up-complete.

(23) For every complete non empty poset L and for every non empty poset S such that S is a retract of L holds S is complete.

(24) Let L be a continuous complete lattice and S be a non empty poset. If S is a retract of L, then S is continuous.

(25) Let L be an up-complete non empty poset and S be a non empty poset. If S is a UPS retract of L, then S is up-complete.

(26) Let L be a complete non empty poset and S be a non empty poset. If S is a UPS retract of L, then S is complete.

(27) Let L be a continuous complete lattice and S be a non empty poset. If S is a UPS retract of L, then S is continuous.

(28) Let L be a relational structure, S be a full relational substructure of L, and R be a relational substructure of S. Then R is full if and only if R is a full relational substructure of L.

(29) Let L be a non empty transitive relational structure and S be a directed-sups-inheriting non empty full relational substructure of L. Then every directed-sups-inheriting non empty relational substructure of S is a directed-sups-inheriting relational substructure of L.

(30) Let L be an up-complete non empty poset and S_1, S_2 be directed-sups-inheriting full non empty relational substructures of L. Suppose S_1 is a relational substructure of S_2. Then S_1 is a directed-sups-inheriting full relational substructure of S_2.

Let X, Y be non empty topological spaces. One can check that every continuous map from X into Y is continuous.

2. Products

Let R be a binary relation. We say that R is poset-yielding if and only if:

(Def. 5) For every set x such that $x \in \operatorname{rng} R$ holds x is a poset.

Let us observe that every binary relation which is poset-yielding is also relational structure yielding and reflexive-yielding.

Let X be a non empty set, let L be a non empty relational structure, let i be an element of X, and let Y be a subset of L^X. Then $\pi_i Y$ is a subset of L.

Let X be a set and let S be a poset. Note that $X \longmapsto S$ is poset-yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is poset-yielding and nonempty.

Let I be a non empty set and let J be a poset-yielding nonempty many sorted set indexed by I. Note that $\prod J$ is transitive and antisymmetric.

Let I be a non empty set, let J be a poset-yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty poset.

Next we state a number of propositions:

(31) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, f be an element of $\prod J$, and X be a subset of $\prod J$. Then $f \geqslant X$ if and only if for every element i of I holds $f(i) \geqslant \pi_i X$.

(32) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, f be an element of $\prod J$, and X be a subset of $\prod J$. Then $f \leqslant X$ if and only if for every element i of I holds $f(i) \leqslant \pi_i X$.

(33) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. Then sup X exists in $\prod J$ if and only if for every element i of I holds sup $\pi_i X$ exists in $J(i)$.

(34) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. Then inf X exists in $\prod J$ if and only if for every element i of I holds inf $\pi_i X$ exists in $J(i)$.

(35) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. If sup X exists in $\prod J$, then for every element i of I holds $(\sup X)(i) = \sup \pi_i X$.

(36) Let I be a non empty set, J be a poset-yielding nonempty many sorted set indexed by I, and X be a subset of $\prod J$. If inf X exists in $\prod J$, then for every element i of I holds $(\inf X)(i) = \inf \pi_i X$.

(37) Let I be a non empty set, J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I, X be a directed subset of $\prod J$, and i be an element of I. Then $\pi_i X$ is directed.

(38) Let I be a non empty set and J, K be relational structure yielding nonempty many sorted sets indexed by I. Suppose that for every element i of I holds $K(i)$ is a relational substructure of $J(i)$. Then $\prod K$ is a relational substructure of $\prod J$.

(39) Let I be a non empty set and J, K be relational structure yielding nonempty many sorted sets indexed by I. Suppose that for every element i of I holds $K(i)$ is a full relational substructure of $J(i)$. Then $\prod K$ is a full relational substructure of $\prod J$.

(40) Let L be a non empty relational structure, S be a non empty relational substructure of L, and X be a set. Then S^X is a relational substructure of L^X.

(41) Let L be a non empty relational structure, S be a full non empty relational substructure of L, and X be a set. Then S^X is a full relational substructure of L^X.

3. Inheritance

Let S, T be non empty relational structures and let X be a set. We say that S inherits sup of X from T if and only if:

(Def. 6) If sup X exists in T, then $\bigsqcup_T X \in$ the carrier of S.

We say that S inherits inf of X from T if and only if:

(Def. 7) If inf X exists in T, then $\bigsqcap_T X \in$ the carrier of S.

Next we state two propositions:

(42) Let T be a non empty transitive relational structure, S be a full non empty relational substructure of T, and X be a subset of S. Then S inherits sup of X from T if and only if if sup X exists in T, then sup X exists in S and $\sup X = \bigsqcup_T X$.

(43) Let T be a non empty transitive relational structure, S be a full non empty relational substructure of T, and X be a subset of S. Then S inherits inf of X from T if and only if if inf X exists in T, then inf X exists in S and $\inf X = \bigsqcap_T X$.

In this article we present several logical schemes. The scheme *ProductSupsInher* deals with a non empty set $\mathcal{A}$, poset-yielding nonempty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and and states that:

For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ holds $\prod \mathcal{C}$ inherits sup of X from $\prod \mathcal{B}$

provided the following conditions are satisfied:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}(i)]$,
- For every element i of $\mathcal{A}$ holds $\mathcal{C}(i)$ is a full relational substructure of $\mathcal{B}(i)$, and
- For every element i of $\mathcal{A}$ and for every subset X of $\mathcal{C}(i)$ such that $\mathcal{P}[X, \mathcal{C}(i)]$ holds $\mathcal{C}(i)$ inherits sup of X from $\mathcal{B}(i)$.

The scheme *PowerSupsInherit* deals with a non empty set $\mathcal{A}$, a non empty poset $\mathcal{B}$, a non empty full relational substructure $\mathcal{C}$ of $\mathcal{B}$, and and states that:

For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ holds $\mathcal{C}^{\mathcal{A}}$ inherits sup of X from $\mathcal{B}^{\mathcal{A}}$

provided the following requirements are met:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}]$, and
- For every subset X of $\mathcal{C}$ such that $\mathcal{P}[X, \mathcal{C}]$ holds $\mathcal{C}$ inherits sup of X from $\mathcal{B}$.

The scheme *ProductInfsInher* deals with a non empty set $\mathcal{A}$, poset-yielding nonempty many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and and states that:

For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ holds $\prod \mathcal{C}$ inherits inf of X from $\prod \mathcal{B}$

provided the parameters meet the following conditions:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,

- For every subset X of $\prod \mathcal{C}$ such that $\mathcal{P}[X, \prod \mathcal{C}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}(i)]$,
- For every element i of $\mathcal{A}$ holds $\mathcal{C}(i)$ is a full relational substructure of $\mathcal{B}(i)$, and
- For every element i of $\mathcal{A}$ and for every subset X of $\mathcal{C}(i)$ such that $\mathcal{P}[X, \mathcal{C}(i)]$ holds $\mathcal{C}(i)$ inherits inf of X from $\mathcal{B}(i)$.

The scheme *PowerInfsInherit* deals with a non empty set $\mathcal{A}$, a non empty poset $\mathcal{B}$, a non empty full relational substructure $\mathcal{C}$ of $\mathcal{B}$, and and states that:

For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ holds $\mathcal{C}^{\mathcal{A}}$ inherits inf of X from $\mathcal{B}^{\mathcal{A}}$

provided the following conditions are satisfied:

- Let L be a non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\mathcal{P}[X, S]$, then $\mathcal{P}[X, L]$,
- For every subset X of $\mathcal{C}^{\mathcal{A}}$ such that $\mathcal{P}[X, \mathcal{C}^{\mathcal{A}}]$ and for every element i of $\mathcal{A}$ holds $\mathcal{P}[\pi_i X, \mathcal{C}]$, and
- For every subset X of $\mathcal{C}$ such that $\mathcal{P}[X, \mathcal{C}]$ holds $\mathcal{C}$ inherits inf of X from $\mathcal{B}$.

Let I be a set, let L be a non empty relational structure, let X be a non empty subset of L^I, and let i be a set. Observe that $\pi_i X$ is non empty.

The following proposition is true

(44) Let L be a non empty poset, S be a directed-sups-inheriting non empty full relational substructure of L, and X be a non empty set. Then S^X is a directed-sups-inheriting relational substructure of L^X.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let X be a non empty subset of $\prod J$, and let i be a set. Observe that $\pi_i X$ is non empty.

One can prove the following proposition

(45) For every non empty set X and for every up-complete non empty poset L holds L^X is up-complete.

Let L be an up-complete non empty poset and let X be a non empty set. Note that L^X is up-complete.

4. Topological Retracts

Let T be a topological space. Note that the topology of T is non empty.

We now state a number of propositions:

(46) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then $\operatorname{rng} f =$ the carrier of S.

(47) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then f is idempotent.

(48) Let T be a non empty topological space and V be an open subset of T. Then $\chi_{V,\text{the carrier of } T}$ is a continuous map from T into the Sierpiński space.

(49) Let T be a non empty topological space and x, y be elements of T. Suppose that for every open subset W of T such that $y \in W$ holds $x \in W$. Then $[0 \longmapsto y, 1 \longmapsto x]$ is a continuous map from the Sierpiński space into T.

(50) Let T be a non empty topological space, x, y be elements of T, and V be an open subset of T. Suppose $x \in V$ and $y \notin V$. Then $\chi_{V,\text{the carrier of } T} \cdot [0 \longmapsto y, 1 \longmapsto x] = \mathrm{id}_{\text{the Sierpiński space}}$.

(51) Let T be a non empty 1-sorted structure, V, W be subsets of T, S be a topological augmentation of $2^1_\subseteq$, and f, g be maps from T into S. Suppose $f = \chi_{V,\text{the carrier of } T}$ and $g = \chi_{W,\text{the carrier of } T}$. Then $V \subseteq W$ if and only if $f \leqslant g$.

(52) Let L be a non empty relational structure, X be a non empty set, and R be a full non empty relational substructure of L^X. Suppose that for every set a holds a is an element of R iff there exists an element x of L such that $a = X \longmapsto x$. Then L and R are isomorphic.

(53) Let S, T be non empty topological spaces. Then S and T are homeomorphic if and only if there exists a continuous map f from S into T and there exists a continuous map g from T into S such that $f \cdot g = \mathrm{id}_T$ and $g \cdot f = \mathrm{id}_S$.

(54) Let T, S, R be non empty topological spaces, f be a map from T into S, g be a map from S into T, and h be a map from S into R. If $f \cdot g = \mathrm{id}_S$ and h is a homeomorphism, then $h \cdot f \cdot (g \cdot h^{-1}) = \mathrm{id}_R$.

(55) Let T, S, R be non empty topological spaces. Suppose S is a topological retract of T and S and R are homeomorphic. Then R is a topological retract of T.

(56) For every non empty topological space T and for every non empty subspace S of T holds $\mathrm{incl}(S,T)$ is continuous.

(57) Let T be a non empty topological space, S be a non empty subspace of T, and f be a continuous map from T into S. If f is a retraction, then $f \cdot \mathrm{incl}(S,T) = \mathrm{id}_S$.

(58) Let T be a non empty topological space and S be a non empty subspace of T. If S is a retract of T, then S is a topological retract of T.

(59) Let R, T be non empty topological spaces. Then R is a topological retract of T if and only if there exists a non empty subspace S of T such that S

is a retract of T and S and R are homeomorphic.

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Technical Preliminaries to Algebraic Specifications

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The papers [15], [3], [11], [5], [6], [7], [8], [4], [10], [13], [2], [9], [12], [1], [16], [17], and [14] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all functions f, g, h such that $\operatorname{dom} f \cap \operatorname{dom} g \subseteq \operatorname{dom} h$ holds $f{+\cdot}g{+\cdot}h = g{+\cdot}f{+\cdot}h$.

(2) For all functions f, g, h such that $f \subseteq g$ and $\operatorname{rng} h \cap \operatorname{dom} g \subseteq \operatorname{dom} f$ holds $g \cdot h = f \cdot h$.

(3) For all functions f, g, h such that $\operatorname{dom} f \subseteq \operatorname{rng} g$ and $\operatorname{dom} f$ misses $\operatorname{rng} h$ and $g^{\circ} \operatorname{dom} h$ misses $\operatorname{dom} f$ holds $f \cdot (g{+\cdot}h) = f \cdot g$.

(4) For all functions f_1, f_2, g_1, g_2 such that $f_1 \approx f_2$ and $g_1 \approx g_2$ holds $f_1 \cdot g_1 \approx f_2 \cdot g_2$.

(5) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \subseteq g$, then $f^* \subseteq g^*$.

(6) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \approx g$, then $f^* \approx g^*$.

Let X be a set and let f be a function. The functor X-indexing f yielding a many sorted set indexed by X is defined as follows:

(Def. 1) X-indexing $f = \mathrm{id}_X{+\cdot}f{\restriction}X$.

We now state a number of propositions:

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(7) For every set X and for every function f holds $\operatorname{rng}(X\text{-indexing}\,f) = (X \setminus \operatorname{dom} f) \cup f^{\circ}X$.

(8) For every non empty set X and for every function f and for every element x of X holds $(X\text{-indexing}\,f)(x) = (\mathrm{id}_X{+\cdot}f)(x)$.

(9) For all sets X, x and for every function f such that $x \in X$ holds if $x \in \operatorname{dom} f$, then $(X\text{-indexing}\,f)(x) = f(x)$ and if $x \notin \operatorname{dom} f$, then $(X\text{-indexing}\,f)(x) = x$.

(10) For every set X and for every function f such that $\operatorname{dom} f = X$ holds $X\text{-indexing}\,f = f$.

(11) For every set X and for every function f holds $X\text{-indexing}(X\text{-indexing}\,f) = X\text{-indexing}\,f$.

(12) For every set X and for every function f holds $X\text{-indexing}(\mathrm{id}_X{+\cdot}f) = X\text{-indexing}\,f$.

(13) For every set X and for every function f such that $f \subseteq \mathrm{id}_X$ holds $X\text{-indexing}\,f = \mathrm{id}_X$.

(14) For every set X holds $X\text{-indexing}\,\emptyset = \mathrm{id}_X$.

(15) For every set X and for every function f holds $X\text{-indexing}\,f{\restriction}X = X\text{-indexing}\,f$.

(16) For every set X and for every function f such that $X \subseteq \operatorname{dom} f$ holds $X\text{-indexing}\,f = f{\restriction}X$.

(17) For every function f holds $\emptyset\text{-indexing}\,f = \emptyset$.

(18) For all sets X, Y and for every function f such that $X \subseteq Y$ holds $(Y\text{-indexing}\,f){\restriction}X = X\text{-indexing}\,f$.

(19) For all sets X, Y and for every function f holds $(X \cup Y)\text{-indexing}\,f = (X\text{-indexing}\,f){+\cdot}(Y\text{-indexing}\,f)$.

(20) For all sets X, Y and for every function f holds $X\text{-indexing}\,f \approx Y\text{-indexing}\,f$.

(21) For all sets X, Y and for every function f holds $(X \cup Y)\text{-indexing}\,f = (X\text{-indexing}\,f) \cup (Y\text{-indexing}\,f)$.

(22) For every non empty set X and for all functions f, g such that $\operatorname{rng} g \subseteq X$ holds $(X\text{-indexing}\,f) \cdot g = (\mathrm{id}_X{+\cdot}f) \cdot g$.

(23) For all functions f, g such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ and $\operatorname{rng} g$ misses $\operatorname{dom} f$ and for every set X holds $f \cdot (X\text{-indexing}\,g) = f{\restriction}X$.

Let f be a function. A function is called a rng-retraction of f if:

(Def. 2) $\operatorname{dom}\mathrm{it} = \operatorname{rng} f$ and $f \cdot \mathrm{it} = \mathrm{id}_{\operatorname{rng} f}$.

We now state several propositions:

(24) For every function f and for every rng-retraction g of f holds $\operatorname{rng} g \subseteq \operatorname{dom} f$.

(25) Let f be a function, g be a rng-retraction of f, and x be a set. If $x \in \operatorname{rng} f$, then $g(x) \in \operatorname{dom} f$ and $f(g(x)) = x$.

(26) For every function f such that f is one-to-one holds f^{-1} is a rng-retraction of f.

(27) For every function f such that f is one-to-one and for every rng-retraction g of f holds $g = f^{-1}$.

(28) Let f_1, f_2 be functions. Suppose $f_1 \approx f_2$. Let g_1 be a rng-retraction of f_1 and g_2 be a rng-retraction of f_2. Then $g_1 {+\cdot} g_2$ is a rng-retraction of $f_1 {+\cdot} f_2$.

(29) Let f_1, f_2 be functions. Suppose $f_1 \subseteq f_2$. Let g_1 be a rng-retraction of f_1. Then there exists a rng-retraction g_2 of f_2 such that $g_1 \subseteq g_2$.

2. Replacement in Signature

Let S be a non empty non void many sorted signature and let f, g be functions. We say that f and g form a replacement in S if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let o_1, o_2 be operation symbols of S. Suppose $(\mathrm{id}_{\text{the operation symbols of } S} {+\cdot} g)(o_1) = (\mathrm{id}_{\text{the operation symbols of } S} {+\cdot} g)(o_2)$. Then

(i) $(\mathrm{id}_{\text{the carrier of } S} {+\cdot} f) \cdot \mathrm{Arity}(o_1) = (\mathrm{id}_{\text{the carrier of } S} {+\cdot} f) \cdot \mathrm{Arity}(o_2)$, and

(ii) $(\mathrm{id}_{\text{the carrier of } S} {+\cdot} f)$(the result sort of o_1) $=$ $(\mathrm{id}_{\text{the carrier of } S} {+\cdot} f)$(the result sort of o_2).

One can prove the following propositions:

(30) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if for all operation symbols o_1, o_2 of S such that ((the operation symbols of S)-indexing g)(o_1) = ((the operation symbols of S)-indexing g)(o_2) holds ((the carrier of S)-indexing f)$\cdot\mathrm{Arity}(o_1)$ = ((the carrier of S)-indexing f)$\cdot$ $\mathrm{Arity}(o_2)$ and ((the carrier of S)-indexing f)(the result sort of o_1) = ((the carrier of S)-indexing f)(the result sort of o_2).

(31) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if (the carrier of S)-indexing f and (the operation symbols of S)-indexing g form a replacement in S.

In the sequel S, S' denote non void signatures and f, g denote functions.

One can prove the following four propositions:

(32) If f and g form morphism between S and S', then f and g form a replacement in S.

(33) f and $\emptyset$ form a replacement in S.

(34) If g is one-to-one and (the operation symbols of S) $\cap$ rng $g \subseteq$ dom g, then f and g form a replacement in S.

(35) If g is one-to-one and rng g misses the operation symbols of S, then f and g form a replacement in S.

Let X be a set, let Y be a non empty set, let a be a function from Y into X^*, and let r be a function from Y into X. Observe that $\langle X, Y, a, r\rangle$ is non void.

Let S be a non empty non void many sorted signature and let f, g be functions. Let us assume that f and g form a replacement in S. The functor S with-replacement(f, g) yields a strict non empty non void many sorted signature and is defined by the conditions (Def. 4).

(Def. 4)(i) (The carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g),

(ii) the carrier of S with-replacement(f, g) = rng((the carrier of S) -indexing f), and

(iii) the operation symbols of S with-replacement(f, g) = rng((the operation symbols of S) -indexing g).

The following propositions are true:

(36) Let S_1, S_2 be non void signatures, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Then $f^* \cdot$ the arity of S_1 = (the arity of S_2) $\cdot g$.

(37) Suppose f and g form a replacement in S. Then (the carrier of S) -indexing f is a function from the carrier of S into the carrier of S with-replacement(f, g).

(38) Suppose f and g form a replacement in S. Let f' be a function from the carrier of S into the carrier of S with-replacement(f, g). Suppose $f' =$ (the carrier of S) -indexing f. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the arity of S with-replacement(f, g) = $f'^* \cdot$ the arity of $S \cdot g'$.

(39) Suppose f and g form a replacement in S. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the result sort of S with-replacement(f, g) = ((the carrier of S) -indexing f) $\cdot$ the result sort of $S \cdot g'$.

(40) If f and g form morphism between S and S', then S with-replacement(f, g) is a subsignature of S'.

(41) f and g form a replacement in S if and only if (the carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g).

(42) Suppose dom $f \subseteq$ the carrier of S and dom $g \subseteq$ the operation symbols of S and f and g form a replacement in S. Then $\mathrm{id}_{\text{the carrier of } S} {+\cdot} f$ and $\mathrm{id}_{\text{the operation symbols of } S} {+\cdot} g$ form morphism be-

tween S and S with-replacement(f, g).

(43) Suppose $\operatorname{dom} f =$ the carrier of S and $\operatorname{dom} g =$ the operation symbols of S and f and g form a replacement in S. Then f and g form morphism between S and S with-replacement(f, g).

(44) If f and g form a replacement in S, then S with-replacement((the carrier of S) -indexing $f, g) = S$ with-replacement(f, g).

(45) If f and g form a replacement in S, then S with-replacement$(f,$ (the operation symbols of S) -indexing $g) = S$ with-replacement(f, g).

3. Signature Extensions

Let S be a signature. A signature is called an extension of S if:

(Def. 5) S is a subsignature of it.

The following propositions are true:

(46) For all signatures S, E holds S is a subsignature of E iff E is an extension of S.

(47) Every signature S is an extension of S.

(48) For every signature S_1 and for every extension S_2 of S_1 holds every extension of S_2 is an extension of S_1.

(49) For all non empty signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 {+\cdot} S_2$ is an extension of S_1.

(50) For all non empty signatures S_1, S_2 holds $S_1 {+\cdot} S_2$ is an extension of S_2.

(51) Let S_1, S_2, S be non empty many sorted signatures and f_1, g_1, f_2, g_2 be functions. Suppose $f_1 \approx f_2$ and f_1 and g_1 form morphism between S_1 and S and f_2 and g_2 form morphism between S_2 and S. Then $f_1 {+\cdot} f_2$ and $g_1 {+\cdot} g_2$ form morphism between $S_1 {+\cdot} S_2$ and S.

(52) Let S_1, S_2, E be non empty signatures. Then E is an extension of S_1 and an extension of S_2 if and only if $S_1 \approx S_2$ and E is an extension of $S_1 {+\cdot} S_2$.

Let S be a non empty signature. One can check that every extension of S is non empty.

Let S be a non void signature. One can verify that every extension of S is non void.

One can prove the following proposition

(53) For all signatures S, T such that S is empty holds T is an extension of S.

Let S be a signature. One can check that there exists an extension of S which is non empty, non void, and strict.

The following three propositions are true:

(54) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then f and g form a replacement in S.

(55) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then E with-replacement(f, g) is an extension of S with-replacement(f, g).

(56) Let S_1, S_2 be non void signatures. Suppose $S_1 \approx S_2$. Let f, g be functions. If f and g form a replacement in $S_1 +\cdot S_2$, then $(S_1 +\cdot S_2)$ with-replacement$(f, g) =$
$(S_1$ with-replacement$(f, g)) +\cdot (S_2$ with-replacement$(f, g))$.

4. Algebras

Algebra is defined by:

(Def. 6) There exists a non void signature S such that it is a feasible algebra over S.

Let S be a signature. An algebra is called an algebra of S if:

(Def. 7) There exists a non void extension E of S such that it is a feasible algebra over E.

One can prove the following propositions:

(57) For every non void signature S holds every feasible algebra over S is an algebra of S.

(58) For every signature S and for every extension E of S holds every algebra of E is an algebra of S.

(59) Let S be a signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Then the carrier of $S \subseteq$ the carrier of E and the operation symbols of $S \subseteq$ the operation symbols of E.

(60) Let S be a non void signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Let o be an operation symbol of S. Then (the characteristics of $A)(o)$ is a function from (the sorts of $A)^{\#}(\mathrm{Arity}(o))$ into (the sorts of A)(the result sort of o).

(61) Let S be a non empty signature, A be an algebra of S, and E be a non empty many sorted signature. If A is an algebra over E, then A is an algebra over $E +\cdot S$.

(62) Let S_1, S_2 be non empty signatures and A be an algebra over S_1. Suppose A is an algebra over S_2. Then the carrier of $S_1 =$ the carrier of S_2 and the operation symbols of $S_1 =$ the operation symbols of S_2.

(63) For every non void signature S and for every non-empty disjoint algebra A over S holds the sorts of A are one-to-one.

(64) Let S be a non void signature, A be a disjoint algebra over S, and C_1, C_2 be components of the sorts of A. Then $C_1 = C_2$ or C_1 misses C_2.

(65) Let S, S' be non void signatures and A be a non-empty disjoint algebra over S. Suppose A is an algebra over S'. Then the many sorted signature of $S =$ the many sorted signature of S'.

(66) Let S' be a non void signature and A be a non-empty disjoint algebra over S. If A is an algebra of S', then S is an extension of S'.

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Multivariate Polynomials with Arbitrary Number of Variables[1]

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Summary. The goal of this article is to define multivariate polynomials in arbitrary number of indeterminates and then to prove that they constitute a ring (over appropriate structure of coefficients).

The introductory section includes quite a number of auxiliary lemmas related to many different parts of the MML. The second section characterizes the sequence flattening operation, introduced in [7], but so far lacking theorems about its fundamental properties.

We first define formal power series in arbitrary number of variables. The auxiliary concept on which the construction of formal power series is based is the notion of a bag. A bag of a set X is a natural function on X which is zero almost everywhere. The elements of X play the role of formal variables and a bag gives their exponents thus forming a power product. Series are defined for an ordered set of variables (we use ordinal numbers). A series in o variables over a structure S is a function assigning an element of the carrier of S (coefficient) to each bag of o.

We define the operations of addition, complement and multiplication for formal power series and prove their properties which depend on assumed properties of the structure from which the coefficients are taken. (We would like to note that proving associativity of multiplication turned out to be technically complicated.)

Polynomial is defined as a formal power series with finite number of non zero coefficients. In conclusion, the ring of polynomials is defined.

MML Identifier: `POLYNOM1`.

The terminology and notation used in this paper are introduced in the following articles: [24], [23], [10], [35], [1], [3], [7], [6], [11], [31], [15], [25], [12], [13], [8],

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[38], [29], [22], [5], [18], [2], [30], [33], [4], [28], [9], [36], [37], [32], [19], [26], [34], [27], [16], [21], [20], [17], and [14].

1. Basics

The following propositions are true:

(1) For all natural numbers i, j holds $\cdot_{\mathbb{N}}(i, j) = i \cdot j$.

(2) Let X be a set, A be a non empty set, F be a binary operation on A, f be a function from X into A, and x be an element of A. Then $\mathrm{dom}(F^{\circ}(f, x)) = X$.

(3) For all natural numbers a, b, c holds $a -' b -' c = a -' (b + c)$.

(4) For every set X and for every binary relation R such that $\mathrm{field}\, R \subseteq X$ holds R is a binary relation on X.

(5) Let K be a non empty loop structure and p_1, p_2 be finite sequences of elements of the carrier of K. If $\mathrm{dom}\, p_1 = \mathrm{dom}\, p_2$, then $\mathrm{dom}(p_1 + p_2) = \mathrm{dom}\, p_1$.

(6) For every function f and for all sets x, y holds $\mathrm{rng}(f +\cdot (x, y)) \subseteq \mathrm{rng}\, f \cup \{y\}$.

Let A, B be sets, let f be a function from A into B, let x be a set, and let y be an element of B. Then $f +\cdot (x, y)$ is a function from A into B.

Let X be a set, let f be a many sorted set indexed by X, and let x, y be sets. Then $f +\cdot (x, y)$ is a many sorted set indexed by X.

Next we state the proposition

(7) For every one-to-one function f holds $\overline{\overline{(f \textbf{ qua } \mathrm{set})}} = \overline{\overline{\mathrm{rng}\, f}}$.

Let A be a non empty set, let F, G be binary operations on A, and let z, u be elements of A. Observe that $\langle A, F, G, z, u \rangle$ is non empty.

Let A be a set, let X be a set, let D be a non empty set of finite sequences of A, let p be a partial function from X to D, and let i be a set. Then $\pi_i p$ is an element of D.

Let X be a set and let S be a 1-sorted structure.

(Def. 1) A function from X into the carrier of S is said to be a function from X into S.

Let X be a set. Note that there exists an order in X which is linear-order and well-ordering.

The following propositions are true:

(8) Let X be a non empty set, A be a non empty finite subset of X, R be an order in X, and x be an element of X. Suppose $x \in A$ and R linearly orders A and for every element y of X such that $y \in A$ holds $\langle x, y \rangle \in R$. Then $\pi_1\, \mathrm{SgmX}(R, A) = x$.

(9) Let X be a non empty set, A be a non empty finite subset of X, R be an order in X, and x be an element of X. Suppose $x \in A$ and R linearly orders A and for every element y of X such that $y \in A$ holds $\langle y, x\rangle \in R$. Then $\pi_{\mathrm{len}\,\mathrm{SgmX}(R,A)}\,\mathrm{SgmX}(R,A) = x$.

Let X be a non empty set, let A be a non empty finite subset of X, and let R be linear-order order in X. One can verify that $\mathrm{SgmX}(R, A)$ is non empty and one-to-one.

Let us observe that $\emptyset$ is finite sequence yielding.

Let us observe that there exists a finite sequence which is finite sequence yielding.

Let F, G be finite sequence yielding finite sequences. Then $F \frown G$ is a finite sequence yielding finite sequence.

Let D be a set. Note that every finite sequence of elements of D^* is finite sequence yielding.

Let i be a natural number and let f be a finite sequence. Note that $i \mapsto f$ is finite sequence yielding.

Let us observe that every function which is finite sequence yielding is also function yielding.

Let F be a finite sequence yielding finite sequence and let x be a set. Note that $F(x)$ is finite sequence-like.

Let F be a finite sequence. Observe that $\overline{\overline{F}}$ is finite sequence-like.

Let us observe that there exists a finite sequence which is cardinal yielding.

We now state the proposition

(10) Let f be a function. Then f is cardinal yielding if and only if for every set y such that $y \in \mathrm{rng}\, f$ holds y is a cardinal number.

Let F, G be cardinal yielding finite sequences. Note that $F \frown G$ is cardinal yielding.

Let us note that every finite sequence of elements of $\mathbb{N}$ is cardinal yielding.

Let us observe that there exists a finite sequence of elements of $\mathbb{N}$ which is cardinal yielding.

Let D be a set and let F be a finite sequence of elements of D^*. Then $\overline{\overline{F}}$ is a cardinal yielding finite sequence of elements of $\mathbb{N}$.

Let F be a finite sequence of elements of $\mathbb{N}$ and let i be a natural number. Observe that $F{\restriction}i$ is cardinal yielding.

We now state the proposition

(11) For every function F and for every set X holds $\overline{\overline{F{\restriction}X}} = \overline{\overline{F}}{\restriction}X$.

Let F be an empty function. One can verify that $\overline{\overline{F}}$ is empty.

Next we state two propositions:

(12) For every set p holds $\overline{\overline{\langle p\rangle}} = \langle\overline{\overline{p}}\rangle$.

(13) For all finite sequences F, G holds $\overline{\overline{F \frown G}} = \overline{\overline{F}} \frown \overline{\overline{G}}$.

Let X be a set. Note that ε_X is finite sequence yielding.

Let f be a finite sequence. Observe that $\langle f \rangle$ is finite sequence yielding.

One can prove the following proposition

(14) Let f be a function. Then f is finite sequence yielding if and only if for every set y such that $y \in \operatorname{rng} f$ holds y is a finite sequence.

Let F, G be finite sequence yielding finite sequences. One can verify that $F \frown G$ is finite sequence yielding.

Next we state four propositions:

(15) Let L be a non empty loop structure and F be a finite sequence of elements of (the carrier of $L)^*$. Then $\operatorname{dom} \sum F = \operatorname{dom} F$.

(16) Let L be a non empty loop structure and F be a finite sequence of elements of (the carrier of $L)^*$. Then $\sum(\varepsilon_{(\text{the carrier of } L)^*}) = \varepsilon_{(\text{the carrier of } L)}$.

(17) For every non empty loop structure L and for every element p of (the carrier of $L)^*$ holds $\langle \sum p \rangle = \sum \langle p \rangle$.

(18) Let L be a non empty loop structure and F, G be finite sequences of elements of (the carrier of $L)^*$. Then $\sum(F \frown G) = (\sum F) \frown \sum G$.

Let L be a non empty groupoid, let a be an element of the carrier of L, and let p be a finite sequence of elements of the carrier of L. The functor $a \cdot p$ yielding a finite sequence of elements of the carrier of L is defined by:

(Def. 2) $\operatorname{dom}(a \cdot p) = \operatorname{dom} p$ and for every set i such that $i \in \operatorname{dom} p$ holds $\pi_i(a \cdot p) = a \cdot \pi_i p$.

The functor $p \cdot a$ yielding a finite sequence of elements of the carrier of L is defined as follows:

(Def. 3) $\operatorname{dom}(p \cdot a) = \operatorname{dom} p$ and for every set i such that $i \in \operatorname{dom} p$ holds $\pi_i(p \cdot a) = \pi_i p \cdot a$.

The following propositions are true:

(19) Let L be a non empty groupoid and a be an element of the carrier of L. Then $a \cdot \varepsilon_{(\text{the carrier of } L)} = \varepsilon_{(\text{the carrier of } L)}$.

(20) Let L be a non empty groupoid and a be an element of the carrier of L. Then $\varepsilon_{(\text{the carrier of } L)} \cdot a = \varepsilon_{(\text{the carrier of } L)}$.

(21) For every non empty groupoid L and for all elements a, b of the carrier of L holds $a \cdot \langle b \rangle = \langle a \cdot b \rangle$.

(22) For every non empty groupoid L and for all elements a, b of the carrier of L holds $\langle b \rangle \cdot a = \langle b \cdot a \rangle$.

(23) Let L be a non empty groupoid, a be an element of the carrier of L, and p, q be finite sequences of elements of the carrier of L. Then $a \cdot (p \frown q) = (a \cdot p) \frown (a \cdot q)$.

(24) Let L be a non empty groupoid, a be an element of the carrier of L, and p, q be finite sequences of elements of the carrier of L. Then $(p \frown q) \cdot a =$

$(p \cdot a) \frown (q \cdot a)$.

We now state two propositions:

(25) Let L be an add-associative right zeroed right complementable left-distributive non empty double loop structure and x be an element of the carrier of L. Then $0_L \cdot x = 0_L$.

(26) Let L be an add-associative right zeroed right complementable right-distributive non empty double loop structure and x be an element of the carrier of L. Then $x \cdot 0_L = 0_L$.

One can verify that every non empty multiplicative loop with zero structure which is non degenerated is also non trivial.

Let us mention that there exists a non empty strict multiplicative loop with zero structure which is unital.

Let us observe that there exists a non empty strict double loop structure which is Abelian, add-associative, right zeroed, right complementable, associative, commutative, distributive, unital, and non trivial.

Next we state three propositions:

(27) Let L be an add-associative right zeroed right complementable unital right-distributive non empty double loop structure. If $0_L = 1_L$, then L is trivial.

(28) Let L be an add-associative right zeroed right complementable unital distributive non empty double loop structure, a be an element of the carrier of L, and p be a finite sequence of elements of the carrier of L. Then $\sum(a \cdot p) = a \cdot \sum p$.

(29) Let L be an add-associative right zeroed right complementable unital distributive non empty double loop structure, a be an element of the carrier of L, and p be a finite sequence of elements of the carrier of L. Then $\sum(p \cdot a) = \sum p \cdot a$.

2. Sequence Flattening

Let D be a set and let F be an empty finite sequence of elements of D^*. Observe that $\mathrm{Flat}(F)$ is empty.

One can prove the following propositions:

(30) For every set D and for every finite sequence F of elements of D^* holds $\mathrm{len}\,\mathrm{Flat}(F) = \sum \overline{\overline{F}}$.

(31) Let D, E be sets, F be a finite sequence of elements of D^*, and G be a finite sequence of elements of E^*. If $\overline{\overline{F}} = \overline{\overline{G}}$, then $\mathrm{len}\,\mathrm{Flat}(F) = \mathrm{len}\,\mathrm{Flat}(G)$.

(32) Let D be a set, F be a finite sequence of elements of D^*, and k be a set. Suppose $k \in \operatorname{dom} \operatorname{Flat}(F)$. Then there exist natural numbers i, j such that $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F(i)$ and $k = \sum \overline{\overline{F \restriction (i -' 1)}} + j$ and $F(i)(j) = \operatorname{Flat}(F)(k)$.

(33) Let D be a set, F be a finite sequence of elements of D^*, and i, j be natural numbers. If $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F(i)$, then $\sum \overline{\overline{F \restriction (i -' 1)}} + j \in \operatorname{dom} \operatorname{Flat}(F)$ and $F(i)(j) = \operatorname{Flat}(F)(\sum \overline{\overline{F \restriction (i -' 1)}} + j)$.

(34) Let L be an add-associative right zeroed right complementable non empty loop structure and F be a finite sequence of elements of (the carrier of $L)^*$. Then $\sum \operatorname{Flat}(F) = \sum \sum F$.

(35) Let X, Y be non empty sets, f be a finite sequence of elements of X^*, and v be a function from X into Y. Then $(\operatorname{dom} f \longmapsto v) \circ f$ is a finite sequence of elements of Y^*.

(36) Let X, Y be non empty sets, f be a finite sequence of elements of X^*, and v be a function from X into Y. Then there exists a finite sequence F of elements of Y^* such that $F = (\operatorname{dom} f \longmapsto v) \circ f$ and $v \cdot \operatorname{Flat}(f) = \operatorname{Flat}(F)$.

3. Functions Yielding Natural Numbers

Let us note that $\emptyset$ is natural-yielding.

One can check that there exists a function which is natural-yielding.

Let f be a natural-yielding function and let x be a set. Then $f(x)$ is a natural number.

Let f be a natural-yielding function, let x be a set, and let n be a natural number. Observe that $f +\cdot (x, n)$ is natural-yielding.

Let X be a set. One can check that every function from X into $\mathbb{N}$ is natural-yielding.

Let X be a set. Observe that there exists a many sorted set indexed by X which is natural-yielding.

Let X be a set and let b_1, b_2 be natural-yielding many sorted sets indexed by X. The functor $b_1 + b_2$ yields a many sorted set indexed by X and is defined as follows:

(Def. 5)[2] For every set x holds $(b_1 + b_2)(x) = b_1(x) + b_2(x)$.

Let us note that the functor $b_1 + b_2$ is commutative. The functor $b_1 -' b_2$ yields a many sorted set indexed by X and is defined by:

(Def. 6) For every set x holds $(b_1 -' b_2)(x) = b_1(x) -' b_2(x)$.

Next we state two propositions:

[2]The definition (Def. 4) has been removed.

(37) Let X be a set and b, b_1, b_2 be natural-yielding many sorted sets indexed by X. If for every set x such that $x \in X$ holds $b(x) = b_1(x) + b_2(x)$, then $b = b_1 + b_2$.

(38) Let X be a set and b, b_1, b_2 be natural-yielding many sorted sets indexed by X. If for every set x such that $x \in X$ holds $b(x) = b_1(x) -' b_2(x)$, then $b = b_1 -' b_2$.

Let X be a set and let b_1, b_2 be natural-yielding many sorted sets indexed by X. Observe that $b_1 + b_2$ is natural-yielding and $b_1 -' b_2$ is natural-yielding.

The following two propositions are true:

(39) For every set X and for all natural-yielding many sorted sets b_1, b_2, b_3 indexed by X holds $(b_1 + b_2) + b_3 = b_1 + (b_2 + b_3)$.

(40) For every set X and for all natural-yielding many sorted sets b, c, d indexed by X holds $b -' c -' d = b -' (c + d)$.

4. The Support of a Function

Let f be a function. The functor support f is defined as follows:

(Def. 7) For every set x holds $x \in \operatorname{support} f$ iff $f(x) \neq 0$.

One can prove the following proposition

(41) For every function f holds $\operatorname{support} f \subseteq \operatorname{dom} f$.

Let f be a function. We say that f is finite-support if and only if:

(Def. 8) support f is finite.

We introduce f has finite-support as a synonym of f is finite-support.

Let us mention that $\emptyset$ is finite-support.

Let us note that every function which is finite is also finite-support.

Let us observe that there exists a function which is natural-yielding, finite-support, and non empty.

Let f be a finite-support function. Observe that support f is finite.

Let X be a set. Note that there exists a function from X into $\mathbb{N}$ which is finite-support.

Let f be a finite-support function and let x, y be sets. Observe that $f{+\cdot}(x, y)$ is finite-support.

Let X be a set. One can verify that there exists a many sorted set indexed by X which is natural-yielding and finite-support.

One can prove the following propositions:

(42) For every set X and for all natural-yielding many sorted sets b_1, b_2 indexed by X holds $\operatorname{support}(b_1 + b_2) = \operatorname{support} b_1 \cup \operatorname{support} b_2$.

(43) For every set X and for all natural-yielding many sorted sets b_1, b_2 indexed by X holds $\operatorname{support}(b_1 -' b_2) \subseteq \operatorname{support} b_1$.

Let X be a non empty set, let S be a zero structure, and let f be a function from X into S. The functor Support f yielding a subset of X is defined by:

(Def. 9) For every element x of X holds $x \in \operatorname{Support} f$ iff $f(x) \neq 0_S$.

Let X be a non empty set, let S be a zero structure, and let p be a function from X into S. We say that p is finite-Support if and only if:

(Def. 10) Support p is finite.

We introduce p has finite-Support as a synonym of p is finite-Support.

5. Bags

Let X be a set. A bag of X is a natural-yielding finite-support many sorted set indexed by X.

Let X be a finite set. Observe that every many sorted set indexed by X is finite-support.

Let X be a set and let b_1, b_2 be bag of X. Note that $b_1 + b_2$ is finite-support and $b_1 -' b_2$ is finite-support.

The following proposition is true

(44) For every set X holds $X \longmapsto 0$ is a bag of X.

Let n be an ordinal number and let p, q be bag of n. The predicate $p < q$ is defined as follows:

(Def. 11) There exists an ordinal number k such that $p(k) < q(k)$ and for every ordinal number l such that $l \in k$ holds $p(l) = q(l)$.

Let us note that the predicate $p < q$ is antisymmetric.

Next we state the proposition

(45) For every ordinal number n and for all bag p, q, r of n such that $p < q$ and $q < r$ holds $p < r$.

Let n be an ordinal number and let p, q be bag of n. The predicate $p \leqslant q$ is defined as follows:

(Def. 12) $p < q$ or $p = q$.

Let us note that the predicate $p \leqslant q$ is reflexive.

The following propositions are true:

(46) For every ordinal number n and for all bag p, q, r of n such that $p \leqslant q$ and $q \leqslant r$ holds $p \leqslant r$.

(47) For every ordinal number n and for all bag p, q, r of n such that $p < q$ and $q \leqslant r$ holds $p < r$.

(48) For every ordinal number n and for all bag p, q, r of n such that $p \leqslant q$ and $q < r$ holds $p < r$.

(49) For every ordinal number n and for all bag p, q of n holds $p \leqslant q$ or $q \leqslant p$.

Let X be a set and let d, b be bag of X. We say that d divides b if and only if:

(Def. 13) For every set k holds $d(k) \leqslant b(k)$.

Let us note that the predicate d divides b is reflexive.

One can prove the following propositions:

(50) For every set n and for all bag d, b of n such that for every set k such that $k \in n$ holds $d(k) \leqslant b(k)$ holds d divides b.

(51) For every ordinal number n and for all bag b_1, b_2 of n such that b_1 divides b_2 holds $(b_2 -' b_1) + b_1 = b_2$.

(52) For every set X and for all bag b_1, b_2 of X holds $(b_2 + b_1) -' b_1 = b_2$.

(53) For every ordinal number n and for all bag d, b of n such that d divides b holds $d \leqslant b$.

(54) For every set n and for all bag b, b_1, b_2 of n such that $b = b_1 + b_2$ holds b_1 divides b.

Let X be a set. The functor $\mathrm{Bags}\,X$ is defined as follows:

(Def. 14) For every set x holds $x \in \mathrm{Bags}\,X$ iff x is a bag of X.

Let X be a set. Then $\mathrm{Bags}\,X$ is a subset of $\mathrm{Bags}\,X$.

One can prove the following proposition

(55) $\mathrm{Bags}\,\emptyset = \{\emptyset\}$.

Let X be a set. Note that $\mathrm{Bags}\,X$ is non empty.

Let X be a set and let B be a non empty subset of $\mathrm{Bags}\,X$. We see that the element of B is a bag of X.

Let n be a set, let L be a non empty 1-sorted structure, let p be a function from $\mathrm{Bags}\,n$ into L, and let x be a bag of n. Then $p(x)$ is an element of L.

Let X be a set. The functor $\mathrm{EmptyBag}\,X$ yielding an element of $\mathrm{Bags}\,X$ is defined by:

(Def. 15) $\mathrm{EmptyBag}\,X = X \longmapsto 0$.

The following propositions are true:

(56) For all sets X, x holds $(\mathrm{EmptyBag}\,X)(x) = 0$.

(57) For every set X and for every bag b of X holds $b + \mathrm{EmptyBag}\,X = b$.

(58) For every set X and for every bag b of X holds $b -' \mathrm{EmptyBag}\,X = b$.

(59) For every set X and for every bag b of X holds $\mathrm{EmptyBag}\,X -' b = \mathrm{EmptyBag}\,X$.

(60) For every set X and for every bag b of X holds $b -' b = \mathrm{EmptyBag}\,X$.

(61) For every set n and for all bag b_1, b_2 of n such that b_1 divides b_2 and $b_2 -' b_1 = \mathrm{EmptyBag}\,n$ holds $b_2 = b_1$.

(62) For every set n and for every bag b of n such that b divides $\mathrm{EmptyBag}\,n$ holds $\mathrm{EmptyBag}\,n = b$.

(63) For every set n and for every bag b of n holds $\mathrm{EmptyBag}\,n$ divides b.

(64) For every ordinal number n and for every bag b of n holds $\mathrm{EmptyBag}\, n \leqslant b$.

Let n be an ordinal number. The functor $\mathrm{BagOrder}\, n$ yields an order in $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 16) For all bag p, q of n holds $\langle p, q\rangle \in \mathrm{BagOrder}\, n$ iff $p \leqslant q$.

Let n be an ordinal number. Note that $\mathrm{BagOrder}\, n$ is linear-order.

Let X be a set and let f be a function from X into $\mathbb{N}$. The functor $\mathrm{NatMinor}\, f$ yielding a subset of $\mathbb{N}^X$ is defined by the condition (Def. 17).

(Def. 17) Let g be a natural-yielding many sorted set indexed by X. Then $g \in \mathrm{NatMinor}\, f$ if and only if for every set x such that $x \in X$ holds $g(x) \leqslant f(x)$.

Next we state the proposition

(65) For every set X and for every function f from X into $\mathbb{N}$ holds $f \in \mathrm{NatMinor}\, f$.

Let X be a set and let f be a function from X into $\mathbb{N}$. Observe that $\mathrm{NatMinor}\, f$ is non empty and functional.

Let X be a set and let f be a function from X into $\mathbb{N}$. One can verify that every element of $\mathrm{NatMinor}\, f$ is natural-yielding.

The following proposition is true

(66) For every set X and for every finite-support function f from X into $\mathbb{N}$ holds $\mathrm{NatMinor}\, f \subseteq \mathrm{Bags}\, X$.

Let X be a set and let f be a finite-support function from X into $\mathbb{N}$. Then $\mathrm{support}\, f$ is an element of $\mathrm{Fin}\, X$.

The following proposition is true

(67) For every non empty set X and for every finite-support function f from X into $\mathbb{N}$ holds $\overline{\overline{\mathrm{NatMinor}\, f}} = \cdot_{\mathbb{N}}\text{-}\sum_{\mathrm{support}\, f}(+_{\mathbb{N}})^{\circ}(f, 1)$.

Let X be a set and let f be a finite-support function from X into $\mathbb{N}$. One can verify that $\mathrm{NatMinor}\, f$ is finite.

Let n be an ordinal number and let b be a bag of n. The functor $\mathrm{divisors}\, b$ yields a finite sequence of elements of $\mathrm{Bags}\, n$ and is defined by the condition (Def. 18).

(Def. 18) There exists a non empty finite subset S of $\mathrm{Bags}\, n$ such that $\mathrm{divisors}\, b = \mathrm{SgmX}(\mathrm{BagOrder}\, n, S)$ and for every bag p of n holds $p \in S$ iff p divides b.

Let n be an ordinal number and let b be a bag of n. One can check that $\mathrm{divisors}\, b$ is non empty and one-to-one.

The following four propositions are true:

(68) Let n be an ordinal number, i be a natural number, and b be a bag of n. If $i \in \mathrm{dom}\, \mathrm{divisors}\, b$, then $\pi_i\, \mathrm{divisors}\, b$ **qua** element of $\mathrm{Bags}\, n$ divides b.

(69) For every ordinal number n and for every bag b of n holds π_1 divisors $b =$ EmptyBag n and $\pi_{\mathrm{len\,divisors}\,b}$ divisors $b = b$.

(70) Let n be an ordinal number, i be a natural number, and b, b_1, b_2 be bag of n. If $i > 1$ and $i <$ len divisors b, then π_i divisors $b \neq$ EmptyBag n and π_i divisors $b \neq b$.

(71) For every ordinal number n holds divisors EmptyBag $n = \langle$EmptyBag $n\rangle$.

Let n be an ordinal number and let b be a bag of n. The functor decomp b yields a finite sequence of elements of $(\mathrm{Bags}\,n)^2$ and is defined as follows:

(Def. 19) dom decomp b = dom divisors b and for every natural number i and for every bag p of n such that $i \in$ dom decomp b and $p = \pi_i$ divisors b holds π_i decomp $b = \langle p, b -' p\rangle$.

One can prove the following propositions:

(72) Let n be an ordinal number, i be a natural number, and b be a bag of n. If $i \in$ dom decomp b, then there exist bag b_1, b_2 of n such that π_i decomp $b = \langle b_1, b_2\rangle$ and $b = b_1 + b_2$.

(73) Let n be an ordinal number and b, b_1, b_2 be bag of n. If $b = b_1 + b_2$, then there exists a natural number i such that $i \in$ dom decomp b and π_i decomp $b = \langle b_1, b_2\rangle$.

(74) Let n be an ordinal number, i be a natural number, and b, b_1, b_2 be bag of n. If $i \in$ dom decomp b and π_i decomp $b = \langle b_1, b_2\rangle$, then $b_1 = \pi_i$ divisors b.

Let n be an ordinal number and let b be a bag of n. Note that decomp b is non empty one-to-one and finite sequence yielding.

Let n be an ordinal number and let b be an element of Bags n. One can verify that decomp b is non empty one-to-one and finite sequence yielding.

Next we state four propositions:

(75) For every ordinal number n and for every bag b of n holds π_1 decomp $b = \langle$EmptyBag $n, b\rangle$ and $\pi_{\mathrm{len\,decomp}\,b}$ decomp $b = \langle b,$ EmptyBag $n\rangle$.

(76) Let n be an ordinal number, i be a natural number, and b, b_1, b_2 be bag of n. If $i > 1$ and $i <$ len decomp b and π_i decomp $b = \langle b_1, b_2\rangle$, then $b_1 \neq$ EmptyBag n and $b_2 \neq$ EmptyBag n.

(77) For every ordinal number n holds decomp EmptyBag $n = \langle\langle$EmptyBag n, EmptyBag $n\rangle\rangle$.

(78) Let n be an ordinal number, b be a bag of n, and f, g be finite sequences of elements of $((\mathrm{Bags}\,n)^3)^*$. Suppose that

(i) dom f = dom decomp b,

(ii) dom g = dom decomp b,

(iii) for every natural number k such that $k \in$ dom f holds $f(k) =$ (decomp($\pi_1\pi_k$ decomp b **qua** element of Bags n))$^\frown$(len decomp($\pi_1\pi_k$ decomp b **qua** element of Bags n) $\mapsto \langle\pi_2\pi_k$ decomp $b\rangle$), and

(iv) for every natural number k such that $k \in \operatorname{dom} g$ holds $g(k) = (\operatorname{len} \operatorname{decomp}(\pi_2 \pi_k \operatorname{decomp} b$ **qua** element of $\operatorname{Bags} n) \mapsto \langle \pi_1 \pi_k \operatorname{decomp} b \rangle) ^\frown \operatorname{decomp}(\pi_2 \pi_k \operatorname{decomp} b$ **qua** element of $\operatorname{Bags} n)$.

Then there exists a permutation p of $\operatorname{dom} \operatorname{Flat}(f)$ such that $\operatorname{Flat}(g) = \operatorname{Flat}(f) \cdot p$.

6. Formal Power Series

Let X be a set and let S be a 1-sorted structure.

(Def. 20) A function from $\operatorname{Bags} X$ into S is said to be a Series of X, S.

Let n be a set, let L be a right zeroed non empty loop structure, and let p, q be Series of n, L. The functor $p + q$ yielding a Series of n, L is defined as follows:

(Def. 21) For every bag x of n holds $(p + q)(x) = p(x) + q(x)$.

One can prove the following proposition

(79) Let n be a set, L be a right zeroed non empty loop structure, and p, q be Series of n, L. Then $\operatorname{Support} p + q \subseteq \operatorname{Support} p \cup \operatorname{Support} q$.

Let n be a set, let L be an Abelian right zeroed non empty loop structure, and let p, q be Series of n, L. Let us notice that the functor $p+q$ is commutative.

Next we state the proposition

(80) Let n be a set, L be an add-associative right zeroed non empty double loop structure, and p, q, r be Series of n, L. Then $(p+q)+r = p+(q+r)$.

Let n be a set, let L be an add-associative right zeroed right complementable non empty loop structure, and let p be a Series of n, L. The functor $-p$ yields a Series of n, L and is defined by:

(Def. 22) For every bag x of n holds $(-p)(x) = -p(x)$.

Let n be a set, let L be an add-associative right zeroed right complementable non empty loop structure, and let p, q be Series of n, L. The functor $p-q$ yields a Series of n, L and is defined by:

(Def. 23) $p - q = p + -q$.

Let n be a set and let S be a non empty zero structure. The functor $0\_(n, S)$ yields a Series of n, S and is defined by:

(Def. 24) $0\_(n, S) = \operatorname{Bags} n \longmapsto 0_S$.

One can prove the following propositions:

(81) For every set n and for every non empty zero structure S and for every bag b of n holds $(0\_(n, S))(b) = 0_S$.

(82) For every set n and for every right zeroed non empty loop structure L and for every Series p of n, L holds $p + 0\_(n, L) = p$.

Let n be a set and let L be a unital non empty multiplicative loop with zero structure. The functor $1\_(n, L)$ yielding a Series of n, L is defined as follows:

(Def. 25) $1\_(n, L) = 0\_(n, L) +\cdot (\mathrm{EmptyBag}\, n, 1_L)$.

We now state two propositions:

(83) Let n be a set, L be an add-associative right zeroed right complementable non empty loop structure, and p be a Series of n, L. Then $p-p = 0\_(n, L)$.

(84) Let n be a set and L be a unital non empty multiplicative loop with zero structure. Then $(1\_(n, L))(\mathrm{EmptyBag}\, n) = 1_L$ and for every bag b of n such that $b \neq \mathrm{EmptyBag}\, n$ holds $(1\_(n, L))(b) = 0_L$.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed non empty double loop structure, and let p, q be Series of n, L. The functor $p * q$ yields a Series of n, L and is defined by the condition (Def. 26).

(Def. 26) Let b be a bag of n. Then there exists a finite sequence s of elements of the carrier of L such that

(i) $(p * q)(b) = \sum s$,

(ii) $\mathrm{len}\, s = \mathrm{len}\, \mathrm{decomp}\, b$, and

(iii) for every natural number k such that $k \in \mathrm{dom}\, s$ there exist bag b_1, b_2 of n such that $\pi_k \mathrm{decomp}\, b = \langle b_1, b_2 \rangle$ and $\pi_k s = p(b_1) \cdot q(b_2)$.

One can prove the following two propositions:

(85) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable distributive associative non empty double loop structure, and p, q, r be Series of n, L. Then $p * (q + r) = p * q + p * r$.

(86) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable unital distributive associative non empty double loop structure, and p, q, r be Series of n, L. Then $(p * q) * r = p * (q * r)$.

Let n be an ordinal number, let L be an Abelian add-associative right zeroed right complementable commutative non empty double loop structure, and let p, q be Series of n, L. Let us note that the functor $p * q$ is commutative.

One can prove the following three propositions:

(87) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and p be a Series of n, L. Then $p * 0\_(n, L) = 0\_(n, L)$.

(88) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive unital non trivial non empty double loop structure, and p be a Series of n, L. Then $p * 1\_(n, L) = p$.

(89) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive unital non trivial non empty double loop structure, and p be a Series of n, L. Then $1\_(n, L) * p = p$.

7. Polynomials

Let n be a set and let S be a non empty zero structure. Note that there exists a Series of n, S which is finite-Support.

Let n be an ordinal number and let S be a non empty zero structure. A Polynomial of n, S is a finite-Support Series of n, S.

Let n be an ordinal number, let L be a right zeroed non empty loop structure, and let p, q be Polynomial of n, L. Observe that $p+q$ is finite-Support.

Let n be an ordinal number, let L be an add-associative right zeroed right complementable non empty loop structure, and let p be a Polynomial of n, L. Note that $-p$ is finite-Support.

Let n be a natural number, let L be an add-associative right zeroed right complementable non empty loop structure, and let p, q be Polynomial of n, L. Note that $p-q$ is finite-Support.

Let n be an ordinal number and let S be a non empty zero structure. Observe that $0\_(n,S)$ is finite-Support.

Let n be an ordinal number and let L be an add-associative right zeroed right complementable unital right-distributive non trivial non empty double loop structure. Observe that $1\_(n,L)$ is finite-Support.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and let p, q be Polynomial of n, L. One can check that $p*q$ is finite-Support.

8. The Ring of Polynomials

Let n be an ordinal number and let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure. The functor Polynom-Ring(n,L) yields a strict non empty double loop structure and is defined by the conditions (Def. 27).

(Def. 27)(i) For every set x holds $x \in$ the carrier of Polynom-Ring(n,L) iff x is a Polynomial of n, L,

(ii) for all elements x, y of Polynom-Ring(n,L) and for all Polynomial p, q of n, L such that $x=p$ and $y=q$ holds $x+y=p+q$,

(iii) for all elements x, y of Polynom-Ring(n,L) and for all Polynomial p, q of n, L such that $x=p$ and $y=q$ holds $x \cdot y = p*q$,

(iv) $0_{\text{Polynom-Ring}(n,L)} = 0\_(n,L)$, and

(v) $1_{\text{Polynom-Ring}(n,L)} = 1\_(n,L)$.

Let n be an ordinal number and let L be an Abelian right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure. One can check that Polynom-Ring(n,L) is Abelian.

Let n be an ordinal number and let L be an add-associative right zeroed right complementable unital distributive non trivial non empty double loop structure. Observe that Polynom-Ring(n, L) is add-associative.

Let n be an ordinal number and let L be a right zeroed add-associative right complementable unital distributive non trivial non empty double loop structure. Note that Polynom-Ring(n, L) is right zeroed.

Let n be an ordinal number and let L be a right complementable right zeroed add-associative unital distributive non trivial non empty double loop structure. Observe that Polynom-Ring(n, L) is right complementable.

Let n be an ordinal number and let L be an Abelian add-associative right zeroed right complementable commutative unital distributive non trivial non empty double loop structure. Note that Polynom-Ring(n, L) is commutative.

Let n be an ordinal number and let L be an Abelian add-associative right zeroed right complementable unital distributive associative non trivial non empty double loop structure. Note that Polynom-Ring(n, L) is associative.

Let n be an ordinal number and let L be a right zeroed Abelian add-associative right complementable unital distributive associative non trivial non empty double loop structure. One can check that Polynom-Ring(n, L) is unital and right-distributive.

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Continuous Lattices between $\mathbf{T}_0$ Spaces[1]

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Summary. Formalization of [17, pp. 128–130], chapter II, section 4 (4.1 – 4.9).

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The terminology and notation used in this paper have been introduced in the following articles: [29], [16], [12], [13], [11], [1], [2], [32], [18], [30], [24], [25], [26], [27], [3], [9], [34], [35], [33], [28], [15], [21], [37], [10], [31], [20], [23], [5], [14], [6], [22], [8], [4], [19], [36], and [7].

Let I be a set and let J be a relational structure yielding many sorted set indexed by I. We introduce $I\text{-prod}_{\mathrm{POS}}\, J$ as a synonym of $\prod J$.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. One can check that $I\text{-prod}_{\mathrm{POS}}\, J$ is constituted functions.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. We introduce $I\text{-prod}_{\mathrm{TOP}}\, J$ as a synonym of $\prod J$.

Let X, Y be non empty topological spaces. The functor $[X \to Y]$ yields a non empty strict relational structure and is defined as follows:

(Def. 1) $[X \to Y] = [X \to \Omega Y]$.

Let X, Y be non empty topological spaces. Observe that $[X \to Y]$ is reflexive transitive and constituted functions.

Let X be a non empty topological space and let Y be a non empty T_0 topological space. Observe that $[X \to Y]$ is antisymmetric.

We now state three propositions:

(1) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into ΩY.

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(2) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into Y.

(3) Let X, Y be non empty topological spaces, a, b be elements of $[X \to Y]$, and f, g be maps from X into ΩY. If $a = f$ and $b = g$, then $a \leqslant b$ iff $f \leqslant g$.

Let X, Y be non empty topological spaces, let x be a point of X, and let A be a subset of the carrier of $([X \to Y])$. Then $\pi_x A$ is a subset of ΩY.

Let X, Y be non empty topological spaces, let x be a set, and let A be a non empty subset of the carrier of $([X \to Y])$. Observe that $\pi_x A$ is non empty.

We now state three propositions:

(4) Ω(the Sierpiński space) is a topological augmentation of $2^1_{\subseteq}$.

(5) Let X be a non empty topological space. Then there exists a map f from $\langle$the topology of X, $\subseteq\rangle$ into $[X \to$ the Sierpiński space$]$ such that f is isomorphic and for every open subset V of X holds $f(V) = \chi_{V,\text{the carrier of } X}$.

(6) Let X be a non empty topological space. Then $\langle$the topology of X, $\subseteq\rangle$ and $[X \to$ the Sierpiński space$]$ are isomorphic.

Let X, Y, Z be non empty topological spaces and let f be a continuous map from Y into Z. The functor $[X \to f]$ yields a map from $[X \to Y]$ into $[X \to Z]$ and is defined by:

(Def. 2) For every continuous map g from X into Y holds $([X \to f])(g) = f \cdot g$.

The functor $[f \to X]$ yields a map from $[Z \to X]$ into $[Y \to X]$ and is defined by:

(Def. 3) For every continuous map g from Z into X holds $([f \to X])(g) = g \cdot f$.

The following propositions are true:

(7) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \to Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

(8) For every non empty topological space X and for every monotone convergence T_0-space Y holds $[X \to Y]$ is up-complete.

(9) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[X \to f]$ is monotone.

(10) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[X \to f]$ is idempotent.

(11) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[f \to X]$ is monotone.

(12) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[f \to X]$ is idempotent.

(13) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of X, and A be a subset of $[X \to Y]$.

Then $\pi_x([X \to f])^\circ A = f^\circ \pi_x A$.

(14) Let X be a non empty topological space, Y, Z be monotone convergence T_0-spaces, and f be a continuous map from Y into Z. Then $[X \to f]$ is directed-sups-preserving.

(15) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of Y, and A be a subset of $[Z \to X]$. Then $\pi_x([f \to X])^\circ A = \pi_{f(x)} A$.

(16) Let Y, Z be non empty topological spaces, X be a monotone convergence T_0-space, and f be a continuous map from Y into Z. Then $[f \to X]$ is directed-sups-preserving.

(17) Let X, Z be non empty topological spaces and Y be a non empty subspace of Z. Then $[X \to Y]$ is a full relational substructure of $[X \to Z]$.

(18) Let Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. Suppose f is a retraction. Then ΩY is a directed-sups-inheriting relational substructure of ΩZ.

(19) Let X be a non empty topological space, Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. If f is a retraction, then $[X \to f]$ is a retraction of $[X \to Z]$ into $[X \to Y]$.

(20) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. If Y is a retract of Z, then $[X \to Y]$ is a retract of $[X \to Z]$.

(21) Let X, Y, Z be non empty topological spaces and f be a continuous map from Y into Z. If f is a homeomorphism, then $[X \to f]$ is isomorphic.

(22) Let X, Y, Z be non empty topological spaces. If Y and Z are homeomorphic, then $[X \to Y]$ and $[X \to Z]$ are isomorphic.

(23) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. Suppose Y is a retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(24) Let X be a non empty topological space and Y, Z be monotone convergence T_0-spaces. Suppose Y is a topological retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(25) Let Y be a non trivial T_0-space. Suppose Y is not a T_1 space. Then the Sierpiński space is a topological retract of Y.

(26) Let X be a non empty topological space and Y be a non trivial T_0-space. If $[X \to Y]$ has l.u.b.'s, then Y is not a T_1 space.

One can check that the Sierpiński space is non trivial and monotone convergence.

One can verify that there exists a T_0-space which is injective, monotone convergence, and non trivial.

The following propositions are true:

(27) Let X be a non empty topological space and Y be a monotone convergence non trivial T_0-space. If $[X \to Y]$ is complete and continuous, then $\langle$the topology of X, $\subseteq\rangle$ is continuous.

(28) Let X be a non empty topological space, x be a point of X, and Y be a monotone convergence T_0-space. Then there exists a directed-sups-preserving projection map F from $[X \to Y]$ into $[X \to Y]$ such that

(i) for every continuous map f from X into Y holds $F(f) = X \longmapsto f(x)$, and

(ii) there exists a continuous map h from X into X such that $h = X \longmapsto x$ and $F = [h \to Y]$.

(29) Let X be a non empty topological space and Y be a monotone convergence T_0-space. If $[X \to Y]$ is complete and continuous, then ΩY is complete and continuous.

(30) Let X be a non empty 1-sorted structure, I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, f be a map from X into I-$\mathrm{prod}_{\mathrm{TOP}}\, J$, and i be an element of I. Then $(\mathrm{commute}(f))(i) = \mathrm{proj}(J, i) \cdot f$.

(31) For every 1-sorted structure S and for every set M holds the support of $M \longmapsto S = M \longmapsto$ the carrier of S.

(32) Let X, Y be non empty topological spaces, M be a non empty set, and f be a continuous map from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto Y)$. Then $\mathrm{commute}(f)$ is a function from M into the carrier of $([X \to Y])$.

(33) For all non empty topological spaces X, Y holds the carrier of $([X \to Y]) \subseteq (\text{the carrier of } Y)^{\text{the carrier of } X}$.

(34) Let X, Y be non empty topological spaces, M be a non empty set, and f be a function from M into the carrier of $([X \to Y])$. Then $\mathrm{commute}(f)$ is a continuous map from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto Y)$.

(35) Let X be a non empty topological space and M be a non empty set. Then there exists a map F from $[X \to M$-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto$ the Sierpiński space$)]$ into M-$\mathrm{prod}_{\mathrm{POS}}(M \longmapsto ([X \to$ the Sierpiński space$]))$ such that F is isomorphic and for every continuous map f from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto$ the Sierpiński space$)$ holds $F(f) = \mathrm{commute}(f)$.

(36) Let X be a non empty topological space and M be a non empty set. Then $[X \to M$-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto$ the Sierpiński space$)]$ and M-$\mathrm{prod}_{\mathrm{POS}}(M \longmapsto ([X \to$ the Sierpiński space$]))$ are isomorphic.

(37) Let X be a non empty topological space. Suppose $\langle$the topology of X, $\subseteq\rangle$ is continuous. Let Y be an injective T_0-space. Then $[X \to Y]$ is complete

and continuous.

Let us observe that there exists a top-lattice which is non trivial, complete, and Scott.

Next we state the proposition

(38) Let X be a non empty topological space and L be a non trivial complete Scott top-lattice. Then $[X \to L]$ is complete and continuous if and only if $\langle$the topology of X, $\subseteq\rangle$ is continuous and L is continuous.

Let f be a function. Observe that $\mathrm{Union\,disjoint}\, f$ is relation-like.

Let f be a function. The functor G_f yields a binary relation and is defined as follows:

(Def. 4) $G_f = (\mathrm{Union\,disjoint}\, f)^{\smile}$.

In the sequel x, y are sets and f is a function.

We now state three propositions:

(39) $\langle x, y\rangle \in G_f$ iff $x \in \mathrm{dom}\, f$ and $y \in f(x)$.

(40) For every finite set X holds $\pi_1(X)$ is finite and $\pi_2(X)$ is finite.

(41) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and f be a map from X into S. If G_f is an open subset of $[\!: X, Y :\!]$, then f is continuous.

Let W be a binary relation and let X be a set. The functor $\Theta_X(W)$ yielding a function is defined by:

(Def. 5) $\mathrm{dom}\, \Theta_X(W) = X$ and for every x such that $x \in X$ holds $(\Theta_X(W))(x) = W^{\circ}\{x\}$.

One can prove the following proposition

(42) For every binary relation W and for every set X such that $\mathrm{dom}\, W \subseteq X$ holds $G_{\Theta_X(W)} = W$.

Let X, Y be topological spaces. Observe that every subset of the carrier of $[\!: X, Y :\!]$ is relation-like and every element of the topology of $[\!: X, Y :\!]$ is relation-like.

Next we state four propositions:

(43) Let X, Y be non empty topological spaces, W be an open subset of $[\!: X, Y :\!]$, and x be a point of X. Then $W^{\circ}\{x\}$ is an open subset of Y.

(44) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W be an open subset of $[\!: X, Y :\!]$. Then $\Theta_{\text{the carrier of } X}(W)$ is a continuous map from X into S.

(45) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W_1, W_2 be open subsets of $[\!: X, Y :\!]$. Suppose $W_1 \subseteq W_2$. Let f_1, f_2 be elements of $[X \to S]$. If $f_1 = \Theta_{\text{the carrier of } X}(W_1)$ and $f_2 = \Theta_{\text{the carrier of } X}(W_2)$, then $f_1 \leqslant f_2$.

(46) Let X, Y be non empty topological spaces and S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$. Then there exists a map F from $\langle$the topology of $[\!: X, Y :\!]$, $\subseteq\rangle$ into $[X \to S]$ such that F is monotone and for every open subset W of $[\!: X, Y :\!]$ holds $F(W) = \Theta_{\text{the carrier of } X}(W)$.

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Predicate Calculus for Boolean Valued Functions. Part VI

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: BVFUNC14.

The articles [4], [6], [1], [8], [7], [2], [3], [5], [11], [10], and [9] provide the terminology and notation for this paper.

1. Preliminaries

In this paper Y denotes a non empty set.

We now state several propositions:

(1) For every element z of Y and for all partitions P_1, P_2 of Y holds $\mathrm{EqClass}(z, P_1 \wedge P_2) = \mathrm{EqClass}(z, P_1) \cap \mathrm{EqClass}(z, P_2)$.

(2) Let G be a subset of PARTITIONS(Y) and A, B be partitions of Y. If G is a coordinate and $G = \{A, B\}$ and $A \neq B$, then $\bigwedge G = A \wedge B$.

(3) Let G be a subset of PARTITIONS(Y) and B, C, D be partitions of Y. Suppose G is a coordinate and $G = \{B, C, D\}$ and $B \neq C$ and $C \neq D$ and $D \neq B$. Then $\bigwedge G = B \wedge C \wedge D$.

(4) Let G be a subset of PARTITIONS(Y) and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\mathrm{CompF}(A, G) = B \wedge C$ and $\mathrm{CompF}(B, G) = C \wedge A$ and $\mathrm{CompF}(C, G) = A \wedge B$.

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(5) Let G be a subset of PARTITIONS(Y) and A, B, C, D be partitions of Y. Suppose $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D$.

(6) Let G be a subset of PARTITIONS(Y) and A, B, C, D be partitions of Y. Suppose $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D$.

(7) Let G be a subset of PARTITIONS(Y) and A, B, C, D be partitions of Y. Suppose $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D$.

(8) Let G be a subset of PARTITIONS(Y) and A, B, C, D be partitions of Y. Suppose $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$. Then $\mathrm{CompF}(D, G) = A \wedge C \wedge B$.

2. Predicate Calculus

We adopt the following rules: a is an element of BVF(Y), G is a subset of PARTITIONS(Y), and A, B, C are partitions of Y.

One can prove the following propositions:

(9) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(10) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\forall_{a,C}G,A}G,B}G = \forall_{\forall_{\forall_{a,C}G,B}G,A}G$.

(11) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\exists_{a,C}G,A}G,B}G = \forall_{\forall_{\exists_{a,C}G,B}G,A}G$.

(12) Let G be a subset of PARTITIONS(Y), B, C, D be partitions of Y, h be a function, and b, c, d be sets. Suppose $B \neq C$ and $C \neq D$ and $D \neq B$ and $h = (B \longmapsto b) + \cdot (C \longmapsto c) + \cdot (D \longmapsto d)$. Then $\mathrm{dom}\, h = \{B, C, D\}$ and $h(B) = b$ and $h(C) = c$ and $h(D) = d$ and $\mathrm{rng}\, h = \{h(B), h(C), h(D)\}$.

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Predicate Calculus for Boolean Valued Functions. Part VII

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC15`.

The articles [6], [1], [2], [4], [3], and [5] provide the terminology and notation for this paper.

In this paper Y is a non empty set.

Next we state a number of propositions:

(1) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C be partitions of Y, and z, u be elements of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$ and $\mathrm{EqClass}(z, C) = \mathrm{EqClass}(u, C)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

(2) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(3) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(4) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

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(5) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(6) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

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(10) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\forall_{\exists_{a,C}G,A}G,B}G \Subset \forall_{\exists_{\exists_{a,C}G,B}G,A}G$.

(11) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\exists_{\forall_{a,C}G,A}G,B}G = \exists_{\exists_{\forall_{a,C}G,B}G,A}G$.

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(14) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{\exists_{a,C}G,A}G,B}G \Subset \exists_{\forall_{\exists_{a,C}G,B}G,A}G$.

(15) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{\forall_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\forall_{a,C}G,B}G,A}G$.

(16) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{\exists_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\exists_{a,C}G,B}G,A}G$.

(17) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{\forall_{a,C}G,A}G,B}G \Subset \forall_{\exists_{\forall_{a,C}G,B}G,A}G$.

(18) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\forall_{\exists_{a,C}G,A}G,B}G \Subset \forall_{\exists_{\exists_{a,C}G,B}G,A}G$.

(19) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\exists_{\forall_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\forall_{a,C}G,B}G,A}G$.

(20) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\exists_{\exists_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\exists_{a,C}G,B}G,A}G$.

(21) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\forall_{\forall_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\forall_{a,C}G,B}G,A}G$.

(22) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\forall_{\exists_{a,C}G,A}G,B}G \Subset \exists_{\exists_{\exists_{a,C}G,B}G,A}G$.

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Predicate Calculus for Boolean Valued Functions. Part VIII

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC16`.

The terminology and notation used here are introduced in the following articles: [1], [2], [3], [4], and [5].

In this paper Y is a non empty set.

We now state a number of propositions:

(1) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B of Y holds $\neg\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(2) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(3) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

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(5) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

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(6) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(7) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B, C of Y holds $\forall_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(8) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B, C of Y holds $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(9) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\exists_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(10) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(11) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(12) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(13) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\exists_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(14) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\forall_{\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

(15) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

(16) For every element a of $\mathrm{BVF}(Y)$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for all partitions A, B, C of Y holds $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\forall_{\exists_{a,B}G,A}G$.

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[1] Shunichi Kobayashi and Kui Jia. A theory of Boolean valued functions and partitions. *Formalized Mathematics*, 7(**2**):249–254, 1998.

[2] Shunichi Kobayashi and Yatsuka Nakamura. A theory of Boolean valued functions and quantifiers with respect to partitions. *Formalized Mathematics*, 7(**2**):307–312, 1998.

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The terminology and notation used in this paper are introduced in the following papers: [1], [2], [3], [4], and [5].

In this paper Y is a non empty set.

The following propositions are true:

(1) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\exists_{\exists_{a,B}G,A}G$.

(2) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\forall_{\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(3) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\exists_{\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(4) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\forall_{\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(5) Let a be an element of BVF(Y), G be a subset of PARTITIONS(Y), and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

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(6) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg \exists_{\forall_{a,A} G, B} G \Subset \exists_{\neg \forall_{a,B} G, A} G$.

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(10) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg \exists_{\exists_{a,A} G, B} G \Subset \forall_{\neg \forall_{a,B} G, A} G$.

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(16) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg \exists_{\exists_{a,A} G, B} G \Subset \forall_{\exists_{\neg a,B} G, A} G$.

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Asymptotic Notation. Part I: Theory[1]

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Summary. The widely used textbook by Brassard and Bratley [2] includes a chapter devoted to asymptotic notation (Chapter 3, pp. 79–97). We have attempted to test how suitable the current version of Mizar is for recording this type of material in its entirety. A more detailed report on this experiment will be available separately. This article presents the development of notions and a follow-up article [9] includes examples and solutions to problems. The preliminaries introduce a number of properties of real sequences, some operations on real sequences, and a characterization of convergence. The remaining sections in this article correspond to sections of Chapter 3 of [2]. Section 2 defines the O notation and proves the threshold, maximum, and limit rules. Section 3 introduces the Ω and Θ notations and their analogous rules. Conditional asymptotic notation is defined in Section 4 where smooth functions are also discussed. Section 5 defines some operations on asymptotic notation (we have decided not to introduce the asymptotic notation for functions of several variables as it is a straightforward generalization of notions for unary functions).

MML Identifier: ASYMPT_0.

The terminology and notation used in this paper have been introduced in the following articles: [13], [11], [3], [4], [8], [1], [10], [5], [14], [7], [6], and [12].

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1. Preliminaries

In this paper c, d denote real numbers and n, N denote natural numbers.

In this article we present several logical schemes. The scheme *FinSegRng1* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a non empty set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

$\{\mathcal{F}(i); i$ ranges over natural numbers: $\mathcal{A} \leqslant i \ \wedge \ i \leqslant \mathcal{B}\}$ is a finite non empty subset of $\mathcal{C}$

provided the parameters meet the following requirement:

- $\mathcal{A} \leqslant \mathcal{B}$.

The scheme *FinImInit1* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

$\{\mathcal{F}(n); n$ ranges over natural numbers: $n \leqslant \mathcal{A}\}$ is a finite non empty subset of $\mathcal{B}$

for all values of the parameters.

The scheme *FinImInit2* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

$\{\mathcal{F}(n); n$ ranges over natural numbers: $n < \mathcal{A}\}$ is a finite non empty subset of $\mathcal{B}$

provided the parameters meet the following requirement:

- $\mathcal{A} > 0$.

Let c be a real number. We say that c is positive if and only if:

(Def. 1) $c > 0$.

We say that c is negative if and only if:

(Def. 2) $c < 0$.

We say that c is logbase if and only if:

(Def. 3) $c > 0$ and $c \neq 1$.

One can check the following observations:

* there exists a real number which is positive,
* there exists a real number which is negative,
* there exists a real number which is logbase,
* there exists a real number which is non negative,
* there exists a real number which is non positive, and
* there exists a real number which is non logbase.

Let f be a sequence of real numbers. We say that f is eventually-nonnegative if and only if:

(Def. 4) There exists N such that for every n such that $n \geqslant N$ holds $f(n) \geqslant 0$.

We say that f is positive if and only if:

(Def. 5) For every n holds $f(n) > 0$.

We say that f is eventually-positive if and only if:

(Def. 6) There exists N such that for every n such that $n \geqslant N$ holds $f(n) > 0$.

We say that f is eventually-nonzero if and only if:

(Def. 7) There exists N such that for every n such that $n \geqslant N$ holds $f(n) \neq 0$.

We say that f is eventually-nondecreasing if and only if:

(Def. 8) There exists N such that for every n such that $n \geqslant N$ holds $f(n) \leqslant f(n+1)$.

Let us mention that there exists a sequence of real numbers which is eventually-nonnegative, eventually-nonzero, positive, eventually-positive, and eventually-nondecreasing.

One can verify the following observations:

* every sequence of real numbers which is positive is also eventually-positive,
* every sequence of real numbers which is eventually-positive is also eventually-nonnegative and eventually-nonzero, and
* every sequence of real numbers which is eventually-nonnegative and eventually-nonzero is also eventually-positive.

Let f, g be eventually-nonnegative sequences of real numbers. Note that $f + g$ is eventually-nonnegative.

Let f be a sequence of real numbers and let c be a real number. The functor $c + f$ yields a sequence of real numbers and is defined by:

(Def. 9) For every n holds $(c + f)(n) = c + f(n)$.

We introduce $f + c$ as a synonym of $c + f$.

Let f be an eventually-nonnegative sequence of real numbers and let c be a positive real number. One can check that $c\,f$ is eventually-nonnegative.

Let f be an eventually-nonnegative sequence of real numbers and let c be a non negative real number. Note that $c + f$ is eventually-nonnegative.

Let f be an eventually-nonnegative sequence of real numbers and let c be a positive real number. One can check that $c + f$ is eventually-positive.

Let f, g be sequences of real numbers. The functor $\max(f, g)$ yielding a sequence of real numbers is defined as follows:

(Def. 10) For every n holds $(\max(f, g))(n) = \max(f(n), g(n))$.

Let us notice that the functor $\max(f, g)$ is commutative.

Let f be a sequence of real numbers and let g be an eventually-nonnegative sequence of real numbers. One can check that $\max(f, g)$ is eventually-nonnegative.

Let f be a sequence of real numbers and let g be an eventually-positive sequence of real numbers. One can verify that $\max(f, g)$ is eventually-positive.

Let f, g be sequences of real numbers. We say that g majorizes f if and only if:

(Def. 11) There exists N such that for every n such that $n \geqslant N$ holds $f(n) \leqslant g(n)$.

The following propositions are true:

(1) Let f be a sequence of real numbers and N be a natural number. Suppose that for every n such that $n \geqslant N$ holds $f(n) \leqslant f(n+1)$. Let n, m be natural numbers. If $N \leqslant n$ and $n \leqslant m$, then $f(n) \leqslant f(m)$.

(2) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) \neq 0$, then g/f is convergent and $\lim(g/f) = (\lim(f/g))^{-1}$.

(3) For every eventually-nonnegative sequence f of real numbers such that f is convergent holds $0 \leqslant \lim f$.

(4) Let f, g be sequences of real numbers. If f is convergent and g is convergent and g majorizes f, then $\lim f \leqslant \lim g$.

(5) Let f be a sequence of real numbers and g be an eventually-nonzero sequence of real numbers. If f/g is divergent to $+\infty$, then g/f is convergent and $\lim(g/f) = 0$.

2. A Notation for "the order of"

Let f be an eventually-nonnegative sequence of real numbers. The functor $O(f)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined by:

(Def. 12) $O(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,N}$ $(c > 0 \ \wedge \ \bigwedge_n (n \geqslant N \ \Rightarrow t(n) \leqslant c \cdot f(n) \ \wedge \ t(n) \geqslant 0))\}$.

The following propositions are true:

(6) Let x be a set and f be an eventually-nonnegative sequence of real numbers. Suppose $x \in O(f)$. Then x is an eventually-nonnegative sequence of real numbers.

(7) Let f be a positive sequence of real numbers and t be an eventually-nonnegative sequence of real numbers. Then $t \in O(f)$ if and only if there exists c such that $c > 0$ and for every n holds $t(n) \leqslant c \cdot f(n)$.

(8) Let f be an eventually-positive sequence of real numbers, t be an eventually-nonnegative sequence of real numbers, and N be a natural number. Suppose $t \in O(f)$ and for every n such that $n \geqslant N$ holds $f(n) > 0$. Then there exists c such that $c > 0$ and for every n such that $n \geqslant N$ holds $t(n) \leqslant c \cdot f(n)$.

(9) For all eventually-nonnegative sequences f, g of real numbers holds $O(f+g) = O(\max(f,g))$.

(10) For every eventually-nonnegative sequence f of real numbers holds $f \in O(f)$.

(11) For all eventually-nonnegative sequences f, g of real numbers such that $f \in O(g)$ holds $O(f) \subseteq O(g)$.

(12) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in O(g)$ and $g \in O(h)$ holds $f \in O(h)$.

(13) Let f be an eventually-nonnegative sequence of real numbers and c be a positive real number. Then $O(f) = O(c\,f)$.

(14) Let c be a non negative real number and x, f be eventually-nonnegative sequences of real numbers. If $x \in O(f)$, then $x \in O(c+f)$.

(15) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $O(f) = O(g)$.

(16) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $f \in O(g)$ and $g \notin O(f)$.

(17) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $f \notin O(g)$ and $g \in O(f)$.

3. Other Asymptotic Notation

Let f be an eventually-nonnegative sequence of real numbers. The functor $\Omega(f)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined by:

(Def. 13) $\Omega(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{d,N}\ (d > 0\ \wedge\ \bigwedge_n\ (n \geqslant N\ \Rightarrow\ t(n) \geqslant d \cdot f(n)\ \wedge\ t(n) \geqslant 0))\}$.

The following propositions are true:

(18) Let x be a set and f be an eventually-nonnegative sequence of real numbers. Suppose $x \in \Omega(f)$. Then x is an eventually-nonnegative sequence of real numbers.

(19) For all eventually-nonnegative sequences f, g of real numbers holds $f \in \Omega(g)$ iff $g \in O(f)$.

(20) For every eventually-nonnegative sequence f of real numbers holds $f \in \Omega(f)$.

(21) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in \Omega(g)$ and $g \in \Omega(h)$ holds $f \in \Omega(h)$.

(22) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $\Omega(f) = \Omega(g)$.

(23) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $g \in \Omega(f)$ and $f \notin \Omega(g)$.

(24) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $g \notin \Omega(f)$ and $f \in \Omega(g)$.

(25) Let f, t be positive sequences of real numbers. Then $t \in \Omega(f)$ if and only if there exists d such that $d > 0$ and for every n holds $d \cdot f(n) \leqslant t(n)$.

(26) For all eventually-nonnegative sequences f, g of real numbers holds $\Omega(f+g) = \Omega(\max(f, g))$.

Let f be an eventually-nonnegative sequence of real numbers. The functor $\Theta(f)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined as follows:

(Def. 14) $\Theta(f) = O(f) \cap \Omega(f)$.

Next we state several propositions:

(27) Let f be an eventually-nonnegative sequence of real numbers. Then $\Theta(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,d,N} (c > 0 \wedge d > 0 \wedge \bigwedge_n (n \geqslant N \Rightarrow d \cdot f(n) \leqslant t(n) \wedge t(n) \leqslant c \cdot f(n)))\}$.

(28) For every eventually-nonnegative sequence f of real numbers holds $f \in \Theta(f)$.

(29) For all eventually-nonnegative sequences f, g of real numbers such that $f \in \Theta(g)$ holds $g \in \Theta(f)$.

(30) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in \Theta(g)$ and $g \in \Theta(h)$ holds $f \in \Theta(h)$.

(31) Let f, t be positive sequences of real numbers. Then $t \in \Theta(f)$ if and only if there exist c, d such that $c > 0$ and $d > 0$ and for every n holds $d \cdot f(n) \leqslant t(n)$ and $t(n) \leqslant c \cdot f(n)$.

(32) For all eventually-nonnegative sequences f, g of real numbers holds $\Theta(f+g) = \Theta(\max(f, g))$.

(33) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $f \in \Theta(g)$.

(34) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $f \in O(g)$ and $f \notin \Theta(g)$.

(35) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $f \in \Omega(g)$ and $f \notin \Theta(g)$.

4. Conditional Asymptotic Notation

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $O(f|X)$ yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined as follows:

(Def. 15) $O(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,N} (c > 0 \wedge \bigwedge_n (n \geqslant N \wedge n \in X \Rightarrow t(n) \leqslant c \cdot f(n) \wedge t(n) \geqslant 0))\}$.

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $\Omega(f|X)$ yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined by:

(Def. 16) $\Omega(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{d,N} (d > 0 \wedge \bigwedge_n (n \geqslant N \wedge n \in X \Rightarrow t(n) \geqslant d \cdot f(n) \wedge t(n) \geqslant 0))\}$.

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $\Theta(f|X)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined by the condition (Def. 17).

(Def. 17) $\Theta(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,d,N} (c > 0 \ \wedge \ d > 0 \ \wedge \ \bigwedge_n (n \geqslant N \ \wedge \ n \in X \ \Rightarrow \ d \cdot f(n) \leqslant t(n) \ \wedge \ t(n) \leqslant c \cdot f(n)))\}$.

Next we state the proposition

(36) For every eventually-nonnegative sequence f of real numbers and for every set X holds $\Theta(f|X) = O(f|X) \cap \Omega(f|X)$.

Let f be a sequence of real numbers and let b be a natural number. The functor f_b yielding a sequence of real numbers is defined by:

(Def. 18) For every n holds $f_b(n) = f(b \cdot n)$.

Let f be an eventually-nonnegative sequence of real numbers and let b be a natural number. We say that f is smooth w.r.t. b if and only if:

(Def. 19) f is eventually-nondecreasing and $f_b \in O(f)$.

Let f be an eventually-nonnegative sequence of real numbers. We say that f is smooth if and only if:

(Def. 20) For every natural number b such that $b \geqslant 2$ holds f is smooth w.r.t. b.

We now state four propositions:

(37) Let f be an eventually-nonnegative sequence of real numbers. Given a natural number b such that $b \geqslant 2$ and f is smooth w.r.t. b. Then f is smooth.

(38) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geqslant 2$ and $t \in O(f|\{b^n : n$ ranges over natural numbers$\})$. Then $t \in O(f)$.

(39) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geqslant 2$ and $t \in \Omega(f|\{b^n : n$ ranges over natural numbers$\})$. Then $t \in \Omega(f)$.

(40) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geqslant 2$ and $t \in \Theta(f|\{b^n : n$ ranges over natural numbers$\})$. Then $t \in \Theta(f)$.

5. Operations on Asymptotic Notation

Let X be a non empty set and let F, G be non empty sets of functions from X to $\mathbb{R}$. The functor $F + G$ yields a non empty set of functions from X to $\mathbb{R}$ and is defined by the condition (Def. 21).

(Def. 21) $F+G = \{t; t$ ranges over elements of $\mathbb{R}^X$: $\bigvee_{f,g\,:\,\text{element of }\mathbb{R}^X} (f \in F \wedge g \in G \ \wedge \ \bigwedge_{n\,:\,\text{element of }X} \ t(n) = f(n) + g(n))\}$.

Let X be a non empty set and let F, G be non empty sets of functions from X to $\mathbb{R}$. The functor $\max(F, G)$ yields a non empty set of functions from X to $\mathbb{R}$ and is defined by the condition (Def. 22).

(Def. 22) $\max(F, G) = \{t; t$ ranges over elements of $\mathbb{R}^X$: $\bigvee_{f,g\,:\,\text{element of } \mathbb{R}^X} (f \in F \wedge g \in G \wedge \bigwedge_{n\,:\,\text{element of } X} t(n) = \max(f(n), g(n)))\}$.

Next we state two propositions:

(41) For all eventually-nonnegative sequences f, g of real numbers holds $O(f) + O(g) = O(f + g)$.

(42) For all eventually-nonnegative sequences f, g of real numbers holds $\max(O(f), O(g)) = O(\max(f, g))$.

Let F, G be non empty sets of functions from $\mathbb{N}$ to $\mathbb{R}$. The functor F^G yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined by the condition (Def. 23).

(Def. 23) $F^G = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{f,g\,:\,\text{element of } \mathbb{R}^{\mathbb{N}}} \bigvee_{N\,:\,\text{element of } \mathbb{N}} (f \in F \wedge g \in G \wedge \bigwedge_{n\,:\,\text{element of } \mathbb{N}} (n \geqslant N \Rightarrow t(n) = f(n)^{g(n)}))\}$.

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Asymptotic Notation. Part II: Examples and Problems[1]

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Summary. The widely used textbook by Brassard and Bratley [2] includes a chapter devoted to asymptotic notation (Chapter 3, pp. 79–97). We have attempted to test how suitable the current version of Mizar is for recording this type of material in its entirety. This article is a follow-up to [11] in which we introduced the basic notions and general theory. This article presents a Mizar formalization of examples and solutions to problems from Chapter 3 of [2] (some of the examples and solved problems are also in [11]). Not all problems have been solved as some required solutions not amenable for formalization.

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The articles [11], [10], [14], [15], [3], [4], [17], [1], [12], [13], [6], [19], [8], [9], [7], [16], [18], and [5] provide the terminology and notation for this paper.

1. Examples from the Text

We adopt the following rules: c, e denote real numbers, k, n, m, N, n_1, M denote natural numbers, and x denotes a set.

One can prove the following two propositions:

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(1) Let t, t_1 be sequences of real numbers. Suppose that

(i) $t(0) = 0$,

(ii) for every n such that $n > 0$ holds $t(n) = (12 \cdot n^3 \cdot \log_2 n - 5 \cdot n^{\mathbf{2}}) + (\log_2 n)^{\mathbf{2}} + 36$,

(iii) $t_1(0) = 0$, and

(iv) for every n such that $n > 0$ holds $t_1(n) = n^3 \cdot \log_2 n$.

Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = t$ and $s_1 = t_1$ and $s \in O(s_1)$.

(2) Let a, b be logbase real numbers and f, g be sequences of real numbers. Suppose $a > 1$ and $b > 1$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_a n$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = \log_b n$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $O(s) = O(s_1)$.

Let a, b, c be real numbers. The functor $\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined by:

(Def. 1) $(\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}})(n) = a^{b \cdot n + c}$.

Let a be a positive real number and let b, c be real numbers. One can verify that $\{a^{b \cdot n + c)}\}_{n \in \mathbb{N}}$ is eventually-positive.

The following proposition is true

(3) For all positive real numbers a, b such that $a < b$ holds $\{b^{1 \cdot n + 0)}\}_{n \in \mathbb{N}} \notin O(\{a^{1 \cdot n + 0)}\}_{n \in \mathbb{N}})$.

The sequence $\{\log_2 n\}_{n \in \mathbb{N}}$ of real numbers is defined as follows:

(Def. 2) $\{\log_2 n\}_{n \in \mathbb{N}}(0) = 0$ and for every n such that $n > 0$ holds $\{\log_2 n\}_{n \in \mathbb{N}}(n) = \log_2 n$.

Let a be a real number. The functor $\{n^a\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined as follows:

(Def. 3) $\{n^a\}_{n \in \mathbb{N}}(0) = 0$ and for every n such that $n > 0$ holds $\{n^a\}_{n \in \mathbb{N}}(n) = n^a$.

Let us mention that $\{\log_2 n\}_{n \in \mathbb{N}}$ is eventually-positive.

Let a be a real number. Observe that $\{n^a\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state several propositions:

(4) Let f, g be eventually-nonnegative sequences of real numbers. Then $O(f) \subseteq O(g)$ and $O(f) \neq O(g)$ if and only if $f \in O(g)$ and $f \notin \Omega(g)$.

(5) $O(\{\log_2 n\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$ and $O(\{\log_2 n\}_{n \in \mathbb{N}}) \neq O(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(6) $\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}} \in \Omega(\{\log_2 n\}_{n \in \mathbb{N}})$ and $\{\log_2 n\}_{n \in \mathbb{N}} \notin \Omega(\{n^{(\frac{1}{2})}\}_{n \in \mathbb{N}})$.

(7) For every sequence f of real numbers and for every natural number k such that for every n holds $f(n) = \sum_{\kappa=0}^{n}(\{n^k\}_{n \in \mathbb{N}})(\kappa)$ holds $f \in \Theta(\{n^{(k+1)}\}_{n \in \mathbb{N}})$.

(8) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and for every

n such that $n > 0$ holds $f(n) = n^{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and s is not smooth.

Let b be a real number. The functor $\{b\}_{n \in \mathbb{N}}$ yields a sequence of real numbers and is defined as follows:

(Def. 4) $\{b\}_{n \in \mathbb{N}} = \mathbb{N} \longmapsto b$.

Let us note that $\{1\}_{n \in \mathbb{N}}$ is eventually-nonnegative.

One can prove the following proposition

(9) Let f be an eventually-nonnegative sequence of real numbers. Then there exists a non empty set F of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n \in \mathbb{N}}\}$ and $f \in F^{O(\{1\}_{n \in \mathbb{N}})}$ iff there exist N, c, k such that $c > 0$ and for every n such that $n \geqslant N$ holds $1 \leqslant f(n)$ and $f(n) \leqslant c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$.

2. Problem 3.1

One can prove the following proposition

(10) For every sequence f of real numbers such that for every n holds $f(n) = (3 \cdot 10^6 - 18 \cdot 10^3 \cdot n) + 27 \cdot n^{\mathbf{2}}$ holds $f \in O(\{n^2\}_{n \in \mathbb{N}})$.

3. Problem 3.5

We now state three propositions:

(11) $\{n^2\}_{n \in \mathbb{N}} \in O(\{n^3\}_{n \in \mathbb{N}})$.

(12) $\{n^2\}_{n \in \mathbb{N}} \notin \Omega(\{n^3\}_{n \in \mathbb{N}})$.

(13) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{1 \cdot n+1)}\}_{n \in \mathbb{N}}$ and $\{2^{1 \cdot n+0)}\}_{n \in \mathbb{N}} \in \Theta(s)$.

Let a be a natural number. The functor $\{(n+a)!\}_{n \in \mathbb{N}}$ yielding a sequence of real numbers is defined by:

(Def. 5) $\{(n+a)!\}_{n \in \mathbb{N}}(n) = (n+a)!$.

Let a be a natural number. Observe that $\{(n+a)!\}_{n \in \mathbb{N}}$ is eventually-positive.

We now state the proposition

(14) $\{(n+0)!\}_{n \in \mathbb{N}} \notin \Theta(\{(n+1)!\}_{n \in \mathbb{N}})$.

4. Problem 3.6

The following proposition is true

(15) For every sequence f of real numbers such that $f \in O(\{n^1\}_{n \in \mathbb{N}})$ holds $f\, f \in O(\{n^2\}_{n \in \mathbb{N}})$.

5. Problem 3.7

We now state the proposition

(16) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $2\,\{n^1\}_{n \in \mathbb{N}} \in O(\{n^1\}_{n \in \mathbb{N}})$ and $\{2^{2 \cdot n + 0)}\}_{n \in \mathbb{N}} \notin O(s)$.

6. Problem 3.8

One can prove the following proposition

(17) If $\log_2 3 < \frac{159}{100}$, then $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \in O(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \notin \Omega(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$ and $\{n^{(\log_2 3)}\}_{n \in \mathbb{N}} \notin \Theta(\{n^{(\frac{159}{100})}\}_{n \in \mathbb{N}})$.

7. Problem 3.11

We now state the proposition

(18) Let f, g be sequences of real numbers. Suppose for every n holds $f(n) = n \bmod 2$ and for every n holds $g(n) = (n+1) \bmod 2$. Then there exist eventually-nonnegative sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $s \notin O(s_1)$ and $s_1 \notin O(s)$.

8. Problem 3.19

We now state two propositions:

(19) For all eventually-nonnegative sequences f, g of real numbers holds $O(f) = O(g)$ iff $f \in \Theta(g)$.

(20) For all eventually-nonnegative sequences f, g of real numbers holds $f \in \Theta(g)$ iff $\Theta(f) = \Theta(g)$.

9. Problem 3.21

The following propositions are true:

(21) Let e be a real number and f be a sequence of real numbers. Suppose $0 < e$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n \cdot \log_2 n$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^{(1+e)}\}_{n \in \mathbb{N}})$.

(22) Let e be a real number and g be a sequence of real numbers. Suppose $0 < e$ and $e < 1$ and $g(0) = 0$ and $g(1) = 0$ and for every n such that $n > 1$ holds $g(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^{(1+e)}\}_{n \in \mathbb{N}}) \neq O(s)$.

(23) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and $f(1) = 0$ and for every n such that $n > 1$ holds $f(n) = \frac{n^2}{\log_2 n}$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and $O(s) \subseteq O(\{n^8\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{n^8\}_{n \in \mathbb{N}})$.

(24) Let g be a sequence of real numbers. Suppose that for every n holds $g(n) = ((n^{\mathbf{2}} - n) + 1)^4$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{n^8\}_{n \in \mathbb{N}}) = O(s)$.

(25) Let e be a real number. Suppose $0 < e$ and $e < 1$. Then there exists an eventually-positive sequence s of real numbers such that $s = \{1 + e^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(\{n^8\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{n^8\}_{n \in \mathbb{N}}) \neq O(s)$.

10. Problem 3.22

One can prove the following propositions:

(26) Let f, g be sequences of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{\log_2 n}$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = g$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(27) Let f be a sequence of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{\sqrt{n}}$. Then there exist eventually-positive sequences s, s_1 of real numbers such that $s = f$ and $s_1 = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(28) There exist eventually-positive sequences s, s_1 of real numbers such that $s = \{2^{1 \cdot n + 0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{1 \cdot n + 1)}\}_{n \in \mathbb{N}}$ and $O(s) = O(s_1)$.

(29) There exist eventually-positive sequences s, s_1 of real numbers such that $s = \{2^{1 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $s_1 = \{2^{2 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(s_1)$ and $O(s) \neq O(s_1)$.

(30) There exists an eventually-positive sequence s of real numbers such that $s = \{2^{2 \cdot n+0)}\}_{n \in \mathbb{N}}$ and $O(s) \subseteq O(\{(n+0)!\}_{n \in \mathbb{N}})$ and $O(s) \neq O(\{(n+0)!\}_{n \in \mathbb{N}})$.

(31) $O(\{(n+0)!\}_{n \in \mathbb{N}}) \subseteq O(\{(n+1)!\}_{n \in \mathbb{N}})$ and $O(\{(n+0)!\}_{n \in \mathbb{N}}) \neq O(\{(n+1)!\}_{n \in \mathbb{N}})$.

(32) Let g be a sequence of real numbers. Suppose $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \subseteq O(s)$ and $O(\{(n+1)!\}_{n \in \mathbb{N}}) \neq O(s)$.

11. Problem 3.23

One can prove the following proposition

(33) Let given n. Suppose $n \geqslant 1$. Let f be a sequence of real numbers and k be a natural number. If for every n holds $f(n) = \sum_{\kappa=0}^{n}(\{n^k\}_{n \in \mathbb{N}})(\kappa)$, then $f(n) \geqslant \frac{n^{k+1}}{k+1}$.

12. Problem 3.24

One can prove the following proposition

(34) Let f, g be sequences of real numbers. Suppose $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n \cdot \log_2 n$ and for every n holds $f(n) = \log_2(n!)$. Then there exists an eventually-nonnegative sequence s of real numbers such that $s = g$ and $f \in \Theta(s)$.

13. Problem 3.26

The following proposition is true

(35) Let f be an eventually-nondecreasing eventually-nonnegative sequence of real numbers and t be a sequence of real numbers. Suppose that for every n holds if $n \bmod 2 = 0$, then $t(n) = 1$ and if $n \bmod 2 = 1$, then $t(n) = n$. Then $t \notin \Theta(f)$.

14. Problem 3.28

Let f be a function from $\mathbb{N}$ into $\mathbb{R}^*$ and let n be a natural number. Then $f(n)$ is a finite sequence of elements of $\mathbb{R}$.

Let n be a natural number and let a, b be positive real numbers. The functor $\mathrm{Prob28}(n, a, b)$ yields a real number and is defined by:

(Def. 6)(i) $\mathrm{Prob28}(n, a, b) = 0$ if $n = 0$,

(ii) there exists a natural number l and there exists a function p_{28} from $\mathbb{N}$ into $\mathbb{R}^*$ such that $l+1 = n$ and $\mathrm{Prob28}(n, a, b) = \pi_n p_{28}(l)$ and $p_{28}(0) = \langle a \rangle$ and for every natural number n there exists a natural number n_1 such that $n_1 = \lceil \frac{n+1+1}{2} \rceil$ and $p_{28}(n + 1) = p_{28}(n) \frown \langle 4 \cdot \pi_{n_1} p_{28}(n) + b \cdot (n + 1 + 1) \rangle$, otherwise.

Let a, b be positive real numbers. The functor $\{\mathrm{Prob28}(n, a, b)\}_{n\in\mathbb{N}}$ yields a sequence of real numbers and is defined by:

(Def. 7) $(\{\mathrm{Prob28}(n, a, b)\}_{n\in\mathbb{N}})(n) = \mathrm{Prob28}(n, a, b)$.

The following proposition is true

(36) For all positive real numbers a, b holds $\{\mathrm{Prob28}(n, a, b)\}_{n\in\mathbb{N}}$ is eventually-nondecreasing.

15. Problem 3.30

The non empty subset $\{2^n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is defined by:

(Def. 8) $\{2^n : n \in \mathbb{N}\} = \{2^n : n$ ranges over natural numbers$\}$.

Next we state three propositions:

(37) Let f be a sequence of real numbers. Suppose that for every n holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $f(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $f(n) = 2^n$. Then $f \in \Theta(\{n^1\}_{n\in\mathbb{N}}|\{2^n : n \in \mathbb{N}\})$ and $f \notin \Theta(\{n^1\}_{n\in\mathbb{N}})$ and $\{n^1\}_{n\in\mathbb{N}}$ is smooth and f is not eventually-nondecreasing.

(38) Let f, g be sequences of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = n^{2^{\lfloor \log_2 n \rfloor}}$ and $g(0) = 0$ and for every n such that $n > 0$ holds $g(n) = n^n$. Then there exists an eventually-positive sequence s of real numbers such that

(i) $s = g$,

(ii) $f \in \Theta(s|\{2^n : n \in \mathbb{N}\})$,

(iii) $f \notin \Theta(s)$,

(iv) f is eventually-nondecreasing,

(v) s is eventually-nondecreasing, and

(vi) s is not smooth w.r.t. 2.

(39) Let g be a sequence of real numbers. Suppose that for every n holds if $n \in \{2^n : n \in \mathbb{N}\}$, then $g(n) = n$ and if $n \notin \{2^n : n \in \mathbb{N}\}$, then $g(n) = n^2$. Then there exists an eventually-positive sequence s of real numbers such that $s = g$ and $\{n^1\}_{n \in \mathbb{N}} \in \Theta(s|\{2^n : n \in \mathbb{N}\})$ and $\{n^1\}_{n \in \mathbb{N}} \notin \Theta(s)$ and $s_2 \in O(s)$ and $\{n^1\}_{n \in \mathbb{N}}$ is eventually-nondecreasing and s is not eventually-nondecreasing.

16. Problem 3.31

Let x be a natural number. The functor $x¡$ yielding a natural number is defined as follows:

(Def. 9)(i) There exists n such that $n! \leqslant x$ and $x < (n+1)!$ and $x¡ = n!$ if $x \neq 0$,

(ii) $x¡ = 0$, otherwise.

Next we state the proposition

(40) Let f be a sequence of real numbers. Suppose that for every n holds $f(n) = n¡$. Then there exists an eventually-positive sequence s of real numbers such that $s = f$ and f is eventually-nondecreasing and for every n holds $f(n) \leqslant \{n^1\}_{n \in \mathbb{N}}(n)$ and s is not smooth.

17. Problem 3.34

Let us mention that $\{n^1\}_{n \in \mathbb{N}} - \{1\}_{n \in \mathbb{N}}$ is eventually-positive.

One can prove the following proposition

(41) $\Theta(\{n^1\}_{n \in \mathbb{N}} - \{1\}_{n \in \mathbb{N}}) + \Theta(\{n^1\}_{n \in \mathbb{N}}) = \Theta(\{n^1\}_{n \in \mathbb{N}})$.

18. Problem 3.35

One can prove the following proposition

(42) There exists a non empty set F of functions from $\mathbb{N}$ to $\mathbb{R}$ such that $F = \{\{n^1\}_{n \in \mathbb{N}}\}$ and for every n holds $\{n^{(-1)}\}_{n \in \mathbb{N}}(n) \leqslant \{n^1\}_{n \in \mathbb{N}}(n)$ and $\{n^{(-1)}\}_{n \in \mathbb{N}} \notin F^{O(\{1\}_{n \in \mathbb{N}})}$.

19. Addition

The following proposition is true

(43) Let c be a non negative real number and x, f be eventually-nonnegative sequences of real numbers. Given e, N such that $e > 0$ and for every n such that $n \geqslant N$ holds $f(n) \geqslant e$. If $x \in O(c+f)$, then $x \in O(f)$.

20. Potentatially Useful

The following propositions are true:

(44) $2^2 = 4$.

(45) $2^3 = 8$.

(46) $2^4 = 16$.

(47) $2^5 = 32$.

(48) $2^6 = 64$.

(49) $2^{12} = 4096$.

(50) For every n such that $n \geqslant 3$ holds $n^{\mathbf{2}} > 2 \cdot n + 1$.

(51) For every n such that $n \geqslant 10$ holds $2^{n-1} > (2 \cdot n)^{\mathbf{2}}$.

(52) For every n such that $n \geqslant 9$ holds $(n+1)^6 < 2 \cdot n^6$.

(53) For every n such that $n \geqslant 30$ holds $2^n > n^6$.

(54) For every real number x such that $x > 9$ holds $2^x > (2 \cdot x)^{\mathbf{2}}$.

(55) There exists N such that for every n such that $n \geqslant N$ holds $\sqrt{n} - \log_2 n > 1$.

(56) For all real numbers a, b, c such that $a > 0$ and $c > 0$ and $c \neq 1$ holds $a^b = c^{b \cdot \log_c a}$.

(57) $(4+1)! = 120$.

(58) $5^5 = 3125$.

(59) $4^4 = 256$.

(60) For every n holds $(n^{\mathbf{2}} - n) + 1 > 0$.

(61) For every n such that $n \geqslant 2$ holds $n! > 1$.

(62) For all n_1, n such that $n \leqslant n_1$ holds $n! \leqslant n_1!$.

(63) For every k such that $k \geqslant 1$ there exists n such that $n! \leqslant k$ and $k < (n+1)!$ and for every m such that $m! \leqslant k$ and $k < (m+1)!$ holds $m = n$.

(64) For every n such that $n \geqslant 2$ holds $\lceil \frac{n}{2} \rceil < n$.

(65) For every n such that $n \geqslant 3$ holds $n! > n$.

(66) For all natural numbers m, n such that $m > 0$ holds m^n is a natural number.

(67) For every n such that $n \geqslant 2$ holds $2^n > n + 1$.

(68) Let a be a logbase real number and f be a sequence of real numbers. Suppose $a > 1$ and $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_a n$. Then f is eventually-positive.

(69) For all eventually-nonnegative sequences f, g of real numbers holds $f \in O(g)$ and $g \in O(f)$ iff $O(f) = O(g)$.

(70) For all real numbers a, b, c such that $0 < a$ and $a \leqslant b$ and $c \geqslant 0$ holds $a^c \leqslant b^c$.

(71) For every n such that $n \geqslant 4$ holds $2 \cdot n + 3 < 2^n$.

(72) For every n such that $n \geqslant 6$ holds $(n + 1)^{\mathbf{2}} < 2^n$.

(73) For every real number c such that $c > 6$ holds $c^{\mathbf{2}} < 2^c$.

(74) Let e be a positive real number and f be a sequence of real numbers. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = \log_2(n^e)$. Then $f/\{n^e\}_{n \in \mathbb{N}}$ is convergent and $\lim(f/\{n^e\}_{n \in \mathbb{N}}) = 0$.

(75) For every real number e such that $e > 0$ holds $\{\log_2 n\}_{n \in \mathbb{N}}/\{n^e\}_{n \in \mathbb{N}}$ is convergent and $\lim(\{\log_2 n\}_{n \in \mathbb{N}}/\{n^e\}_{n \in \mathbb{N}}) = 0$.

(76) For every sequence f of real numbers and for every N such that for every n such that $n \leqslant N$ holds $f(n) \geqslant 0$ holds $\sum_{\kappa=0}^{N} f(\kappa) \geqslant 0$.

(77) For all sequences f, g of real numbers and for every N such that for every n such that $n \leqslant N$ holds $f(n) \leqslant g(n)$ holds $\sum_{\kappa=0}^{N} f(\kappa) \leqslant \sum_{\kappa=0}^{N} g(\kappa)$.

(78) Let f be a sequence of real numbers and b be a real number. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = b$. Let N be a natural number. Then $\sum_{\kappa=0}^{N} f(\kappa) = b \cdot N$.

(79) For all sequences f, g of real numbers and for all natural numbers N, M holds $\sum_{\kappa=N+1}^{M} f(\kappa) + f(N + 1) = \sum_{\kappa=N+1+1}^{M} f(\kappa)$.

(80) Let f, g be sequences of real numbers, M be a natural number, and given N. Suppose $N \geqslant M + 1$. If for every n such that $M + 1 \leqslant n$ and $n \leqslant N$ holds $f(n) \leqslant g(n)$, then $\sum_{\kappa=N+1}^{M} f(\kappa) \leqslant \sum_{\kappa=N+1}^{M} g(\kappa)$.

(81) For every n holds $\lceil \frac{n}{2} \rceil \leqslant n$.

(82) Let f be a sequence of real numbers, b be a real number, and N be a natural number. Suppose $f(0) = 0$ and for every n such that $n > 0$ holds $f(n) = b$. Let M be a natural number. Then $\sum_{\kappa=N+1}^{M} f(\kappa) = b \cdot (N - M)$.

(83) Let f, g be sequences of real numbers, N be a natural number, and c be a real number. Suppose f is convergent and $\lim f = c$ and for every n such that $n \geqslant N$ holds $f(n) = g(n)$. Then g is convergent and $\lim g = c$.

(84) For every n such that $n \geqslant 1$ holds $(n^{\mathbf{2}} - n) + 1 \leqslant n^{\mathbf{2}}$.

(85) For every n such that $n \geqslant 1$ holds $n^{\mathbf{2}} \leqslant 2 \cdot ((n^{\mathbf{2}} - n) + 1)$.

(86) For every real number e such that $0 < e$ and $e < 1$ there exists N such that for every n such that $n \geqslant N$ holds $n \cdot \log_2(1+e) - 8 \cdot \log_2 n > 8 \cdot \log_2 n$.

(87) For every n such that $n \geqslant 10$ holds $\frac{2^{2 \cdot n}}{n!} < \frac{1}{2^{n-9}}$.

(88) For every n such that $n \geqslant 3$ holds $2 \cdot (n-2) \geqslant n-1$.

(89) For every real number c such that $c \geqslant 0$ holds $c^{\frac{1}{2}} = \sqrt{c}$.

(90) There exists N such that for every n such that $n \geqslant N$ holds $n - \sqrt{n} \cdot \log_2 n > \frac{n}{2}$.

(91) For every sequence s of real numbers such that for every n holds $s(n) = (1 + \frac{1}{n+1})^{n+1}$ holds s is non-decreasing.

(92) For every n such that $n \geqslant 1$ holds $(\frac{n+1}{n})^n \leqslant (\frac{n+2}{n+1})^{n+1}$.

(93) For all k, n such that $k \leqslant n$ holds $\binom{n}{k} \geqslant \frac{\binom{n+1}{k}}{n+1}$.

(94) For every sequence f of real numbers such that for every n holds $f(n) = \log_2(n!)$ and for every n holds $f(n) = \sum_{\kappa=0}^{n}(\{\log_2 n\}_{n \in \mathbb{N}})(\kappa)$.

(95) For every n such that $n \geqslant 4$ holds $n \cdot \log_2 n \geqslant 2 \cdot n$.

(96) Let a, b be positive real numbers. Then $\mathrm{Prob28}(0, a, b) = 0$ and $\mathrm{Prob28}(1, a, b) = a$ and for every n such that $n \geqslant 2$ there exists n_1 such that $n_1 = \lceil \frac{n}{2} \rceil$ and $\mathrm{Prob28}(n, a, b) = 4 \cdot \mathrm{Prob28}(n_1, a, b) + b \cdot n$.

(97) For every n such that $n \geqslant 2$ holds $n^{\mathbf{2}} > n + 1$.

(98) For every n such that $n \geqslant 1$ holds $2^{n+1} - 2^n > 1$.

(99) For every n such that $n \geqslant 2$ holds $2^n - 1 \notin \{2^n : n \in \mathbb{N}\}$.

(100) For all n, k such that $k \geqslant 1$ and $n! \leqslant k$ and $k < (n+1)!$ holds $k¡ = n!$.

(101) For all real numbers a, b, c such that $a > 1$ and $b \geqslant a$ and $c \geqslant 1$ holds $\log_a c \geqslant \log_b c$.

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Predicate Calculus for Boolean Valued Functions. Part X

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC18`.

The notation and terminology used here are introduced in the following articles: [1], [2], [3], [4], and [5].

In this paper Y is a non empty set.

One can prove the following propositions:

(1) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg \exists_{\exists_{a,A}G,B}G \Subset \exists_{\forall_{\neg a,B}G,A}G$.

(2) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\neg \exists_{\exists_{a,A}G,B}G \Subset \forall_{\forall_{\neg a,B}G,A}G$.

(3) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg \exists_{a,A}G,B}G \Subset \neg \exists_{\forall_{a,B}G,A}G$.

(4) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg \exists_{a,A}G,B}G \Subset \neg \exists_{\forall_{a,B}G,A}G$.

(5) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg \exists_{a,A}G,B}G \Subset \neg \forall_{\exists_{a,B}G,A}G$.

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(6) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\exists_{\exists_{a,B}G,A}G$.

(7) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(8) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(9) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(10) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(11) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\exists_{\neg\exists_{a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(12) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\exists_{a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(13) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\exists_{a,A}G,B}G \Subset \exists_{\neg\exists_{a,B}G,A}G$.

(14) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C be partitions of Y. Suppose G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$. Then $\forall_{\neg\exists_{a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.

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Predicate Calculus for Boolean Valued Functions. Part XI

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of usual predicate logic.

MML Identifier: `BVFUNC19`.

The terminology and notation used in this paper have been introduced in the following articles: [1], [2], [3], [4], and [5].

For simplicity, we adopt the following rules: Y is a non empty set, a is an element of $\mathrm{BVF}(Y)$, G is a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C are partitions of Y.

One can prove the following propositions:

(1) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\neg \exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(2) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\neg \exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(3) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\neg \exists_{a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(4) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\neg \exists_{a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(5) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\neg \exists_{a,A}G,B}G \Subset \exists_{\forall_{\neg a,B}G,A}G$.

(6) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\neg \exists_{a,A}G,B}G \Subset \forall_{\forall_{\neg a,B}G,A}G$.

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(7) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.
(8) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.
(9) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\exists_{a,B}G,A}G$.
(10) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \neg\exists_{\exists_{a,B}G,A}G$.
(11) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\exists_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.
(12) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.
(13) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.
(14) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.
(15) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.
(16) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.
(17) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\neg\exists_{a,B}G,A}G$.
(18) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.
(19) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\exists_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.
(21)[1] If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.
(22) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.
(23) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.
(24) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.
(25) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\forall_{\neg a,B}G,A}G$.
(26) If G is a coordinate and $G = \{A, B, C\}$ and $A \neq B$ and $B \neq C$ and $C \neq A$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\forall_{\neg a,B}G,A}G$.

[1]The proposition (20) has been removed.

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Four Variable Predicate Calculus for Boolean Valued Functions. Part I

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of ordinary predicate logic.

MML Identifier: `BVFUNC20`.

The terminology and notation used here have been introduced in the following articles: [10], [4], [6], [1], [8], [7], [2], [3], [5], [11], and [9].

1. Preliminaries

For simplicity, we follow the rules: Y is a non empty set, a is an element of $\mathrm{BVF}(Y)$, G is a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C, D are partitions of Y.

One can prove the following propositions:

(1) Let h be a function and A', B', C', D' be sets. Suppose $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$ and $h = (B \dot{\longmapsto} B') {+\cdot} (C \dot{\longmapsto} C') {+\cdot} (D \dot{\longmapsto} D') {+\cdot} (A \dot{\longmapsto} A')$. Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$.

(2) Let A, B, C, D be sets, h be a function, and A', B', C', D' be sets. If $h = (B \dot{\longmapsto} B') {+\cdot} (C \dot{\longmapsto} C') {+\cdot} (D \dot{\longmapsto} D') {+\cdot} (A \dot{\longmapsto} A')$, then $\mathrm{dom}\, h = \{A, B, C, D\}$.

(3) For every function h and for all sets A', B', C', D' such that $G = \{A, B, C, D\}$ and $h = (B \dot{\longmapsto} B') {+\cdot} (C \dot{\longmapsto} C') {+\cdot} (D \dot{\longmapsto} D') {+\cdot} (A \dot{\longmapsto} A')$ holds $\mathrm{rng}\, h = \{h(A), h(B), h(C), h(D)\}$.

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(4) Let z, u be elements of Y and h be a function. Suppose G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(5) Let z, u be elements of Y. Suppose G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$ and $\mathrm{EqClass}(z, C \wedge D) = \mathrm{EqClass}(u, C \wedge D)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

2. Four Variable Predicate Calculus

Next we state a number of propositions:

(6) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\forall_{a,B}G,A}G$.

(7) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(8) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(9) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\exists_{a,B}G,A}G \Subset \exists_{\exists_{a,A}G,B}G$.

(10) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\exists_{a,B}G,A}G = \exists_{\exists_{a,A}G,B}G$.

(11) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(12) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(13) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(14) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(15) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(16) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(17) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(18) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(19) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(20) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(21) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(22) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(23) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\exists_{\neg a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(24) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

(25) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

(26) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\exists_{\exists_{a,B}G,A}G$.

(27) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(28) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(29) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(30) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(31) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(32) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(33) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(34) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(35) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \forall_{\neg\forall_{a,B}G,A}G$.

(36) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \exists_{\neg\exists_{a,B}G,A}G$.

(37) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.

(38) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(39) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(40) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\forall_{\exists_{a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(41) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

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Four Variable Predicate Calculus for Boolean Valued Functions. Part II

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of ordinary predicate logic.

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The notation and terminology used here have been introduced in the following papers: [1], [2], [4], [3], and [5].

For simplicity, we use the following convention: Y is a non empty set, a is an element of $\mathrm{BVF}(Y)$, G is a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C, D are partitions of Y.

Next we state a number of propositions:

(1) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \exists_{\forall_{\neg a,B}G,A}G$.

(2) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\neg\exists_{\exists_{a,A}G,B}G \Subset \forall_{\forall_{\neg a,B}G,A}G$.

(3) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

(4) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\exists_{\forall_{a,B}G,A}G$.

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(5) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\forall_{\exists_{a,B}G,A}G$.

(6) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \neg\exists_{\exists_{a,B}G,A}G$.

(7) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(8) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\forall_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(9) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

(10) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \exists_{\neg\forall_{a,B}G,A}G$.

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(14) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.

(15) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(16) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg\exists_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(17) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\neg\exists_{a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(18) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$

and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\neg \exists_{a,A} G, B} G \Subset \forall_{\exists_{\neg a,B} G, A} G$.

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(21) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\forall_{\neg a,A} G, B} G \Subset \neg \exists_{\forall_{a,B} G, A} G$.

(22) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A} G, B} G \Subset \neg \exists_{\forall_{a,B} G, A} G$.

(23) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A} G, B} G \Subset \neg \forall_{\exists_{a,B} G, A} G$.

(24) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A} G, B} G \Subset \neg \exists_{\exists_{a,B} G, A} G$.

(25) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\exists_{\neg a,A} G, B} G \Subset \exists_{\neg \forall_{a,B} G, A} G$.

(26) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\exists_{\neg a,A} G, B} G \Subset \exists_{\neg \forall_{a,B} G, A} G$.

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(30) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A} G, B} G \Subset \forall_{\neg \forall_{a,B} G, A} G$.

(31) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A} G, B} G \Subset$

$\exists_{\neg\exists_{a,B}G,A}G$.

(32) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\neg\exists_{a,B}G,A}G$.

(33) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\exists_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(34) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\exists_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(35) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(36) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(37) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\exists_{\forall_{\neg a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(38) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\exists_{\neg a,B}G,A}G$.

(39) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \exists_{\forall_{\neg a,B}G,A}G$.

(40) If G is a coordinate and $G = \{A, B, C, D\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $B \neq C$ and $B \neq D$ and $C \neq D$, then $\forall_{\forall_{\neg a,A}G,B}G \Subset \forall_{\forall_{\neg a,B}G,A}G$.

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Function Spaces in the Category of Directed Suprema Preserving Maps[1]

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Summary. Formalization of [15, pp. 115–117], chapter II, section 2 (2.5 – 2.10).

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The notation and terminology used here are introduced in the following papers: [33], [2], [10], [11], [9], [1], [26], [3], [31], [16], [29], [23], [24], [27], [4], [34], [35], [32], [28], [14], [30], [17], [19], [22], [8], [6], [13], [7], [25], [21], [5], [18], [36], [20], and [12].

1. Currying, Uncurrying and Commuting Functions

Let F be a function. We say that F is uncurrying if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function yielding function, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{uncurry} f$.

We say that F is currying if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function and $\pi_1(x)$ is a binary relation, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{curry} f$.

We say that F is commuting if and only if the conditions (Def. 3) are satisfied.

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(Def. 3)(i) For every set x such that $x \in \operatorname{dom} F$ holds x is a function yielding function, and

(ii) for every function f such that $f \in \operatorname{dom} F$ holds $F(f) = \operatorname{commute}(f)$.

Let us note that every function which is empty is also uncurrying, currying, and commuting.

Let us mention that there exists a function which is uncurrying, currying, and commuting.

Let F be an uncurrying function and let X be a set. Observe that $F{\restriction}X$ is uncurrying.

Let F be a currying function and let X be a set. Note that $F{\restriction}X$ is currying.

The following propositions are true:

(1) Let X, Y, Z, D be sets. Suppose $D \subseteq (Z^Y)^X$. Then there exists a many sorted set F indexed by D such that F is uncurrying and $\operatorname{rng} F \subseteq Z^{[:X,Y:]}$.

(2) Let X, Y, Z, D be sets. Suppose $D \subseteq Z^{[:X,Y:]}$. Then there exists a many sorted set F indexed by D such that F is currying and if if $Y = \emptyset$, then $X = \emptyset$, then $\operatorname{rng} F \subseteq (Z^Y)^X$.

Let X, Y, Z be sets. Note that there exists a many sorted set indexed by $(Z^Y)^X$ which is uncurrying and there exists a many sorted set indexed by $Z^{[:X,Y:]}$ which is currying.

Next we state several propositions:

(3) Let A, B be non empty sets, C be a set, and f, g be commuting functions. If $\operatorname{dom} f \subseteq (C^B)^A$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

(4) Let B be a non empty set, A, C be sets, f be an uncurrying function, and g be a currying function. If $\operatorname{dom} f \subseteq (C^B)^A$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

(5) Let A, B, C be sets, f be a currying function, and g be an uncurrying function. If $\operatorname{dom} f \subseteq C^{[:A,B:]}$ and $\operatorname{rng} f \subseteq \operatorname{dom} g$, then $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$.

(6) For every function yielding function f and for all sets i, A such that $i \in \operatorname{dom}\operatorname{commute}(f)$ holds $(\operatorname{commute}(f))(i)^\circ A \subseteq \pi_i f^\circ A$.

(7) Let f be a function yielding function and i, A be sets. If for every function g such that $g \in f^\circ A$ holds $i \in \operatorname{dom} g$, then $\pi_i f^\circ A \subseteq (\operatorname{commute}(f))(i)^\circ A$.

(8) For all sets X, Y and for every function f such that $\operatorname{rng} f \subseteq Y^X$ and for all sets i, A such that $i \in X$ holds $(\operatorname{commute}(f))(i)^\circ A = \pi_i f^\circ A$.

(9) For every function f and for all sets i, A such that $[:A, \{i\}:] \subseteq \operatorname{dom} f$ holds $\pi_i(\operatorname{curry} f)^\circ A = f^\circ[:A, \{i\}:]$.

Let X be a set and let Y be a non empty functional set. One can verify that every function from X into Y is function yielding.

Let T be a constituted functions 1-sorted structure. Observe that the carrier of T is functional.

Let X be a set and let L be a non empty relational structure. One can check that L^X is constituted functions.

One can verify that there exists a lattice which is constituted functions, complete, and strict and there exists a 1-sorted structure which is constituted functions and non empty.

Let T be a constituted functions non empty relational structure. Note that every non empty relational substructure of T is constituted functions.

Next we state four propositions:

(10) Let S, T be complete lattices, f be an idempotent map from T into T, and h be a map from S into $\mathrm{Im}\, f$. Then $f \cdot h = h$.

(11) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a relational substructure of T_1. Let f be a map from S into T and f_1 be a map from S into T_1. If f is monotone and $f = f_1$, then f_1 is monotone.

(12) Let S be a non empty relational structure and T, T_1 be non empty relational structures. Suppose T is a full relational substructure of T_1. Let f be a map from S into T and f_1 be a map from S into T_1. If f_1 is monotone and $f = f_1$, then f is monotone.

(13) For every set X and for every subset V of X holds $(\chi_{V,X})^{-1}(\{1\}) = V$ and $(\chi_{V,X})^{-1}(\{0\}) = X \setminus V$.

2. Maps of Power Posets

Let X be a non empty set, let T be a non empty relational structure, let f be an element of T^X, and let x be an element of X. Then $f(x)$ is an element of T.

Next we state several propositions:

(14) Let X be a non empty set, T be a non empty relational structure, and f, g be elements of T^X. Then $f \leqslant g$ if and only if for every element x of X holds $f(x) \leqslant g(x)$.

(15) Let X be a set and L, S be non empty relational structures. Suppose the relational structure of $L =$ the relational structure of S. Then $L^X = S^X$.

(16) Let S_1, S_2, T_1, T_2 be non empty topological spaces. Suppose that

(i) the topological structure of $S_1 =$ the topological structure of S_2, and

(ii) the topological structure of $T_1 =$ the topological structure of T_2.

Then $[S_1 \to T_1] = [S_2 \to T_2]$.

(17) Let X be a set. Then there exists a map f from $2^X_\subseteq$ into $(2^1_\subseteq)^X$ such that f is isomorphic and for every subset Y of X holds $f(Y) = \chi_{Y,X}$.

(18) For every set X holds $2^X_\subseteq$ and $(2^1_\subseteq)^X$ are isomorphic.

(19) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^X)^Y$, S_2 be a full non empty relational substructure of $(T^Y)^X$, and F be a map from S_1 into S_2. If F is commuting, then F is monotone.

(20) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[:X, Y:]}$, and F be a map from S_1 into S_2. If F is uncurrying, then F is monotone.

(21) Let X, Y be non empty sets, T be a non empty poset, S_1 be a full non empty relational substructure of $(T^Y)^X$, S_2 be a full non empty relational substructure of $T^{[:X, Y:]}$, and F be a map from S_2 into S_1. If F is currying, then F is monotone.

3. Posets of Directed Suprema Preserving Maps

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. The functor $\mathrm{UPS}(S,T)$ yielding a strict relational structure is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{UPS}(S,T)$ is a full relational substructure of $T^{\text{the carrier of } S}$, and

(ii) for every set x holds $x \in$ the carrier of $\mathrm{UPS}(S,T)$ iff x is a directed-sups-preserving map from S into T.

Let S be a non empty relational structure and let T be a non empty reflexive antisymmetric relational structure. Observe that $\mathrm{UPS}(S,T)$ is non empty reflexive antisymmetric and constituted functions.

Let S be a non empty relational structure and let T be a non empty poset. One can verify that $\mathrm{UPS}(S,T)$ is transitive.

We now state the proposition

(22) Let S be a non empty relational structure and T be a non empty reflexive antisymmetric relational structure. Then the carrier of $\mathrm{UPS}(S,T) \subseteq$ (the carrier of $T)^{\text{the carrier of } S}$.

Let S be a non empty relational structure, let T be a non empty reflexive antisymmetric relational structure, let f be an element of $\mathrm{UPS}(S,T)$, and let s be an element of S. Then $f(s)$ is an element of T.

Next we state three propositions:

(23) Let S be a non empty relational structure, T be a non empty reflexive antisymmetric relational structure, and f, g be elements of $\mathrm{UPS}(S,T)$. Then $f \leqslant g$ if and only if for every element s of S holds $f(s) \leqslant g(s)$.

(24) For all complete Scott top-lattices S, T holds $\mathrm{UPS}(S,T) = \mathrm{SCMaps}(S,T)$.

(25) Let S, S' be non empty relational structures and T, T' be non empty reflexive antisymmetric relational structures. Suppose that
(i) the relational structure of $S =$ the relational structure of S', and
(ii) the relational structure of $T =$ the relational structure of T'.
Then $\mathrm{UPS}(S, T) = \mathrm{UPS}(S', T')$.

Let S, T be complete lattices. Note that $\mathrm{UPS}(S, T)$ is complete.

The following propositions are true:

(26) Let S, T be complete lattices. Then $\mathrm{UPS}(S, T)$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

(27) For all complete lattices S, T and for every subset A of $\mathrm{UPS}(S, T)$ holds $\sup A = \bigsqcup_{(T^{\text{the carrier of } S})} A$.

Let S_1, S_2, T_1, T_2 be non empty reflexive antisymmetric relational structures and let f be a map from S_1 into S_2. Let us assume that f is directed-sups-preserving. Let g be a map from T_1 into T_2. Let us assume that g is directed-sups-preserving. The functor $\mathrm{UPS}(f, g)$ yields a map from $\mathrm{UPS}(S_2, T_1)$ into $\mathrm{UPS}(S_1, T_2)$ and is defined by:

(Def. 5) For every directed-sups-preserving map h from S_2 into T_1 holds $(\mathrm{UPS}(f, g))(h) = g \cdot h \cdot f$.

Next we state a number of propositions:

(28) Let S_1, S_2, S_3, T_1, T_2, T_3 be non empty posets, f_1 be a directed-sups-preserving map from S_2 into S_3, f_2 be a directed-sups-preserving map from S_1 into S_2, g_1 be a directed-sups-preserving map from T_1 into T_2, and g_2 be a directed-sups-preserving map from T_2 into T_3. Then $\mathrm{UPS}(f_2, g_2) \cdot \mathrm{UPS}(f_1, g_1) = \mathrm{UPS}(f_1 \cdot f_2, g_2 \cdot g_1)$.

(29) For all non empty reflexive antisymmetric relational structures S, T holds $\mathrm{UPS}(\mathrm{id}_S, \mathrm{id}_T) = \mathrm{id}_{\mathrm{UPS}(S,T)}$.

(30) Let S_1, S_2, T_1, T_2 be complete lattices, f be a directed-sups-preserving map from S_1 into S_2, and g be a directed-sups-preserving map from T_1 into T_2. Then $\mathrm{UPS}(f, g)$ is directed-sups-preserving.

(31) Ω(the Sierpiński space) is Scott.

(32) For every complete Scott top-lattice S holds $[S \to \text{the Sierpiński space}] = \mathrm{UPS}(S, 2^1_\subseteq)$.

(33) Let S be a complete lattice. Then there exists a map F from $\mathrm{UPS}(S, 2^1_\subseteq)$ into $\langle \sigma(S), \subseteq \rangle$ such that F is isomorphic and for every directed-sups-preserving map f from S into $2^1_\subseteq$ holds $F(f) = f^{-1}(\{1\})$.

(34) For every complete lattice S holds $\mathrm{UPS}(S, 2^1_\subseteq)$ and $\langle \sigma(S), \subseteq \rangle$ are isomorphic.

(35) Let S_1, S_2, T_1, T_2 be complete lattices, f be a map from S_1 into S_2, and g be a map from T_1 into T_2. If f is isomorphic and g is isomorphic, then $\mathrm{UPS}(f, g)$ is isomorphic.

(36) Let S_1, S_2, T_1, T_2 be complete lattices. Suppose S_1 and S_2 are isomorphic and T_1 and T_2 are isomorphic. Then $\mathrm{UPS}(S_2, T_1)$ and $\mathrm{UPS}(S_1, T_2)$ are isomorphic.

(37) Let S, T be complete lattices and f be a directed-sups-preserving projection map from T into T. Then $\mathrm{Im}\,\mathrm{UPS}(\mathrm{id}_S, f) = \mathrm{UPS}(S, \mathrm{Im}\,f)$.

(38) Let X be a non empty set, S, T be non empty posets, f be a directed-sups-preserving map from S into T^X, and i be an element of X. Then $(\mathrm{commute}(f))(i)$ is a directed-sups-preserving map from S into T.

(39) Let X be a non empty set, S, T be non empty posets, and f be a directed-sups-preserving map from S into T^X. Then $\mathrm{commute}(f)$ is a function from X into the carrier of $\mathrm{UPS}(S, T)$.

(40) Let X be a non empty set, S, T be non empty posets, and f be a function from X into the carrier of $\mathrm{UPS}(S, T)$. Then $\mathrm{commute}(f)$ is a directed-sups-preserving map from S into T^X.

(41) For every non empty set X and for all non empty posets S, T holds there exists a map from $\mathrm{UPS}(S, T^X)$ into $\mathrm{UPS}(S, T)^X$ which is commuting and isomorphic.

(42) For every non empty set X and for all non empty posets S, T holds $\mathrm{UPS}(S, T^X)$ and $(\mathrm{UPS}(S, T))^X$ are isomorphic.

(43) For all continuous complete lattices S, T holds $\mathrm{UPS}(S, T)$ is continuous.

(44) For all algebraic complete lattices S, T holds $\mathrm{UPS}(S, T)$ is algebraic.

(45) Let R, S, T be complete lattices and f be a directed-sups-preserving map from R into $\mathrm{UPS}(S, T)$. Then uncurry f is a directed-sups-preserving map from $[\!: R, S :\!]$ into T.

(46) Let R, S, T be complete lattices and f be a directed-sups-preserving map from $[\!: R, S :\!]$ into T. Then curry f is a directed-sups-preserving map from R into $\mathrm{UPS}(S, T)$.

(47) For all complete lattices R, S, T holds there exists a map from $\mathrm{UPS}(R, \mathrm{UPS}(S, T))$ into $\mathrm{UPS}([\!: R, S :\!], T)$ which is uncurrying and isomorphic.

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Property of Complex Functions

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Summary. This article introduces properties of complex function, calculations of them, boundedness and constant.

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The articles [11], [2], [1], [9], [3], [4], [5], [12], [6], [7], [10], and [8] provide the terminology and notation for this paper.

1. Definitions of Complex Functions

For simplicity, we adopt the following convention: X, Y are sets, C is a non empty set, c is an element of C, f, f_1, f_2, f_3, g, g_1 are partial functions from C to $\mathbb{C}$, p is a real number, and r, q are elements of $\mathbb{C}$.

A Complex is an element of $\mathbb{C}$.

Let us consider C, f_1, f_2. The functor $\frac{f_1}{f_2}$ yields a partial function from C to $\mathbb{C}$ and is defined as follows:

(Def. 1) $\mathrm{dom}(\frac{f_1}{f_2}) = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$ and for every c such that $c \in \mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

Let us consider C, f. The functor $\frac{1}{f}$ yields a partial function from C to $\mathbb{C}$ and is defined by:

(Def. 2) $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\mathbb{C}}\})$ and for every c such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})_c = (f_c)^{-1}$.

Next we state a number of propositions:

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(3)[1] $\mathrm{dom}(f_1+f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1+f_2)$ holds $(f_1+f_2)_c = (f_1)_c + (f_2)_c$.

(4) $\mathrm{dom}(f_1-f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1-f_2)$ holds $(f_1-f_2)_c = (f_1)_c - (f_2)_c$.

(5) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c = (f_1)_c \cdot (f_2)_c$.

(6) $\mathrm{dom}(\frac{f_1}{f_2}) = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$ and for every c such that $c \in \mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

(7) $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)_c = r \cdot f_c$.

(9)[2] $\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(-f)$ holds $(-f)_c = -f_c$.

(10) $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\mathbb{C}}\})$ and for every c such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})_c = (f_c)^{-1}$.

(15)[3] $\mathrm{dom}(\frac{1}{g}) \subseteq \mathrm{dom}\, g$ and $\mathrm{dom}\, g \cap (\mathrm{dom}\, g \setminus g^{-1}(\{0_{\mathbb{C}}\})) = \mathrm{dom}\, g \setminus g^{-1}(\{0_{\mathbb{C}}\})$.

(16) $\mathrm{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1}(\{0_{\mathbb{C}}\}) = (\mathrm{dom}\, f_1 \setminus {f_1}^{-1}(\{0_{\mathbb{C}}\})) \cap (\mathrm{dom}\, f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$.

(17) If $c \in \mathrm{dom}(\frac{1}{f})$, then $f_c \neq 0_{\mathbb{C}}$.

(18) $(\frac{1}{f})^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$.

(19) $|f|^{-1}(\{0\}) = f^{-1}(\{0_{\mathbb{C}}\})$ and $(-f)^{-1}(\{0_{\mathbb{C}}\}) = f^{-1}(\{0_{\mathbb{C}}\})$.

(20) $\mathrm{dom}(\frac{1}{\frac{1}{f}}) = \mathrm{dom}(f \restriction \mathrm{dom}(\frac{1}{f}))$.

(21) If $r \neq 0_{\mathbb{C}}$, then $(r\, f)^{-1}(\{0_{\mathbb{C}}\}) = f^{-1}(\{0_{\mathbb{C}}\})$.

2. Basic Properties of Operations

The following propositions are true:

(22) $(f_1+f_2)+f_3 = f_1+(f_2+f_3)$.

(23) $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(24) $(f_1+f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(25) $f_3\, (f_1+f_2) = f_3\, f_1 + f_3\, f_2$.

(26) $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(27) $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(28) $(f_1-f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

[1]The propositions (1) and (2) have been removed.
[2]The proposition (8) has been removed.
[3]The propositions (11)–(14) have been removed.

(29) $f_3\, f_1 - f_3\, f_2 = f_3\,(f_1 - f_2)$.

(30) $r\,(f_1 + f_2) = r\, f_1 + r\, f_2$.

(31) $(r \cdot q)\, f = r\,(q\, f)$.

(32) $r\,(f_1 - f_2) = r\, f_1 - r\, f_2$.

(33) $f_1 - f_2 = (-1_{\mathbb{C}})\,(f_2 - f_1)$.

(34) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(35) $1_{\mathbb{C}}\, f = f$.

(36) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(37) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(38) $|f_1\, f_2| = |f_1|\,|f_2|$.

(39) $|r\, f| = |r|\,|f|$.

(40) $-f = (-1_{\mathbb{C}})\, f$.

(41) $--f = f$.

(42) $f_1 - f_2 = f_1 + -f_2$.

(43) $f_1 - -f_2 = f_1 + f_2$.

(44) $\frac{1}{\frac{1}{f}} = f \restriction \mathrm{dom}(\frac{1}{f})$.

(45) $\frac{1}{f_1\, f_2} = \frac{1}{f_1}\, \frac{1}{f_2}$.

(46) If $r \neq 0_{\mathbb{C}}$, then $\frac{1}{r\, f} = r^{-1}\, \frac{1}{f}$.

(47) $1_{\mathbb{C}} \neq 0_{\mathbb{C}}$.

(48) $(-1_{\mathbb{C}})^{-1} = -1_{\mathbb{C}}$.

(49) $\frac{1}{-f} = (-1_{\mathbb{C}})\, \frac{1}{f}$.

(50) $\frac{1}{|f|} = |\frac{1}{f}|$.

(51) $\frac{f}{g} = f\, \frac{1}{g}$.

(52) $r\, \frac{g}{f} = \frac{r\, g}{f}$.

(53) $\frac{f}{g}\, g = f \restriction \mathrm{dom}(\frac{1}{g})$.

(54) $\frac{f}{g}\, \frac{f_1}{g_1} = \frac{f\, f_1}{g\, g_1}$.

(55) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2 \restriction \mathrm{dom}(\frac{1}{f_2})}{f_1}$.

(56) $g\, \frac{f_1}{f_2} = \frac{g\, f_1}{f_2}$.

(57) $\frac{g}{\frac{f_1}{f_2}} = \frac{g\, (f_2 \restriction \mathrm{dom}(\frac{1}{f_2}))}{f_1}$.

(58) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(59) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1 + f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1 - f_2}{f}$.

(60) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1\, g + g_1\, f}{f\, g}$.

(61) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f\,(g_1 \restriction \mathrm{dom}(\frac{1}{g_1}))}{g\,f_1}$.

(62) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1\,g - g_1\,f}{f\,g}$.

(63) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(64) $(f_1+f_2)\restriction X = f_1\restriction X + f_2\restriction X$ and $(f_1+f_2)\restriction X = f_1\restriction X + f_2$ and $(f_1+f_2)\restriction X = f_1 + f_2\restriction X$.

(65) $(f_1\,f_2)\restriction X = (f_1\restriction X)\,(f_2\restriction X)$ and $(f_1\,f_2)\restriction X = (f_1\restriction X)\,f_2$ and $(f_1\,f_2)\restriction X = f_1\,(f_2\restriction X)$.

(66) $(-f)\restriction X = -f\restriction X$ and $\frac{1}{f}\restriction X = \frac{1}{f\restriction X}$ and $|f|\restriction X = |f\restriction X|$.

(67) $(f_1-f_2)\restriction X = f_1\restriction X - f_2\restriction X$ and $(f_1-f_2)\restriction X = f_1\restriction X - f_2$ and $(f_1-f_2)\restriction X = f_1 - f_2\restriction X$.

(68) $\frac{f_1}{f_2}\restriction X = \frac{f_1\restriction X}{f_2\restriction X}$ and $\frac{f_1}{f_2}\restriction X = \frac{f_1\restriction X}{f_2}$ and $\frac{f_1}{f_2}\restriction X = \frac{f_1}{f_2\restriction X}$.

(69) $(r\,f)\restriction X = r\,(f\restriction X)$.

3. Total Partial Functions from a Domain, to Complex

We now state a number of propositions:

(70)(i) f_1 is total and f_2 is total iff $f_1 + f_2$ is total,

(ii) f_1 is total and f_2 is total iff $f_1 - f_2$ is total, and

(iii) f_1 is total and f_2 is total iff $f_1\,f_2$ is total.

(71) f is total iff $r\,f$ is total.

(72) f is total iff $-f$ is total.

(73) f is total iff $|f|$ is total.

(74) $\frac{1}{f}$ is total iff $f^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f is total.

(75) f_1 is total and ${f_2}^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f_2 is total iff $\frac{f_1}{f_2}$ is total.

(76) If f_1 is total and f_2 is total, then $(f_1+f_2)_c = (f_1)_c + (f_2)_c$ and $(f_1-f_2)_c = (f_1)_c - (f_2)_c$ and $(f_1\,f_2)_c = (f_1)_c \cdot (f_2)_c$.

(77) If f is total, then $(r\,f)_c = r \cdot f_c$.

(78) If f is total, then $(-f)_c = -f_c$ and $|f|(c) = |f_c|$.

(79) If $\frac{1}{f}$ is total, then $(\frac{1}{f})_c = (f_c)^{-1}$.

(80) If f_1 is total and $\frac{1}{f_2}$ is total, then $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

4. Bounded and Constant Partial Functions from a Domain, to Complex

Let us consider C, f, Y. We say that f is bounded on Y if and only if:

(Def. 3) $|f|$ is bounded on Y.

The following propositions are true:

(81) f is bounded on Y iff there exists a real number p such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $|f_c| \leqslant p$.

(82) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(83) If $X \cap \operatorname{dom} f = \emptyset$, then f is bounded on X.

(84) If f is bounded on Y, then $r\,f$ is bounded on Y.

(85) $|f|$ is lower bounded on X.

(86) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(87) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(88) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1\,f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(89) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(90) Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1\,f_2$ is a constant on $X \cap Y$.

(91) If f is a constant on Y, then $q\,f$ is a constant on Y.

(92) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(93) If f is a constant on Y, then f is bounded on Y.

(94) If f is a constant on Y, then for every r holds $r\,f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(95) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(96) Suppose f_1 is bounded on X and f_2 is a constant on Y. Then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and $f_1\,f_2$ is bounded on $X \cap Y$.

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Property of Complex Sequence and Continuity of Complex Function

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Summary. This article introduces properties of complex sequence and continuity of complex function. The first section shows convergence of complex sequence and constant complex sequence. In the next section, definition of continuity of complex function and properties of continuous complex function are shown.

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The papers [14], [8], [3], [1], [9], [10], [12], [4], [5], [2], [6], [15], [16], [7], [13], and [11] provide the notation and terminology for this paper.

1. Complex Sequence

For simplicity, we adopt the following rules: n, m, k denote natural numbers, x denotes a set, X, X_1 denote sets, g, x_0, x_1, x_2 denote elements of $\mathbb{C}$, s_1, s_2, s_3, s_4, s_5, s_6 denote complex sequences, Y denotes a subset of $\mathbb{C}$, f, f_1, f_2, h, h_1, h_2 denote partial functions from $\mathbb{C}$ to $\mathbb{C}$, r, s denote real numbers, and N_1 denotes an increasing sequence of naturals.

Let us consider h, s_3. Let us assume that $\operatorname{rng} s_3 \subseteq \operatorname{dom} h$. The functor $h \cdot s_3$ yielding a complex sequence is defined by:

(Def. 1) $h \cdot s_3 = (h$ **qua** function$)\ \cdot(s_3)$.

Let us consider f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 2) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

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One can prove the following propositions:

(2)[1] $s_4 = s_5 - s_6$ iff for every n holds $s_4(n) = s_5(n) - s_6(n)$.

(3) $\mathrm{rng}(s_3 \uparrow n) \subseteq \mathrm{rng}\, s_3$.

(4) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, f$, then $s_3(n) \in \mathrm{dom}\, f$.

(5) $x \in \mathrm{rng}\, s_3$ iff there exists n such that $x = s_3(n)$.

(6) $s_3(n) \in \mathrm{rng}\, s_3$.

(7) If s_4 is a subsequence of s_3, then $\mathrm{rng}\, s_4 \subseteq \mathrm{rng}\, s_3$.

(8) If s_4 is a subsequence of s_3 and s_3 is non-zero, then s_4 is non-zero.

(9) $(s_4 + s_5)\, N_1 = s_4\, N_1 + s_5\, N_1$ and $(s_4 - s_5)\, N_1 = s_4\, N_1 - s_5\, N_1$ and $(s_4\, s_5)\, N_1 = s_4\, N_1\, (s_5\, N_1)$.

(10) $(g\, s_3)\, N_1 = g\, (s_3\, N_1)$.

(11) $(-s_3)\, N_1 = -s_3\, N_1$ and $|s_3| \cdot N_1 = |s_3\, N_1|$.

(12) $(s_3\, N_1)^{-1} = {s_3}^{-1}\, N_1$.

(13) $(s_4/s_3)\, N_1 = (s_4\, N_1)/(s_3\, N_1)$.

(14) If for every n holds $s_3(n) \in Y$, then $\mathrm{rng}\, s_3 \subseteq Y$.

(15) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $h \cdot s_3 = (h$ **qua** function$)\ \cdot(s_3)$.

(16) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, f$, then $(f \cdot s_3)(n) = f_{s_3(n)}$.

(17) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, f$, then $(f \cdot s_3) \uparrow n = f \cdot (s_3 \uparrow n)$.

(18) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h_1 \cap \mathrm{dom}\, h_2$, then $(h_1 + h_2) \cdot s_3 = h_1 \cdot s_3 + h_2 \cdot s_3$ and $(h_1 - h_2) \cdot s_3 = h_1 \cdot s_3 - h_2 \cdot s_3$ and $(h_1\, h_2) \cdot s_3 = (h_1 \cdot s_3)\, (h_2 \cdot s_3)$.

(19) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $(g\, h) \cdot s_3 = g\, (h \cdot s_3)$.

(20) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $-h \cdot s_3 = (-h) \cdot s_3$.

(21) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}(\frac{1}{h})$, then $h \cdot s_3$ is non-zero.

(22) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}(\frac{1}{h})$, then $\frac{1}{h} \cdot s_3 = (h \cdot s_3)^{-1}$.

(23) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $\Re((h \cdot s_3)\, N_1) = \Re(h \cdot (s_3\, N_1))$.

(24) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $\Im((h \cdot s_3)\, N_1) = \Im(h \cdot (s_3\, N_1))$.

(25) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}\, h$, then $(h \cdot s_3)\, N_1 = h \cdot (s_3\, N_1)$.

(26) If $\mathrm{rng}\, s_4 \subseteq \mathrm{dom}\, h$ and s_5 is a subsequence of s_4, then $h \cdot s_5$ is a subsequence of $h \cdot s_4$.

(27) If h is total, then $(h \cdot s_3)(n) = h_{s_3(n)}$.

(28) If h is total, then $h \cdot (s_3 \uparrow n) = (h \cdot s_3) \uparrow n$.

(29) If h_1 is total and h_2 is total, then $(h_1 + h_2) \cdot s_3 = h_1 \cdot s_3 + h_2 \cdot s_3$ and $(h_1 - h_2) \cdot s_3 = h_1 \cdot s_3 - h_2 \cdot s_3$ and $(h_1\, h_2) \cdot s_3 = (h_1 \cdot s_3)\, (h_2 \cdot s_3)$.

(30) If h is total, then $(g\, h) \cdot s_3 = g\, (h \cdot s_3)$.

(31) If $\mathrm{rng}\, s_3 \subseteq \mathrm{dom}(h{\restriction}X)$, then $(h{\restriction}X) \cdot s_3 = h \cdot s_3$.

[1]The proposition (1) has been removed.

(32) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h \restriction X)$ and if $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h \restriction Y)$ or $X \subseteq Y$, then $(h \restriction X) \cdot s_3 = (h \restriction Y) \cdot s_3$.

(33) If $\operatorname{rng} s_3 \subseteq \operatorname{dom}(h \restriction X)$ and $h^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $(\frac{1}{h} \restriction X) \cdot s_3 = ((h \restriction X) \cdot s_3)^{-1}$.

Let f be a function. We say that f is constant if and only if:

(Def. 3) For all sets n_1, n_2 such that $n_1 \in \operatorname{dom} f$ and $n_2 \in \operatorname{dom} f$ holds $f(n_1) = f(n_2)$.

Let us consider s_3. Let us observe that s_3 is constant if and only if:

(Def. 4) There exists g such that for every n holds $s_3(n) = g$.

Next we state a number of propositions:

(34) s_3 is constant iff there exists g such that $\operatorname{rng} s_3 = \{g\}$.

(35) s_3 is constant iff for every n holds $s_3(n) = s_3(n+1)$.

(36) s_3 is constant iff for all n, k holds $s_3(n) = s_3(n+k)$.

(37) s_3 is constant iff for all n, m holds $s_3(n) = s_3(m)$.

(38) $s_3 \uparrow k$ is a subsequence of s_3.

(39) If s_4 is a subsequence of s_3 and s_3 is convergent, then s_4 is convergent.

(40) If s_4 is a subsequence of s_3 and s_3 is convergent, then $\lim s_4 = \lim s_3$.

(41) If s_3 is convergent and there exists k such that for every n such that $k \leqslant n$ holds $s_4(n) = s_3(n)$, then s_4 is convergent.

(42) If s_3 is convergent and there exists k such that for every n such that $k \leqslant n$ holds $s_4(n) = s_3(n)$, then $\lim s_3 = \lim s_4$.

(43) If s_3 is convergent, then $s_3 \uparrow k$ is convergent and $\lim(s_3 \uparrow k) = \lim s_3$.

(44) If s_3 is convergent and there exists k such that $s_3 = s_4 \uparrow k$, then s_4 is convergent.

(45) If s_3 is convergent and there exists k such that $s_3 = s_4 \uparrow k$, then $\lim s_4 = \lim s_3$.

(46) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then there exists k such that $s_3 \uparrow k$ is non-zero.

(47) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then there exists s_4 which is a subsequence of s_3 and non-zero.

(48) If s_3 is constant, then s_3 is convergent.

(49) If s_3 is constant and $g \in \operatorname{rng} s_3$ or s_3 is constant and there exists n such that $s_3(n) = g$, then $\lim s_3 = g$.

(50) If s_3 is constant, then for every n holds $\lim s_3 = s_3(n)$.

(51) If s_3 is convergent and $\lim s_3 \neq 0_{\mathbb{C}}$, then for every s_4 such that s_4 is a subsequence of s_3 and non-zero holds $\lim({s_4}^{-1}) = (\lim s_3)^{-1}$.

(52) If s_3 is constant and s_4 is convergent, then $\lim(s_3 + s_4) = s_3(0) + \lim s_4$ and $\lim(s_3 - s_4) = s_3(0) - \lim s_4$ and $\lim(s_4 - s_3) = \lim s_4 - s_3(0)$ and

$\lim(s_3\, s_4) = s_3(0) \cdot \lim s_4$.

The scheme *CompSeqChoice* concerns and states that:

There exists s_1 such that for every n holds $\mathcal{P}[n, s_1(n)]$

provided the following condition is satisfied:

- For every n there exists g such that $\mathcal{P}[n, g]$.

2. Continuity of Complex Sequence

We now state several propositions:

(53) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

(54) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(55) Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 and $f_1\, f_2$ is continuous in x_0.

(56) If f is continuous in x_0, then $g\, f$ is continuous in x_0.

(57) If f is continuous in x_0, then $-f$ is continuous in x_0.

(58) If f is continuous in x_0 and $f_{x_0} \neq 0_{\mathbb{C}}$, then $\frac{1}{f}$ is continuous in x_0.

(59) If f_1 is continuous in x_0 and $(f_1)_{x_0} \neq 0_{\mathbb{C}}$ and f_2 is continuous in x_0, then $\frac{f_2}{f_1}$ is continuous in x_0.

Let us consider f, X. We say that f is continuous on X if and only if:

(Def. 5) $X \subseteq \operatorname{dom} f$ and for every x_0 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

One can prove the following propositions:

(60) Let given X, f. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $f_{\lim s_1} = \lim(f \cdot s_1)$.

(61) f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(62) f is continuous on X iff $f{\restriction}X$ is continuous on X.

(63) If f is continuous on X and $X_1 \subseteq X$, then f is continuous on X_1.

(64) If $x_0 \in \operatorname{dom} f$, then f is continuous on $\{x_0\}$.

(65) Let given X, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X and $f_1\,f_2$ is continuous on X.

(66) Let given X, X_1, f_1, f_2. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$ and $f_1\,f_2$ is continuous on $X \cap X_1$.

(67) For all g, X, f such that f is continuous on X holds $g\,f$ is continuous on X.

(68) If f is continuous on X, then $-f$ is continuous on X.

(69) If f is continuous on X and $f^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(70) If f is continuous on X and $(f{\restriction}X)^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$, then $\frac{1}{f}$ is continuous on X.

(71) If f_1 is continuous on X and ${f_1}^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f_2 is continuous on X, then $\frac{f_2}{f_1}$ is continuous on X.

(72) If f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then f is continuous on $\mathbb{C}$.

Let us consider X. We say that X is compact if and only if:

(Def. 6) For every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and convergent and $\lim s_2 \in X$.

One can prove the following propositions:

(73) For every f such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(74) If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^{\circ}Y$ is compact.

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Scalar Multiple of Riemann Definite Integral

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Summary. The goal of this article is to prove a scalar multiplicity of Riemann definite integral. Therefore, we defined a scalar product to the subset of real space, and we proved some relating lemmas. At last, we proved a scalar multiplicity of Riemann definite integral. As a result, a linearity of Riemann definite integral was proven by unifying the previous article [7].

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The papers [2], [6], [3], [7], [13], [1], [4], [14], [5], [8], [16], [12], [10], [11], [9], and [15] provide the notation and terminology for this paper.

1. Lemmas of Finite Sequence

We adopt the following rules: r, x, y are real numbers, i, j are natural numbers, and p is a finite sequence of elements of $\mathbb{R}$.

The following proposition is true

(1) For every closed-interval subset A of $\mathbb{R}$ and for every x holds $x \in A$ iff $\inf A \leqslant x$ and $x \leqslant \sup A$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is non-decreasing if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let n be a natural number. Suppose $n \in \operatorname{dom} I_1$ and $n+1 \in \operatorname{dom} I_1$. Let r, s be real numbers. If $r = I_1(n)$ and $s = I_1(n+1)$, then $r \leqslant s$.

One can verify that there exists a finite sequence of elements of $\mathbb{R}$ which is non-decreasing.

The following three propositions are true:

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(2) Let p be a non-decreasing finite sequence of elements of $\mathbb{R}$ and given i, j. If $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \leqslant j$, then $p(i) \leqslant p(j)$.

(3) Let given p. Then there exists a non-decreasing finite sequence q of elements of $\mathbb{R}$ such that p and q are fiberwise equipotent.

(4) Let D be a non empty set, f be a finite sequence of elements of D, and k_1, k_2, k_3 be natural numbers. If $1 \leqslant k_1$ and $k_3 \leqslant \operatorname{len} f$ and $k_1 \leqslant k_2$ and $k_2 < k_3$, then $(\operatorname{mid}(f, k_1, k_2)) \frown \operatorname{mid}(f, k_2 + 1, k_3) = \operatorname{mid}(f, k_1, k_3)$.

2. Scalar Product of Real Subset

Let X be a subset of $\mathbb{R}$ and let r be a real number. The functor $r \cdot X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) $r \cdot X = \{r \cdot x : x \in X\}$.

The following propositions are true:

(5) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is upper bounded on X and $Y \subseteq X$, then $f \restriction Y$ is upper bounded on Y.

(6) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is lower bounded on X and $Y \subseteq X$, then $f \restriction Y$ is lower bounded on Y.

(7) For every non empty subset X of $\mathbb{R}$ holds $r \cdot X$ is non empty.

(8) For every subset X of $\mathbb{R}$ holds $r \cdot X = \{r \cdot x : x \in X\}$.

(9) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leqslant r$ holds $r \cdot X$ is upper bounded.

(10) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leqslant 0$ holds $r \cdot X$ is lower bounded.

(11) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leqslant r$ holds $r \cdot X$ is lower bounded.

(12) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leqslant 0$ holds $r \cdot X$ is upper bounded.

(13) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leqslant r$ holds $\sup(r \cdot X) = r \cdot \sup X$.

(14) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leqslant 0$ holds $\inf(r \cdot X) = r \cdot \sup X$.

(15) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leqslant r$ holds $\inf(r \cdot X) = r \cdot \inf X$.

(16) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leqslant 0$ holds $\sup(r \cdot X) = r \cdot \inf X$.

3. Scalar Multiple of Integral

The following propositions are true:

(17) For every non empty set X and for every partial function f from X to $\mathbb{R}$ such that f is total holds $\mathrm{rng}(r\,f) = r \cdot \mathrm{rng}\,f$.

(18) For all non empty sets X, Z and for every partial function f from X to $\mathbb{R}$ holds $\mathrm{rng}(r\,(f \restriction Z)) = r \cdot \mathrm{rng}(f \restriction Z)$.

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of $\mathrm{divs}\,A$. If f is total and bounded on A and $r \geqslant 0$, then $(\mathrm{upper\_sum\_set}\,r\,f)(D) \geqslant r \cdot \inf \mathrm{rng}\,f \cdot \mathrm{vol}(A)$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of $\mathrm{divs}\,A$. If f is total and bounded on A and $r \leqslant 0$, then $(\mathrm{upper\_sum\_set}\,r\,f)(D) \geqslant r \cdot \sup \mathrm{rng}\,f \cdot \mathrm{vol}(A)$.

(21) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of $\mathrm{divs}\,A$. If f is total and bounded on A and $r \geqslant 0$, then $(\mathrm{lower\_sum\_set}\,r\,f)(D) \leqslant r \cdot \sup \mathrm{rng}\,f \cdot \mathrm{vol}(A)$.

(22) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of $\mathrm{divs}\,A$. If f is total and bounded on A and $r \leqslant 0$, then $(\mathrm{lower\_sum\_set}\,r\,f)(D) \leqslant r \cdot \inf \mathrm{rng}\,f \cdot \mathrm{vol}(A)$.

(23) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\,D$ and f is upper bounded on A and total and $r \geqslant 0$. Then $(\mathrm{upper\_volume}(r\,f, D))(i) = r \cdot (\mathrm{upper\_volume}(f, D))(i)$.

(24) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\,D$ and f is upper bounded on A and total and $r \leqslant 0$. Then $(\mathrm{lower\_volume}(r\,f, D))(i) = r \cdot (\mathrm{upper\_volume}(f, D))(i)$.

(25) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\,D$ and f is lower bounded on A and total and $r \geqslant 0$. Then $(\mathrm{lower\_volume}(r\,f, D))(i) = r \cdot (\mathrm{lower\_volume}(f, D))(i)$.

(26) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\,D$ and f is lower bounded on A and total and $r \leqslant 0$. Then $(\mathrm{upper\_volume}(r\,f, D))(i) = r \cdot (\mathrm{lower\_volume}(f, D))(i)$.

(27) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and total and $r \geqslant 0$, then $\mathrm{upper\_sum}(r\,f, D) = r \cdot \mathrm{upper\_sum}(f, D)$.

(28) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and total and $r \leqslant 0$, then $\mathrm{lower\_sum}(r\,f, D) = r \cdot \mathrm{upper\_sum}(f, D)$.

(29) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and total and $r \geqslant 0$, then $\mathrm{lower\_sum}(r\,f, D) = r \cdot \mathrm{lower\_sum}(f, D)$.

(30) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and total and $r \leqslant 0$, then $\mathrm{upper\_sum}(r\,f, D) = r \cdot \mathrm{lower\_sum}(f, D)$.

(31) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. Suppose f is total and bounded on A and f is integrable on A. Then $r\,f$ is integrable on A and $\mathrm{integral}\, r\,f = r \cdot \mathrm{integral}\, f$.

4. Monotoneity of Integral

One can prove the following propositions:

(32) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. Suppose f is total and bounded on A and f is integrable on A and for every x such that $x \in A$ holds $f(x) \geqslant 0$. Then $\mathrm{integral}\, f \geqslant 0$.

(33) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be partial functions from A to $\mathbb{R}$. Suppose that

- (i) f is total and bounded on A,
- (ii) f is integrable on A,
- (iii) g is total and bounded on A, and
- (iv) g is integrable on A.

Then $f - g$ is integrable on A and $\mathrm{integral}\, f - g = \mathrm{integral}\, f - \mathrm{integral}\, g$.

(34) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be partial functions from A to $\mathbb{R}$. Suppose that

- (i) f is total and bounded on A,
- (ii) f is integrable on A,
- (iii) g is total and bounded on A,
- (iv) g is integrable on A, and
- (v) for every x such that $x \in A$ holds $f(x) \geqslant g(x)$.

Then $\mathrm{integral}\, f \geqslant \mathrm{integral}\, g$.

5. Definition of Division Sequence

Next we state two propositions:

(35) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is total and bounded on A, then $\operatorname{rng} \text{upper\_sum\_set} f$ is lower bounded.

(36) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is total and bounded on A, then $\operatorname{rng} \text{lower\_sum\_set} f$ is upper bounded.

Let A be a closed-interval subset of $\mathbb{R}$. A DivSequence of A is a function from $\mathbb{N}$ into $\operatorname{divs} A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let T be a DivSequence of A. The functor δ_T yielding a sequence of real numbers is defined by:

(Def. 3) For every i holds $\delta_T(i) = \delta_{T(i)}$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, and let T be a DivSequence of A. The functor $\text{upper\_sum}(f, T)$ yields a sequence of real numbers and is defined by:

(Def. 4) For every i holds $(\text{upper\_sum}(f, T))(i) = \text{upper\_sum}(f, T(i))$.

The functor $\text{lower\_sum}(f, T)$ yields a sequence of real numbers and is defined as follows:

(Def. 5) For every i holds $(\text{lower\_sum}(f, T))(i) = \text{lower\_sum}(f, T(i))$.

The following propositions are true:

(37) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\operatorname{divs} A$. If $D_1 \leqslant D_2$, then for every j such that $j \in \operatorname{dom} D_2$ there exists i such that $i \in \operatorname{dom} D_1$ and $\operatorname{divset}(D_2, j) \subseteq \operatorname{divset}(D_1, i)$.

(38) For all finite non empty subsets X, Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\max X \leqslant \max Y$.

(39) For all finite non empty subsets X, Y of $\mathbb{R}$ such that there exists y such that $y \in Y$ and $\max X \leqslant y$ holds $\max X \leqslant \max Y$.

(40) For all closed-interval subsets A, B of $\mathbb{R}$ such that $A \subseteq B$ holds $\operatorname{vol}(A) \leqslant \operatorname{vol}(B)$.

(41) For every closed-interval subset A of $\mathbb{R}$ and for all elements D_1, D_2 of $\operatorname{divs} A$ such that $D_1 \leqslant D_2$ holds $\delta_{(D_1)} \geqslant \delta_{(D_2)}$.

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Darboux's Theorem

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Summary. In this article, we have proved the Darboux's theorem. This theorem is important to prove the Riemann integrability. We can replace an upper bound and a lower bound of a function which is the definition of Riemann integration with convergence of sequence by Darboux's theorem.

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The articles [18], [14], [1], [2], [3], [12], [7], [8], [13], [4], [6], [9], [19], [11], [5], [10], [15], [17], and [16] provide the notation and terminology for this paper.

1. Lemmas of Division

We adopt the following convention: x, y are real numbers, i, j, k are natural numbers, and p, q are finite sequences of elements of $\mathbb{R}$.

The following propositions are true:

(1) Let A be a closed-interval subset of $\mathbb{R}$ and D be an element of $\operatorname{divs} A$. If $\operatorname{vol}(A) \neq 0$, then there exists i such that $i \in \operatorname{dom} D$ and $\operatorname{vol}(\operatorname{divset}(D, i)) > 0$.

(2) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of $\operatorname{divs} A$, and given x. If $x \in A$, then there exists j such that $j \in \operatorname{dom} D$ and $x \in \operatorname{divset}(D, j)$.

(3) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\operatorname{divs} A$. Then there exists an element D of $\operatorname{divs} A$ such that $D_1 \leqslant D$ and $D_2 \leqslant D$ and $\operatorname{rng} D = \operatorname{rng} D_1 \cup \operatorname{rng} D_2$.

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(4) Let A be a closed-interval subset of $\mathbb{R}$ and D, D_1 be elements of divs A. Suppose $\delta_{(D_1)} < \min \operatorname{rng} \mathrm{upper\_volume}(\chi_{A,A}, D)$. Let given x, y, i. If $i \in \operatorname{dom} D_1$ and $x \in \operatorname{rng} D \cap \mathrm{divset}(D_1, i)$ and $y \in \operatorname{rng} D \cap \mathrm{divset}(D_1, i)$, then $x = y$.

(5) For all p, q such that $\operatorname{rng} p = \operatorname{rng} q$ and p is increasing and q is increasing holds $p = q$.

(6) Let A be a closed-interval subset of $\mathbb{R}$, D, D_1 be elements of divs A, and given i, j. Suppose $D \leqslant D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leqslant j$. Then $\mathrm{indx}(D_1, D, i) \leqslant \mathrm{indx}(D_1, D, j)$ and $\mathrm{indx}(D_1, D, i) \in \operatorname{dom} D_1$ and $\mathrm{indx}(D_1, D, j) \in \operatorname{dom} D_1$.

(7) Let A be a closed-interval subset of $\mathbb{R}$, D, D_1 be elements of divs A, and given i, j. Suppose $D \leqslant D_1$ and $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i < j$. Then $\mathrm{indx}(D_1, D, i) < \mathrm{indx}(D_1, D, j)$ and $\mathrm{indx}(D_1, D, i) \in \operatorname{dom} D_1$ and $\mathrm{indx}(D_1, D, j) \in \operatorname{dom} D_1$.

(8) For every closed-interval subset A of $\mathbb{R}$ and for every element D of divs A holds $\delta_D \geqslant 0$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, g be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \mathrm{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geqslant 2$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is total and bounded on A. Then $\sum \mathrm{lower\_volume}(g, D_2) - \sum \mathrm{lower\_volume}(g, D_1) \leqslant (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, g be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \mathrm{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{len} D_1 \geqslant 2$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and g is total and bounded on A. Then $\sum \mathrm{upper\_volume}(g, D_1) - \sum \mathrm{upper\_volume}(g, D_2) \leqslant (\sup \operatorname{rng} g - \inf \operatorname{rng} g) \cdot \delta_{(D_1)}$.

(11) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of divs A, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D$ and $j \in \operatorname{dom} D$ and $i \leqslant j$ and $r < (\mathrm{mid}(D, i, j))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ such that $r = \inf B$ and $\sup B = (\mathrm{mid}(D, i, j))(\operatorname{len} \mathrm{mid}(D, i, j))$ and $\operatorname{len} \mathrm{mid}(D, i, j) = (j - i) + 1$ and $\mathrm{mid}(D, i, j)$ is a DivisionPoint of B.

(12) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose $x \in \mathrm{divset}(D_1, \operatorname{len} D_1)$ and $\mathrm{vol}(A) \neq 0$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is total and bounded on A and $x > \inf A$. Then $\sum \mathrm{lower\_volume}(f, D_2) - \sum \mathrm{lower\_volume}(f, D_1) \leqslant (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(13) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D_1, D_2 be elements of divs A, and given x. Suppose

$x \in \operatorname{divset}(D_1, \operatorname{len} D_1)$ and $\operatorname{vol}(A) \neq 0$ and $D_1 \leqslant D_2$ and $\operatorname{rng} D_2 = \operatorname{rng} D_1 \cup \{x\}$ and f is total and bounded on A and $x > \inf A$. Then $\sum \text{upper\_volume}(f, D_1) - \sum \text{upper\_volume}(f, D_2) \leqslant (\sup \operatorname{rng} f - \inf \operatorname{rng} f) \cdot \delta_{(D_1)}$.

(14) Let A be a closed-interval subset of $\mathbb{R}$, D_1, D_2 be elements of $\operatorname{divs} A$, r be a real number, and i, j be natural numbers. Suppose $i \in \operatorname{dom} D_1$ and $j \in \operatorname{dom} D_1$ and $i \leqslant j$ and $D_1 \leqslant D_2$ and $r < (\operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j)))(1)$. Then there exists a closed-interval subset B of $\mathbb{R}$ and there exist elements M_1, M_2 of $\operatorname{divs} B$ such that $r = \inf B$ and $\sup B = M_2(\operatorname{len} M_2)$ and $\sup B = M_1(\operatorname{len} M_1)$ and $M_1 \leqslant M_2$ and $M_1 = \operatorname{mid}(D_1, i, j)$ and $M_2 = \operatorname{mid}(D_2, \operatorname{indx}(D_2, D_1, i), \operatorname{indx}(D_2, D_1, j))$.

(15) Let A be a closed-interval subset of $\mathbb{R}$, D be an element of $\operatorname{divs} A$, and given x. If $x \in \operatorname{rng} D$, then $D(1) \leqslant x$ and $x \leqslant D(\operatorname{len} D)$.

(16) Let p be a finite sequence of elements of $\mathbb{R}$ and given i, j, k. Suppose p is increasing and $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $k \in \operatorname{dom} p$ and $p(i) \leqslant p(k)$ and $p(k) \leqslant p(j)$. Then $p(k) \in \operatorname{rng} \operatorname{mid}(p, i, j)$.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D be an element of $\operatorname{divs} A$, and given i. If f is total and bounded on A and $i \in \operatorname{dom} D$, then $\inf \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \leqslant \sup \operatorname{rng} f$.

(18) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, D be an element of $\operatorname{divs} A$, and given i. If f is total and bounded on A and $i \in \operatorname{dom} D$, then $\sup \operatorname{rng}(f \restriction \operatorname{divset}(D, i)) \geqslant \inf \operatorname{rng} f$.

2. Darboux's Theorem

The following two propositions are true:

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and T be a DivSequence of A. Suppose f is total and bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{lower\_sum}(f, T)$ is convergent and $\lim \text{lower\_sum}(f, T) = \text{lower\_integral}\, f$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and T be a DivSequence of A. Suppose f is total and bounded on A and δ_T is convergent to 0 and $\operatorname{vol}(A) \neq 0$. Then $\text{upper\_sum}(f, T)$ is convergent and $\lim \text{upper\_sum}(f, T) = \text{upper\_integral}\, f$.

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Five Variable Predicate Calculus for Boolean Valued Functions. Part I

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of ordinary predicate logic.

MML Identifier: `BVFUNC22`.

The terminology and notation used here have been introduced in the following articles: [10], [4], [6], [1], [8], [7], [2], [3], [5], [11], and [9].

1. Preliminaries

For simplicity, we follow the rules: Y denotes a non empty set, a denotes an element of $\mathrm{BVF}(Y)$, G denotes a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C, D, E denote partitions of Y.

One can prove the following propositions:

(1) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D \wedge E$.

(2) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D \wedge E$.

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(3) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D \wedge E$.

(4) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{CompF}(D, G) = A \wedge B \wedge C \wedge E$.

(5) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{CompF}(E, G) = A \wedge B \wedge C \wedge D$.

(6) Let A, B, C, D, E be sets, h be a function, and A', B', C', D', E' be sets. Suppose $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (A \dot{\longmapsto} A')$. Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$ and $h(E) = E'$.

(7) Let A, B, C, D, E be sets, h be a function, and A', B', C', D', E' be sets. Suppose $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (A \dot{\longmapsto} A')$. Then $\mathrm{dom}\, h = \{A, B, C, D, E\}$.

(8) Let A, B, C, D, E be sets, h be a function, and A', B', C', D', E' be sets. Suppose $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (A \dot{\longmapsto} A')$. Then $\mathrm{rng}\, h = \{h(A), h(B), h(C), h(D), h(E)\}$.

(9) Let G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D \wedge E) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(10) Let G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E be partitions of Y, and z, u be elements of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$ and $\mathrm{EqClass}(z, C \wedge D \wedge E) = \mathrm{EqClass}(u, C \wedge D \wedge E)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

2. Predicate Calculus

One can prove the following propositions:

(11) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\forall_{a,B}G,A}G$.

(12) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(13) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(14) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\exists_{\exists_{a,B}G,A}G \Subset \exists_{\exists_{a,A}G,B}G$.

(15) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(16) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(17) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(18) $\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(19) $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(20) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(22)[1] Suppose that

[1]The proposition (21) has been removed.

G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\exists_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

(23) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\neg\forall_{\forall_{a,A}G,B}G = \exists_{\neg\forall_{a,B}G,A}G$.

(24) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\neg\forall_{\forall_{a,A}G,B}G = \exists_{\exists_{\neg a,B}G,A}G$.

(25) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\neg\forall_{a,A}G,B}G \Subset \neg\forall_{\forall_{a,B}G,A}G$.

(26) Suppose that
G is a coordinate and $G = \{A, B, C, D, E\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $C \neq D$ and $C \neq E$ and $D \neq E$. Then $\forall_{\neg\forall_{a,A}G,B}G \Subset \exists_{\exists_{\neg a,B}G,A}G$.

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Six Variable Predicate Calculus for Boolean Valued Functions. Part I

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of ordinary predicate logic.

MML Identifier: `BVFUNC23`.

The terminology and notation used in this paper are introduced in the following papers: [10], [4], [6], [1], [8], [7], [2], [3], [5], [11], and [9].

1. Preliminaries

For simplicity, we follow the rules: Y denotes a non empty set, a denotes an element of $\mathrm{BVF}(Y)$, G denotes a subset of $\mathrm{PARTITIONS}(Y)$, and A, B, C, D, E, F denote partitions of Y.

We now state a number of propositions:

(1) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D \wedge E \wedge F$.

(2) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D \wedge E \wedge F$.

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(3) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D \wedge E \wedge F$.

(4) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(D, G) = A \wedge B \wedge C \wedge E \wedge F$.

(5) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(E, G) = A \wedge B \wedge C \wedge D \wedge F$.

(6) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{CompF}(F, G) = A \wedge B \wedge C \wedge D \wedge E$.

(7) Let A, B, C, D, E, F be sets, h be a function, and A', B', C', D', E', F' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (A \dot{\longmapsto} A')$.
Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$ and $h(E) = E'$ and $h(F) = F'$.

(8) Let A, B, C, D, E, F be sets, h be a function, and A', B', C', D', E', F' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (A \dot{\longmapsto} A')$.
Then $\mathrm{dom}\, h = \{A, B, C, D, E, F\}$.

(9) Let A, B, C, D, E, F be sets, h be a function, and A', B', C', D', E', F' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (A \dot{\longmapsto} A')$.

Then $\operatorname{rng} h = \{h(A), h(B), h(C), h(D), h(E), h(F)\}$.

(10) Let G be a subset of PARTITIONS(Y), A, B, C, D, E, F be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D \wedge E \wedge F) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(11) Let G be a subset of PARTITIONS(Y), A, B, C, D, E, F be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$ and $\mathrm{EqClass}(z, C \wedge D \wedge E \wedge F) = \mathrm{EqClass}(u, C \wedge D \wedge E \wedge F)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

2. Predicate Calculus

The following propositions are true:

(12) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\forall_{\forall_{a,A}G,B} G \Subset \forall_{\forall_{a,B}G,A} G$.

(13) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\forall_{\forall_{a,A}G,B} G = \forall_{\forall_{a,B}G,A} G$.

(14) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\exists_{\forall_{a,A}G,B} G \Subset \forall_{\exists_{a,B}G,A} G$.

(15) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\exists_{\exists_{a,B}G,A} G \Subset \exists_{\exists_{a,A}G,B} G$.

(16) Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(17) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(18) $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(19) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(20) $\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(21) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $D \neq E$ and $D \neq F$ and $E \neq F$. Then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

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The Construction and Computation of for-loop Programs for SCMPDS[1]

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Summary. This article defines two for-loop statements for SCMPDS. One is called for-up, which corresponds to "for (i=x; i<0; i+=n) S" in C language. Another is called for-down, which corresponds to "for (i=x; i>0; i-=n) S". Here, we do not present their unconditional halting (called parahalting) property, because we have not found that there exists a useful for-loop statement with unconditional halting, and the proof of unconditional halting is much simpler than that of conditional halting. It is hard to formalize all halting conditions, but some cases can be formalized. We choose loop invariants as halting conditions to prove halting problem of for-up/down statements. When some variables (except the loop control variable) keep undestroyed on a set for the loop invariant, and the loop body is halting for this condition, the corresponding for-up/down is halting and computable under this condition. The computation of for-loop statements can be realized by evaluating its body. At the end of the article, we verify for-down statements by two examples for summing.

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The papers [17], [18], [22], [19], [1], [3], [20], [4], [7], [8], [6], [23], [2], [15], [25], [13], [9], [12], [10], [11], [14], [5], [24], [21], and [16] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x is a set, n is a natural number, a is a Int position, i, j, k are instructions of SCMPDS, s, s_1, s_2 are

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states of SCMPDS, l_1, l are instructions-locations of SCMPDS, and I, J, K are Program-block.

We now state a number of propositions:

(1) For every state s of SCMPDS and for all natural numbers m, n such that $\mathbf{IC}_s = \mathrm{inspos}\, m$ holds $\mathrm{ICplusConst}(s, n-m) = \mathrm{inspos}\, n$.

(2) For all finite partial states P, Q of SCMPDS such that $P \subseteq Q$ holds $\mathrm{ProgramPart}(P) \subseteq \mathrm{ProgramPart}(Q)$.

(3) For all programmed finite partial states P, Q of SCMPDS and for every natural number k such that $P \subseteq Q$ holds $\mathrm{Shift}(P, k) \subseteq \mathrm{Shift}(Q, k)$.

(4) If $\mathbf{IC}_s = \mathrm{inspos}\, 0$, then $\mathrm{Initialized}(s) = s$.

(5) If $\mathbf{IC}_s = \mathrm{inspos}\, 0$, then $s {+}\!\cdot \mathrm{Initialized}(I) = s {+}\!\cdot I$.

(6) $(\mathrm{Computation}(s))(n) \restriction$the instruction locations of SCMPDS $= s \restriction$the instruction locations of SCMPDS.

(7) Let s_1, s_2 be states of SCMPDS. Suppose $\mathbf{IC}_{(s_1)} = \mathbf{IC}_{(s_2)}$ and $s_1 \restriction \text{Data-Loc}_{\mathrm{SCM}} = s_2 \restriction \text{Data-Loc}_{\mathrm{SCM}}$ and $s_1 \restriction$the instruction locations of SCMPDS $= s_2 \restriction$the instruction locations of SCMPDS. Then $s_1 = s_2$.

(8) $l \in \mathrm{dom}\, I$ iff $l \in \mathrm{dom}\, \mathrm{Initialized}(I)$.

(9) If $x \in \mathrm{dom}\, I$, then $I(x) = (s {+}\!\cdot (I {+}\!\cdot \text{Start-At}(l)))(x)$.

(10) If $l_1 \in \mathrm{dom}\, I$, then $(s {+}\!\cdot \mathrm{Initialized}(I))(l_1) = I(l_1)$.

(11) $(s {+}\!\cdot (I {+}\!\cdot \text{Start-At}(l)))(a) = s(a)$.

(12) $(s {+}\!\cdot \text{Start-At}(l_1))(\mathbf{IC}_{\mathrm{SCMPDS}}) = l_1$.

(13) $\mathrm{card}(I;i) = \mathrm{card}\, I + 1$.

(14) $(I;i;j)(\mathrm{inspos}\, \mathrm{card}\, I) = i$.

(15) $(i;I;j);k = i;(I;j;k)$.

(16) $\mathrm{Shift}(J, \mathrm{card}\, I) \subseteq I;J;K$.

(17) $I \subseteq \mathrm{stop}\, I;J$.

(18) If $l_1 \in \mathrm{dom}\, I$, then $(\mathrm{Shift}(\mathrm{stop}\, I, n))(l_1 + n) = (\mathrm{Shift}(I, n))(l_1 + n)$.

(19) If $\mathrm{card}\, I > 0$, then $(\mathrm{Shift}(\mathrm{stop}\, I, n))(\mathrm{inspos}\, n) = (\mathrm{Shift}(I, n))(\mathrm{inspos}\, n)$.

(20) For every state s of SCMPDS and for every instruction i of SCMPDS such that $\mathrm{InsCode}(i) \in \{0, 4, 5, 6\}$ holds $\mathrm{Exec}(i, s) \restriction \text{Data-Loc}_{\mathrm{SCM}} = s \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(21) For all states s, s_3 of SCMPDS holds $(s {+}\!\cdot s_3 \restriction$the instruction locations of SCMPDS$) \restriction \text{Data-Loc}_{\mathrm{SCM}} = s \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(22) For every instruction i of SCMPDS holds $\mathrm{rng}\, \mathrm{Load}(i) = \{i\}$.

(23) If $\mathbf{IC}_{(s_1)} = \mathbf{IC}_{(s_2)}$ and $s_1 \restriction \text{Data-Loc}_{\mathrm{SCM}} = s_2 \restriction \text{Data-Loc}_{\mathrm{SCM}}$, then $\mathbf{IC}_{\mathrm{Exec}(i, s_1)} = \mathbf{IC}_{\mathrm{Exec}(i, s_2)}$ and $\mathrm{Exec}(i, s_1) \restriction \text{Data-Loc}_{\mathrm{SCM}} = \mathrm{Exec}(i, s_2) \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(24) Let s_1, s_2 be states of SCMPDS and I be a Program-block. Suppose I is closed on s_1 and Initialized(stop I) $\subseteq s_1$ and Initialized(stop I) $\subseteq s_2$ and $s_1 \restriction$Data-Loc$_{\rm SCM}$ $= s_2 \restriction$Data-Loc$_{\rm SCM}$. Let i be a natural number. Then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and CurInstr((Computation(s_1))(i)) = CurInstr((Computation(s_2))(i)) and (Computation(s_1))(i)$\restriction$Data-Loc$_{\rm SCM}$ = (Computation(s_2))(i)$\restriction$Data-Loc$_{\rm SCM}$.

(25) Let s_1, s_2 be states of SCMPDS and I be a Program-block. Suppose I is closed on s_1 and halting on s_1 and $s_1 \restriction$Data-Loc$_{\rm SCM}$ $= s_2 \restriction$Data-Loc$_{\rm SCM}$. Let k be a natural number. Then (Computation($s_1 +\cdot$ Initialized(stop I)))(k) and (Computation($s_2 +\cdot$ Initialized(stop I)))(k) are equal outside the instruction locations of SCMPDS and CurInstr((Computation($s_1 +\cdot$ Initialized (stop I)))(k)) = CurInstr((Computation($s_2 +\cdot$ Initialized(stop I)))(k)).

(26) Let I be a Program-block. Suppose that
- (i) I is closed on s_1 and halting on s_1,
- (ii) Initialized(stop I) $\subseteq s_1$,
- (iii) Initialized(stop I) $\subseteq s_2$, and
- (iv) s_1 and s_2 are equal outside the instruction locations of SCMPDS.

Let k be a natural number. Then (Computation(s_1))(k) and (Computation(s_2))(k) are equal outside the instruction locations of SCMPDS and CurInstr((Computation(s_1))(k)) = CurInstr((Computation(s_2))(k)).

(27) Let s_1, s_2 be states of SCMPDS and I be a Program-block. Suppose I is closed on s_1 and halting on s_1 and Initialized(stop I) $\subseteq s_1$ and Initialized(stop I) $\subseteq s_2$ and $s_1 \restriction$Data-Loc$_{\rm SCM}$ $= s_2 \restriction$Data-Loc$_{\rm SCM}$. Then LifeSpan(s_1) = LifeSpan(s_2).

(28) Let I be a Program-block. Suppose that
- (i) I is closed on s_1 and halting on s_1,
- (ii) Initialized(stop I) $\subseteq s_1$,
- (iii) Initialized(stop I) $\subseteq s_2$, and
- (iv) s_1 and s_2 are equal outside the instruction locations of SCMPDS.

Then LifeSpan(s_1) = LifeSpan(s_2) and Result(s_1) and Result(s_2) are equal outside the instruction locations of SCMPDS.

(29) Let s_1, s_2 be states of SCMPDS and I be a Program-block. Suppose I is closed on s_1 and halting on s_1 and $s_1 \restriction$Data-Loc$_{\rm SCM}$ $= s_2 \restriction$Data-Loc$_{\rm SCM}$. Then LifeSpan($s_1 +\cdot$ Initialized(stop I)) = LifeSpan($s_2 +\cdot$ Initialized(stop I)) and Result($s_1 +\cdot$ Initialized(stop I)) and Result($s_2 +\cdot$ Initialized(stop I)) are equal outside the instruction locations of SCMPDS.

(30) Let s_1, s_2 be states of SCMPDS and I be a Program-block. Suppose that
- (i) I is closed on s_1 and halting on s_1,
- (ii) Initialized(stop I) $\subseteq s_1$,

(iii) Initialized(stop I) $\subseteq s_2$, and
(iv) there exists a natural number k such that (Computation(s_1))(k) and s_2 are equal outside the instruction locations of SCMPDS.
Then Result(s_1) and Result(s_2) are equal outside the instruction locations of SCMPDS.

Let I be a Program-block. One can check that Initialized(I) is initial.

The following propositions are true:

(31) Let s be a state of SCMPDS, I be a Program-block, and a be a Int position. If I is halting on s, then $(\mathrm{IExec}(I,s))(a) = (\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop}\,I)))(\mathrm{LifeSpan}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop}\,I)))(a)$.

(32) Let s be a state of SCMPDS, I be a parahalting Program-block, and a be a Int position. Then $(\mathrm{IExec}(I,s))(a) =$ $(\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop}\,I)))(\mathrm{LifeSpan}(s{+\cdot}\,\mathrm{Initialized}(\mathrm{stop}\,I)))(a)$.

(33) Let I be a Program-block and i be a natural number. If Initialized(stop I) $\subseteq s$ and I is closed on s and halting on s and $i <$ LifeSpan(s), then $\mathbf{IC}_{(\mathrm{Computation}(s))(i)} \in \mathrm{dom}\,I$.

(34) Let I be a shiftable Program-block. Suppose Initialized(stop I) $\subseteq s_1$ and I is closed on s_1 and halting on s_1. Let n be a natural number. Suppose Shift$(I,n) \subseteq s_2$ and $\mathrm{card}\,I > 0$ and $\mathbf{IC}_{(s_2)} = \mathrm{inspos}\,n$ and $s_1 \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}} = s_2 \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$. Let i be a natural number. If $i <$ LifeSpan(s_1), then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(i)} + n = \mathbf{IC}_{(\mathrm{Computation}(s_2))(i)}$ and CurInstr((Computation(s_1))(i)) = CurInstr((Computation(s_2))(i)) and $(\mathrm{Computation}(s_1))(i) \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}} = (\mathrm{Computation}(s_2))(i) \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(35) For every No-StopCode Program-block I such that Initialized(stop I) $\subseteq$ s and I is halting on s and $\mathrm{card}\,I > 0$ holds LifeSpan(s) > 0.

(36) Let I be a No-StopCode shiftable Program-block. Suppose Initialized (stop I) $\subseteq s_1$ and I is closed on s_1 and halting on s_1. Let n be a natural number. Suppose Shift$(I,n) \subseteq s_2$ and $\mathrm{card}\,I > 0$ and $\mathbf{IC}_{(s_2)} = \mathrm{inspos}\,n$ and $s_1 \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}} = s_2 \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$. Then $\mathbf{IC}_{(\mathrm{Computation}(s_2))(\mathrm{LifeSpan}(s_1))} = \mathrm{inspos}\,\mathrm{card}\,I + n$ and (Computation(s_1)) $(\mathrm{LifeSpan}(s_1)) \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}} =$ $(\mathrm{Computation}(s_2))(\mathrm{LifeSpan}(s_1)) \restriction \mathrm{Data\text{-}Loc}_{\mathrm{SCM}}$.

(37) Let s be a state of SCMPDS, I be a Program-block, and n be a natural number. If $\mathbf{IC}_{(\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(I)))(n)} = \mathrm{inspos}\,0$, then $(\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(I)))(n){+\cdot}\,\mathrm{Initialized}(I) =$ $(\mathrm{Computation}(s{+\cdot}\,\mathrm{Initialized}(I)))(n)$.

(38) Let I be a Program-block, J be a Program-block, and k be a natural number. Suppose I is closed on s and halting on s and $k \leqslant$ LifeSpan($s{+\cdot}$ Initialized(stop I)). Then (Computation($s{+\cdot}$ Initialized (stop I)))(k) and $(\mathrm{Computation}(s{+\cdot}((I;J){+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{inspos}\,0))))(k)$ are

equal outside the instruction locations of SCMPDS.

(39) Let I, J be Program-block and k be a natural number. Suppose $I \subseteq J$ and I is closed on s and halting on s and $k \leqslant \mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I))$. Then $(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(J)))(k)$ and $(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))(k)$ are equal outside the instruction locations of SCMPDS.

(40) Let I, J be Program-block and k be a natural number. Suppose $k \leqslant \mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I))$ and $I \subseteq J$ and I is closed on s and halting on s. Then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(J)))(k)} \in \mathrm{dom}\, \mathrm{stop}\, I$.

(41) Let I, J be Program-block. Suppose $I \subseteq J$ and I is closed on s and halting on s. Then $\mathrm{CurInstr}((\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(J)))$ $(\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))) = \mathbf{halt}_{\mathrm{SCMPDS}}$ or $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(J)))(\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))} = \mathrm{inspos}\, \mathrm{card}\, I$.

(42) Let I, J be Program-block. Suppose I is halting on s and J is closed on $\mathrm{IExec}(I, s)$ and halting on $\mathrm{IExec}(I, s)$. Then J is closed on $(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))(\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))$ and halting on $(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))$ $(\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))$.

(43) Let I be a Program-block and J be a shiftable Program-block. Suppose I is closed on s and halting on s and J is closed on $\mathrm{IExec}(I, s)$ and halting on $\mathrm{IExec}(I, s)$. Then $I;J$ is closed on s and $I;J$ is halting on s.

(44) Let I be a No-StopCode Program-block and J be a Program-block. If $I \subseteq J$ and I is closed on s and halting on s, then $\mathbf{IC}_{(\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(J)))(\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))} = \mathrm{inspos}\, \mathrm{card}\, I$.

(45) Let I be a Program-block, s be a state of SCMPDS, and k be a natural number. If I is halting on s and $k < \mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I))$, then $\mathrm{CurInstr}((\mathrm{Computation}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)))(k)) \neq \mathbf{halt}_{\mathrm{SCMPDS}}$.

(46) Let I, J be Program-block, s be a state of SCMPDS, and k be a natural number. Suppose I is closed on s and halting on s and $k < \mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I))$. Then $\mathrm{CurInstr}((\mathrm{Computation}(s+\cdot$ $\mathrm{Initialized}(\mathrm{stop}\, I;J)))(k)) \neq \mathbf{halt}_{\mathrm{SCMPDS}}$.

(47) Let I be a No-StopCode Program-block and J be a shiftable Program-block. Suppose I is closed on s and halting on s and J is closed on $\mathrm{IExec}(I, s)$ and halting on $\mathrm{IExec}(I, s)$. Then $\mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I;J)) = \mathrm{LifeSpan}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I)) + \mathrm{LifeSpan}(\mathrm{Result}(s+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, I))+\cdot\, \mathrm{Initialized}(\mathrm{stop}\, J))$.

(48) Let I be a No-StopCode Program-block and J be a shiftable Program-block. Suppose I is closed on s and halting on s and J is closed on $\mathrm{IExec}(I, s)$ and halting on $\mathrm{IExec}(I, s)$. Then $\mathrm{IExec}(I;J, s) = \mathrm{IExec}(J, \mathrm{IExec}(I, s))+\cdot\, \mathrm{Start\text{-}At}(\mathbf{IC}_{\mathrm{IExec}(J, \mathrm{IExec}(I, s))} + \mathrm{card}\, I)$.

(49) Let I be a No-StopCode Program-block and J be a shiftable Program-block. Suppose I is closed on s and halting on s and J is closed on $\mathrm{IExec}(I, s)$ and halting on $\mathrm{IExec}(I, s)$. Then $(\mathrm{IExec}(I;J, s))(a) = (\mathrm{IExec}(J, \mathrm{IExec}(I, s)))(a)$.

(50) Let I be a No-StopCode Program-block and j be a parahalting shiftable instruction of SCMPDS. If I is closed on s and halting on s, then $(\mathrm{IExec}(I;j, s))(a) = (\mathrm{Exec}(j, \mathrm{IExec}(I, s)))(a)$.

2. The Construction of for-up loop Program

Let a be a Int position, let i be an integer, let n be a natural number, and let I be a Program-block. The functor for-up(a, i, n, I) yielding a Program-block is defined by:

(Def. 1) for-up$(a, i, n, I) = ((a, i) >= 0_goto\, \mathrm{card}\, I + 3);I; \mathrm{AddTo}(a, i, n);$ goto $(-(\mathrm{card}\, I + 2))$.

3. The Computation of for-up loop Program

We now state several propositions:

(51) Let a be a Int position, i be an integer, n be a natural number, and I be a Program-block. Then card for-up$(a, i, n, I) = \mathrm{card}\, I + 3$.

(52) Let a be a Int position, i be an integer, n, m be natural numbers, and I be a Program-block. Then $m < \mathrm{card}\, I + 3$ if and only if $\mathrm{inspos}\, m \in$ dom for-up(a, i, n, I).

(53) Let a be a Int position, i be an integer, n be a natural number, and I be a Program-block. Then (for-up(a, i, n, I))(inspos 0) $= (a, i) >= 0\_goto\, \mathrm{card}\, I + 3$ and (for-up(a, i, n, I))(inspos card $I + 1$) $= \mathrm{AddTo}(a, i, n)$ and (for-up(a, i, n, I))(inspos card $I + 2$) $=$ goto $(-(\mathrm{card}\, I + 2))$.

(54) Let s be a state of SCMPDS, I be a Program-block, a be a Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a), i)) \geqslant 0$, then for-up$(a, i, n, I)$ is closed on s and for-up(a, i, n, I) is halting on s.

(55) Let s be a state of SCMPDS, I be a Program-block, a, c be Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a), i)) \geqslant 0$, then IExec(for-up$(a, i, n, I), s) = s +\cdot$ Start-At(inspos card $I + 3$).

(56) Let s be a state of SCMPDS, I be a Program-block, a be a Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a), i)) \geqslant 0$, then $\mathbf{IC}_{\mathrm{IExec}(\text{for-up}(a,i,n,I),s)} = \mathrm{inspos}\, \mathrm{card}\, I + 3$.

(57) Let s be a state of SCMPDS, I be a Program-block, a, b be Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a), i)) \geqslant 0$, then $(\mathrm{IExec}(\text{for-up}(a, i, n, I), s))(b) = s(b)$.

(58) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a be a Int position, i be an integer, n be a natural number, and X be a set. Suppose that

(i) $s(\mathrm{DataLoc}(s(a), i)) < 0$,

(ii) $\mathrm{DataLoc}(s(a), i) \notin X$,

(iii) $n > 0$,

(iv) $\mathrm{card}\, I > 0$,

(v) $a \neq \mathrm{DataLoc}(s(a), i)$, and

(vi) for every state t of SCMPDS such that for every Int position x such that $x \in X$ holds $t(x) = s(x)$ and $t(a) = s(a)$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) = t(\mathrm{DataLoc}(s(a), i))$ and I is closed on t and halting on t and for every Int position y such that $y \in X$ holds $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then $\text{for-up}(a, i, n, I)$ is closed on s and $\text{for-up}(a, i, n, I)$ is halting on s.

(59) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a be a Int position, i be an integer, n be a natural number, and X be a set. Suppose that

(i) $s(\mathrm{DataLoc}(s(a), i)) < 0$,

(ii) $\mathrm{DataLoc}(s(a), i) \notin X$,

(iii) $n > 0$,

(iv) $\mathrm{card}\, I > 0$,

(v) $a \neq \mathrm{DataLoc}(s(a), i)$, and

(vi) for every state t of SCMPDS such that for every Int position x such that $x \in X$ holds $t(x) = s(x)$ and $t(a) = s(a)$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) = t(\mathrm{DataLoc}(s(a), i))$ and I is closed on t and halting on t and for every Int position y such that $y \in X$ holds $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then $\mathrm{IExec}(\text{for-up}(a, i, n, I), s) =$
$\mathrm{IExec}(\text{for-up}(a, i, n, I), \mathrm{IExec}(I; \mathrm{AddTo}(a, i, n), s))$.

Let I be a shiftable Program-block, let a be a Int position, let i be an integer, and let n be a natural number. Observe that $\text{for-up}(a, i, n, I)$ is shiftable.

Let I be a No-StopCode Program-block, let a be a Int position, let i be an integer, and let n be a natural number. Note that $\text{for-up}(a, i, n, I)$ is No-StopCode.

4. The Construction of for-down loop Program

Let a be a Int position, let i be an integer, let n be a natural number, and let I be a Program-block. The functor $\mathrm{for}-\mathrm{down}(a,i,n,I)$ yielding a Program-block is defined as follows:

(Def. 2) $\mathrm{for}-\mathrm{down}(a,i,n,I) = ((a,i) <= 0\_goto\,\mathrm{card}\,I+3);I;\mathrm{AddTo}(a,i,-n);$ goto $(-(\mathrm{card}\,I+2))$.

5. The Computation of for-down loop Program

One can prove the following propositions:

(60) Let a be a Int position, i be an integer, n be a natural number, and I be a Program-block. Then $\mathrm{card}\,\mathrm{for}-\mathrm{down}(a,i,n,I) = \mathrm{card}\,I+3$.

(61) Let a be a Int position, i be an integer, n, m be natural numbers, and I be a Program-block. Then $m < \mathrm{card}\,I+3$ if and only if $\mathrm{inspos}\,m \in \mathrm{dom}\,\mathrm{for}-\mathrm{down}(a,i,n,I)$.

(62) Let a be a Int position, i be an integer, n be a natural number, and I be a Program-block. Then $(\mathrm{for}-\mathrm{down}(a,i,n,I))(\mathrm{inspos}\,0) = (a,i) <= 0\_goto\,\mathrm{card}\,I+3$ and $(\mathrm{for}-\mathrm{down}(a,i,n,I))(\mathrm{inspos}\,\mathrm{card}\,I+1) = \mathrm{AddTo}(a,i,-n)$ and $(\mathrm{for}-\mathrm{down}(a,i,n,I))(\mathrm{inspos}\,\mathrm{card}\,I+2) =$ goto $(-(\mathrm{card}\,I+2))$.

(63) Let s be a state of SCMPDS, I be a Program-block, a be a Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a),i)) \leqslant 0$, then $\mathrm{for}-\mathrm{down}(a,i,n,I)$ is closed on s and $\mathrm{for}-\mathrm{down}(a,i,n,I)$ is halting on s.

(64) Let s be a state of SCMPDS, I be a Program-block, a, c be Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a),i)) \leqslant 0$, then $\mathrm{IExec}(\mathrm{for}-\mathrm{down}(a,i,n,I),s) = s{+\cdot}\,\mathrm{Start\text{-}At}(\mathrm{inspos}\,\mathrm{card}\,I+3)$.

(65) Let s be a state of SCMPDS, I be a Program-block, a be a Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a),i)) \leqslant 0$, then $\mathbf{IC}_{\mathrm{IExec}(\mathrm{for}-\mathrm{down}(a,i,n,I),s)} = \mathrm{inspos}\,\mathrm{card}\,I+3$.

(66) Let s be a state of SCMPDS, I be a Program-block, a, b be Int position, i be an integer, and n be a natural number. If $s(\mathrm{DataLoc}(s(a),i)) \leqslant 0$, then $(\mathrm{IExec}(\mathrm{for}-\mathrm{down}(a,i,n,I),s))(b) = s(b)$.

(67) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a be a Int position, i be an integer, n be a natural number, and X be a set. Suppose that

(i) $s(\mathrm{DataLoc}(s(a),i)) > 0$,

(ii) $\mathrm{DataLoc}(s(a), i) \notin X$,

(iii) $n > 0$,

(iv) $\mathrm{card}\, I > 0$,

(v) $a \neq \mathrm{DataLoc}(s(a), i)$, and

(vi) for every state t of SCMPDS such that for every Int position x such that $x \in X$ holds $t(x) = s(x)$ and $t(a) = s(a)$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) = t(\mathrm{DataLoc}(s(a), i))$ and I is closed on t and halting on t and for every Int position y such that $y \in X$ holds $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then $\mathrm{for} - \mathrm{down}(a, i, n, I)$ is closed on s and $\mathrm{for} - \mathrm{down}(a, i, n, I)$ is halting on s.

(68) Let s be a state of SCMPDS, I be a No-StopCode shiftable Program-block, a be a Int position, i be an integer, n be a natural number, and X be a set. Suppose that

(i) $s(\mathrm{DataLoc}(s(a), i)) > 0$,

(ii) $\mathrm{DataLoc}(s(a), i) \notin X$,

(iii) $n > 0$,

(iv) $\mathrm{card}\, I > 0$,

(v) $a \neq \mathrm{DataLoc}(s(a), i)$, and

(vi) for every state t of SCMPDS such that for every Int position x such that $x \in X$ holds $t(x) = s(x)$ and $t(a) = s(a)$ holds $(\mathrm{IExec}(I, t))(a) = t(a)$ and $(\mathrm{IExec}(I, t))(\mathrm{DataLoc}(s(a), i)) = t(\mathrm{DataLoc}(s(a), i))$ and I is closed on t and halting on t and for every Int position y such that $y \in X$ holds $(\mathrm{IExec}(I, t))(y) = t(y)$.

Then $\mathrm{IExec}(\mathrm{for} - \mathrm{down}(a, i, n, I), s) = \mathrm{IExec}(\mathrm{for} - \mathrm{down}(a, i, n, I),$
$\mathrm{IExec}(I; \mathrm{AddTo}(a, i, -n), s))$.

Let I be a shiftable Program-block, let a be a Int position, let i be an integer, and let n be a natural number. Observe that $\mathrm{for} - \mathrm{down}(a, i, n, I)$ is shiftable.

Let I be a No-StopCode Program-block, let a be a Int position, let i be an integer, and let n be a natural number. Note that $\mathrm{for} - \mathrm{down}(a, i, n, I)$ is No-StopCode.

6. Two Examples for Summing

Let n be a natural number. The functor $\mathrm{sum}\, n$ yielding a Program-block is defined as follows:

(Def. 3) $\mathrm{sum}\, n = (\mathrm{GBP} := 0); ((\mathrm{GBP})_2 := n); ((\mathrm{GBP})_3 := 0); \mathrm{for} - \mathrm{down}(\mathrm{GBP}, 2, 1, \mathrm{Load}(\mathrm{AddTo}(\mathrm{GBP}, 3, 1)))$.

Next we state three propositions:

(69) For every state s of SCMPDS such that $s(\mathrm{GBP}) = 0$ holds $\mathrm{for-down}(\mathrm{GBP}, 2, 1, \mathrm{Load}(\mathrm{AddTo}(\mathrm{GBP}, 3, 1)))$ is closed on s and $\mathrm{for-down}(\mathrm{GBP}, 2, 1, \mathrm{Load}(\mathrm{AddTo}(\mathrm{GBP}, 3, 1)))$ is halting on s.

(70) Let s be a state of SCMPDS and n be a natural number. If $s(\mathrm{GBP}) = 0$ and $s(\mathrm{intpos}\,2) = n$ and $s(\mathrm{intpos}\,3) = 0$, then $(\mathrm{IExec}(\mathrm{for-down}(\mathrm{GBP}, 2, 1, \mathrm{Load}(\mathrm{AddTo}(\mathrm{GBP}, 3, 1))), s))(\mathrm{intpos}\,3) = n$.

(71) For every state s of SCMPDS and for every natural number n holds $(\mathrm{IExec}(\mathrm{sum}\,n, s))(\mathrm{intpos}\,3) = n$.

Let s_4, c_1, r_1, p_1, p_2 be natural numbers. The functor $\mathrm{sum}(s_4, c_1, r_1, p_1, p_2)$ yields a Program-block and is defined as follows:

(Def. 4) $\mathrm{sum}(s_4, c_1, r_1, p_1, p_2) = ((\mathrm{intpos}\,s_4)_{r_1}{:=}0);(\mathrm{intpos}\,p_1{:=}p_2);$ $\mathrm{for-down}(\mathrm{intpos}\,s_4, c_1, 1, \mathrm{AddTo}(\mathrm{intpos}\,s_4, r_1, \mathrm{intpos}\,p_2, 0);$ $\mathrm{AddTo}(\mathrm{intpos}\,p_1, 0, 1))$.

Next we state three propositions:

(72) Let s be a state of SCMPDS and s_4, c_2, r_1, p_1, p_3 be natural numbers. Suppose $s(\mathrm{intpos}\,s_4) > s_4$ and $c_2 < r_1$ and $s(\mathrm{intpos}\,p_1) = p_3$ and $s(\mathrm{intpos}\,s_4) + r_1 < p_1$ and $p_1 < p_3$ and $p_3 < s(\mathrm{intpos}\,p_3)$. Then $\mathrm{for-down}(\mathrm{intpos}\,s_4, c_2, 1, \mathrm{AddTo}(\mathrm{intpos}\,s_4, r_1, \mathrm{intpos}\,p_3, 0);$ $\mathrm{AddTo}(\mathrm{intpos}\,p_1, 0, 1))$ is closed on s and $\mathrm{for-down}(\mathrm{intpos}\,s_4, c_2, 1,$ $\mathrm{AddTo}(\mathrm{intpos}\,s_4, r_1, \mathrm{intpos}\,p_3, 0); \mathrm{AddTo}(\mathrm{intpos}\,p_1, 0, 1))$ is halting on s.

(73) Let s be a state of SCMPDS, s_4, c_2, r_1, p_1, p_3 be natural numbers, and f be a finite sequence of elements of $\mathbb{N}$. Suppose that $s(\mathrm{intpos}\,s_4) > s_4$ and $c_2 < r_1$ and $s(\mathrm{intpos}\,p_1) = p_3$ and $s(\mathrm{intpos}\,s_4) + r_1 < p_1$ and $p_1 < p_3$ and $p_3 < s(\mathrm{intpos}\,p_3)$ and $s(\mathrm{DataLoc}(s(\mathrm{intpos}\,s_4), r_1)) = 0$ and $\mathrm{len}\,f = s(\mathrm{DataLoc}(s(\mathrm{intpos}\,s_4), c_2))$ and for every natural number k such that $k < \mathrm{len}\,f$ holds $f(k + 1) = s(\mathrm{DataLoc}(s(\mathrm{intpos}\,p_3), k))$. Then $(\mathrm{IExec}(\mathrm{for-down}(\mathrm{intpos}\,s_4, c_2, 1, \mathrm{AddTo}(\mathrm{intpos}\,s_4, r_1, \mathrm{intpos}\,p_3, 0);$ $\mathrm{AddTo}(\mathrm{intpos}\,p_1, 0, 1)), s))(\mathrm{DataLoc}(s(\mathrm{intpos}\,s_4), r_1)) = \sum f$.

(74) Let s be a state of SCMPDS, s_4, c_2, r_1, p_1, p_3 be natural numbers, and f be a finite sequence of elements of $\mathbb{N}$. Suppose that $s(\mathrm{intpos}\,s_4) > s_4$ and $c_2 < r_1$ and $s(\mathrm{intpos}\,s_4) + r_1 < p_1$ and $p_1 < p_3$ and $p_3 < s(\mathrm{intpos}\,p_3)$ and $\mathrm{len}\,f = s(\mathrm{DataLoc}(s(\mathrm{intpos}\,s_4), c_2))$ and for every natural number k such that $k < \mathrm{len}\,f$ holds $f(k + 1) = s(\mathrm{DataLoc}(s(\mathrm{intpos}\,p_3), k))$. Then $(\mathrm{IExec}(\mathrm{sum}(s_4, c_2, r_1, p_1, p_3), s))$ $(\mathrm{DataLoc}(s(\mathrm{intpos}\,s_4), r_1)) = \sum f$.

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Predicate Calculus for Boolean Valued Functions. Part XII

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Summary. In this paper, we proved some elementary predicate calculus formulae containing the quantifiers of Boolean valued functions with respect to partitions. Such a theory is an analogy of ordinary predicate logic.

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The terminology and notation used here are introduced in the following articles: [11], [4], [6], [1], [8], [7], [2], [3], [5], [12], [10], and [9].

1. Preliminaries

For simplicity, we adopt the following convention: Y is a non empty set, a is an element of $\mathrm{BVF}(Y)$, G is a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E, F, J, M, N are partitions of Y, and x, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 are sets.

The following propositions are true:

(1) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D \wedge E \wedge F \wedge J$.

(2) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$

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and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D \wedge E \wedge F \wedge J$.

(3) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D \wedge E \wedge F \wedge J$.

(4) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(D, G) = A \wedge B \wedge C \wedge E \wedge F \wedge J$.

(5) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(E, G) = A \wedge B \wedge C \wedge D \wedge F \wedge J$.

(6) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(F, G) = A \wedge B \wedge C \wedge D \wedge E \wedge J$.

(7) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{CompF}(J, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F$.

(8) Let A, B, C, D, E, F, J be sets, h be a function, and A', B', C', D', E', F', J' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (J \dot{\longmapsto} J') +\cdot (A \dot{\longmapsto} A')$. Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$

and $h(E) = E'$ and $h(F) = F'$ and $h(J) = J'$.

(9) Let A, B, C, D, E, F, J be sets, h be a function, and A', B', C', D', E', F', J' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$ and $h = (B \dot{\longmapsto} B') + \cdot (C \dot{\longmapsto} C') + \cdot (D \dot{\longmapsto} D') + \cdot (E \dot{\longmapsto} E') + \cdot (F \dot{\longmapsto} F') + \cdot (J \dot{\longmapsto} J') + \cdot (A \dot{\longmapsto} A')$. Then $\operatorname{dom} h = \{A, B, C, D, E, F, J\}$.

(10) Let A, B, C, D, E, F, J be sets, h be a function, and A', B', C', D', E', F', J' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$ and $h = (B \dot{\longmapsto} B') + \cdot (C \dot{\longmapsto} C') + \cdot (D \dot{\longmapsto} D') + \cdot (E \dot{\longmapsto} E') + \cdot (F \dot{\longmapsto} F') + \cdot (J \dot{\longmapsto} J') + \cdot (A \dot{\longmapsto} A')$. Then $\operatorname{rng} h = \{h(A), h(B), h(C), h(D), h(E), h(F), h(J)\}$.

(11) Let G be a subset of PARTITIONS(Y), A, B, C, D, E, F, J be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D \wedge E \wedge F \wedge J) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(12) Let G be a subset of PARTITIONS(Y), A, B, C, D, E, F, J be partitions of Y, and z, u be elements of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$ and $\mathrm{EqClass}(z, C \wedge D \wedge E \wedge F \wedge J) = \mathrm{EqClass}(u, C \wedge D \wedge E \wedge F \wedge J)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

(13) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M$.

(14) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$

and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M$.

(15) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D \wedge E \wedge F \wedge J \wedge M$.

(16) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(D, G) = A \wedge B \wedge C \wedge E \wedge F \wedge J \wedge M$.

(17) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(E, G) = A \wedge B \wedge C \wedge D \wedge F \wedge J \wedge M$.

(18) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(F, G) = A \wedge B \wedge C \wedge D \wedge E \wedge J \wedge M$.

(19) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(J, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge M$.

(20) Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{CompF}(M, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge J$.

(21) Let A, B, C, D, E, F, J, M be sets, h be a function, and A', B', C', D', E', F', J', M' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (J \dot{\longmapsto} J') +\cdot (M \dot{\longmapsto} M') +\cdot (A \dot{\longmapsto} A')$. Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$ and $h(E) = E'$ and $h(F) = F'$ and $h(J) = J'$ and $h(M) = M'$.

(22) Let A, B, C, D, E, F, J, M be sets, h be a function, and A', B', C', D', E', F', J', M' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (J \dot{\longmapsto} J') +\cdot (M \dot{\longmapsto} M') +\cdot (A \dot{\longmapsto} A')$. Then $\mathrm{dom}\, h = \{A, B, C, D, E, F, J, M\}$.

(23) Let A, B, C, D, E, F, J, M be sets, h be a function, and A', B', C', D', E', F', J', M' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$ and $h = (B \dot{\longmapsto} B') +\cdot (C \dot{\longmapsto} C') +\cdot (D \dot{\longmapsto} D') +\cdot (E \dot{\longmapsto} E') +\cdot (F \dot{\longmapsto} F') +\cdot (J \dot{\longmapsto} J') +\cdot (M \dot{\longmapsto} M') +\cdot (A \dot{\longmapsto} A')$. Then $\mathrm{rng}\, h = \{h(A), h(B), h(C), h(D), h(E), h(F), h(J), h(M)\}$.

(24) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E, F, J, M be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and

$B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(25) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E, F, J, M be partitions of Y, and z, u be elements of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$ and $\mathrm{EqClass}(z, C \wedge D \wedge E \wedge F \wedge J \wedge M) = \mathrm{EqClass}(u, C \wedge D \wedge E \wedge F \wedge J \wedge M)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

The scheme *UI10* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a set $\mathcal{C}$, a set $\mathcal{D}$, a set $\mathcal{E}$, a set $\mathcal{F}$, a set $\mathcal{G}$, a set $\mathcal{H}$, a set $\mathcal{I}$, a set $\mathcal{J}$, and and states that:

$\mathcal{P}[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}]$

provided the following condition is satisfied:

- For all sets x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}
  holds $\mathcal{P}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$.

Let us consider x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9.

The functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ yielding a set is defined as follows:

(Def. 1) $x \in \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ iff $x = x_1$ or $x = x_2$ or $x = x_3$ or $x = x_4$ or $x = x_5$ or $x = x_6$ or $x = x_7$ or $x = x_8$ or $x = x_9$.

We now state a number of propositions:

(26) $x \in \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ iff $x = x_1$ or $x = x_2$ or $x = x_3$ or $x = x_4$ or $x = x_5$ or $x = x_6$ or $x = x_7$ or $x = x_8$ or $x = x_9$.

(27) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1\} \cup \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

(28) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2\} \cup \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

(29) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3\} \cup \{x_4, x_5, x_6, x_7, x_8, x_9\}$.

(30) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3, x_4\} \cup \{x_5, x_6, x_7, x_8, x_9\}$.

(31) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3, x_4, x_5\} \cup \{x_6, x_7, x_8, x_9\}$.

(32) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \{x_7, x_8, x_9\}$.

(33) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \cup \{x_8, x_9\}$.

(34) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \cup \{x_9\}$.

(35) Let G be a subset of $\mathrm{PARTITIONS}(Y)$ and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(A, G) = B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N$.

(36) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(B, G) = A \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N$.

(37) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(C, G) = A \wedge B \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N$.

(38) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(D, G) = A \wedge B \wedge C \wedge E \wedge F \wedge J \wedge M \wedge N$.

(39) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and

$A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(E, G) = A \wedge B \wedge C \wedge D \wedge F \wedge J \wedge M \wedge N$.

(40) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(F, G) = A \wedge B \wedge C \wedge D \wedge E \wedge J \wedge M \wedge N$.

(41) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(J, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge M \wedge N$.

(42) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(M, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge N$.

(43) Let G be a subset of PARTITIONS(Y) and A, B, C, D, E, F, J, M, N be partitions of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and

$A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{CompF}(N, G) = A \wedge B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M$.

(44) Let A, B, C, D, E, F, J, M, N be sets, h be a function, and A', B', C', D', E', F', J', M', N' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$ and $h = (B \longmapsto B') +\cdot (C \longmapsto C') +\cdot (D \longmapsto D') +\cdot (E \longmapsto E') +\cdot (F \longmapsto F') +\cdot (J \longmapsto J') +\cdot (M \longmapsto M') +\cdot (N \longmapsto N') +\cdot (A \longmapsto A')$. Then $h(A) = A'$ and $h(B) = B'$ and $h(C) = C'$ and $h(D) = D'$ and $h(E) = E'$ and $h(F) = F'$ and $h(J) = J'$ and $h(M) = M'$ and $h(N) = N'$.

(45) Let A, B, C, D, E, F, J, M, N be sets, h be a function, and A', B', C', D', E', F', J', M', N' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$ and $h = (B \longmapsto B') +\cdot (C \longmapsto C') +\cdot (D \longmapsto D') +\cdot (E \longmapsto E') +\cdot (F \longmapsto F') +\cdot (J \longmapsto J') +\cdot (M \longmapsto M') +\cdot (N \longmapsto N') +\cdot (A \longmapsto A')$.
Then $\operatorname{dom} h = \{A, B, C, D, E, F, J, M, N\}$.

(46) Let A, B, C, D, E, F, J, M, N be sets, h be a function, and A', B', C', D', E', F', J', M', N' be sets. Suppose that
$A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$ and $h = (B \longmapsto B') +\cdot (C \longmapsto C') +\cdot (D \longmapsto D') +\cdot (E \longmapsto E') +\cdot (F \longmapsto F') +\cdot (J \longmapsto J') +\cdot (M \longmapsto M') +\cdot (N \longmapsto N') +\cdot (A \longmapsto A')$.
Then $\operatorname{rng} h = \{h(A), h(B), h(C), h(D), h(E), h(F), h(J), h(M), h(N)\}$.

(47) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, A,

B, C, D, E, F, J, M, N be partitions of Y, z, u be elements of Y, and h be a function. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\mathrm{EqClass}(u, B \wedge C \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N) \cap \mathrm{EqClass}(z, A) \neq \emptyset$.

(48) Let a be an element of $\mathrm{BVF}(Y)$, G be a subset of $\mathrm{PARTITIONS}(Y)$, A, B, C, D, E, F, J, M, N be partitions of Y, and z, u be elements of Y. Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$ and $\mathrm{EqClass}(z, C \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N) = \mathrm{EqClass}(u, C \wedge D \wedge E \wedge F \wedge J \wedge M \wedge N)$. Then $\mathrm{EqClass}(u, \mathrm{CompF}(A, G)) \cap \mathrm{EqClass}(z, \mathrm{CompF}(B, G)) \neq \emptyset$.

2. Predicate Calculus

We now state a number of propositions:

(49) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\forall_{a,B}G,A}G$.

(50) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(51) Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(52) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\exists_{\exists_{a,B}G,A}G \Subset \exists_{\exists_{a,A}G,B}G$.

(53) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(54) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(55) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(56) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$ and $F \neq J$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(57) $\forall_{\exists_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(58) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $E \neq F$ and $E \neq J$

and $F \neq J$. Then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(59) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\forall_{a,B}G,A}G$.

(60) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(61) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(62) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\exists_{\exists_{a,B}G,A}G \Subset \exists_{\exists_{a,A}G,B}G$.

(63) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(64) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$

and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(66)[1] Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(67) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $F \neq J$ and $F \neq M$ and $J \neq M$. Then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(68) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\forall_{a,B}G,A}G$.

(69) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\forall_{\forall_{a,A}G,B}G = \forall_{\forall_{a,B}G,A}G$.

(70) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$

[1] The proposition (65) has been removed.

and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\exists_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(71) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\exists_{\exists_{a,B}G,A}G \Subset \exists_{\exists_{a,A}G,B}G$.

(72) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\exists_{\exists_{a,A}G,B}G = \exists_{\exists_{a,B}G,A}G$.

(73) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\forall_{a,B}G,A}G$.

(74) $\forall_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

(75) Suppose that
G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\forall_{\forall_{a,A}G,B}G \Subset \forall_{\exists_{a,B}G,A}G$.

(76) Suppose that

G is a coordinate and $G = \{A, B, C, D, E, F, J, M, N\}$ and $A \neq B$ and $A \neq C$ and $A \neq D$ and $A \neq E$ and $A \neq F$ and $A \neq J$ and $A \neq M$ and $A \neq N$ and $B \neq C$ and $B \neq D$ and $B \neq E$ and $B \neq F$ and $B \neq J$ and $B \neq M$ and $B \neq N$ and $C \neq D$ and $C \neq E$ and $C \neq F$ and $C \neq J$ and $C \neq M$ and $C \neq N$ and $D \neq E$ and $D \neq F$ and $D \neq J$ and $D \neq M$ and $D \neq N$ and $E \neq F$ and $E \neq J$ and $E \neq M$ and $E \neq N$ and $F \neq J$ and $F \neq M$ and $F \neq N$ and $J \neq M$ and $J \neq N$ and $M \neq N$. Then $\exists_{\forall_{a,A}G,B}G \Subset \exists_{\exists_{a,B}G,A}G$.

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