FORMALIZED MATHEMATICS
Volume 6, Number 4, 1997
University of Białystok

Some Properties of Real Maps

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Summary. The main goal of the paper is to show logical equivalence of the two definitions of the *open subset*: one from [2] and the other from [23]. This has been used to show that the other two definitions are equivalent: the continuity of the map as in [20] and in [22]. We used this to show that continuous and one-to-one maps are monotone (see theorems 16 and 17 for details).

MML Identifier: `JORDAN5A`.

The terminology and notation used here are introduced in the following articles: [26], [13], [27], [28], [4], [5], [24], [22], [17], [18], [10], [3], [23], [6], [25], [29], [16], [14], [19], [11], [20], [8], [7], [9], [15], [21], [2], [1], and [12].

1. Preliminaries

One can prove the following four propositions:

(1) For all points p, q of $\mathcal{E}^2_{\rm T}$ and for every subset P of $\mathcal{E}^2_{\rm T}$ such that P is an arc from p to q holds P is compact.

(2) For every real number r holds $0 \leqslant r$ and $r \leqslant 1$ iff $r \in$ the carrier of $\mathbb{I}$.

(3) For all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ and for all real numbers r_1, r_2 such that $(1-r_1)\cdot p_1 + r_1 \cdot p_2 = (1-r_2)\cdot p_1 + r_2 \cdot p_2$ holds $r_1 = r_2$ or $p_1 = p_2$.

(4) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $p_1 \neq p_2$. Then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T}) \restriction \mathcal{L}(p_1, p_2)$ such that for every real number x such that $x \in [0,1]$ holds $f(x) = (1-x)\cdot p_1 + x \cdot p_2$ and f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$.

[1]This paper was written while the author visited the Shinshu University in the winter of 1997.

ISSN 1426–2630

One can verify that $\mathcal{E}^2_{\mathrm{T}}$ is arcwise connected.

One can check that there exists a subspace of $\mathcal{E}^2_{\mathrm{T}}$ which is compact and non empty.

The following proposition is true

(5) Let a, b be points of $\mathcal{E}^2_{\mathrm{T}}$, f be a path from a to b, P be a non empty compact subspace of $\mathcal{E}^2_{\mathrm{T}}$, and g be a map from $\mathbb{I}$ into P. If f is one-to-one and $g = f$ and $\Omega_P = \mathrm{rng}\, f$, then g is a homeomorphism.

2. Equivalence of analytical and topological definitions of continuity

We now state a number of propositions:

(6) Let X be a subset of $\mathbb{R}$. Then $X \in$ the open set family of the metric space of real numbers if and only if X is open.

(7) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$, x be a point of $\mathbb{R}^{\mathbf{1}}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_1 be a real number. If f is continuous at x and $f = g$ and $x = x_1$, then g is continuous in x_1.

(8) Let f be a continuous map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If $f = g$, then g is continuous on $\mathbb{R}$.

(9) Let f be a continuous one-to-one map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Then

(i) for all points x, y of $\mathbb{I}$ and for all real numbers p, q, f_1, f_2 such that $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$ holds $f_1 < f_2$, or

(ii) for all points x, y of $\mathbb{I}$ and for all real numbers p, q, f_1, f_2 such that $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$ holds $f_1 > f_2$.

(10) Let r, g_1, a, b be real numbers and x be an element of the carrier of $[a,\, b]_{\mathrm{M}}$. If $a \leqslant b$ and $x = r$ and $g_1 > 0$ and $]r - g_1, r + g_1[\ \subseteq [a, b]$, then $]r - g_1, r + g_1[\ = \mathrm{Ball}(x, g_1)$.

(11) Let a, b be real numbers and X be a subset of $\mathbb{R}$. Suppose $a < b$ and $a \notin X$ and $b \notin X$. If $X \in$ the open set family of $[a,\, b]_{\mathrm{M}}$, then X is open.

(12) For every open subset X of $\mathbb{R}$ and for all real numbers a, b such that $X \subseteq [a, b]$ holds $a \notin X$ and $b \notin X$.

(13) Let a, b be real numbers, X be a subset of $\mathbb{R}$, and V be a subset of the carrier of $[a,\, b]_{\mathrm{M}}$. Suppose $a \leqslant b$ and $V = X$. If X is open, then $V \in$ the open set family of $[a,\, b]_{\mathrm{M}}$.

(14) Let a, b, c, d, x_1 be real numbers, f be a map from $[a,\, b]_{\mathrm{T}}$ into $[c,\, d]_{\mathrm{T}}$, x be a point of $[a,\, b]_{\mathrm{T}}$, and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $c < d$ and f is continuous at x and $f(a) = c$ and $f(b) = d$ and f is one-to-one and $f = g$ and $x = x_1$. Then $g \restriction [a, b]$ is continuous in x_1.

(15) Let a, b, c, d be real numbers, f be a map from $[a,\ b]_{\mathrm{T}}$ into $[c,\ d]_{\mathrm{T}}$, and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is continuous and one-to-one and $a < b$ and $c < d$ and $f = g$ and $f(a) = c$ and $f(b) = d$. Then g is continuous on $[a, b]$.

3. On the monotonicity of continuous maps

One can prove the following propositions:

(16) Let a, b, c, d be real numbers and f be a map from $[a,\ b]_{\mathrm{T}}$ into $[c,\ d]_{\mathrm{T}}$. Suppose $a < b$ and $c < d$ and f is continuous and one-to-one and $f(a) = c$ and $f(b) = d$. Let x, y be points of $[a,\ b]_{\mathrm{T}}$ and p, q, f_1, f_2 be real numbers. If $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$, then $f_1 < f_2$.

(17) Let f be a continuous one-to-one map from $\mathbb{I}$ into $\mathbb{I}$. Suppose $f(0) = 0$ and $f(1) = 1$. Let x, y be points of $\mathbb{I}$ and p, q, f_1, f_2 be real numbers. If $x = p$ and $y = q$ and $p < q$ and $f_1 = f(x)$ and $f_2 = f(y)$, then $f_1 < f_2$.

(18) Let a, b, c, d be real numbers, f be a map from $[a,\ b]_{\mathrm{T}}$ into $[c,\ d]_{\mathrm{T}}$, P be a non empty subset of $[a,\ b]_{\mathrm{T}}$, and P_1, Q_1 be subsets of $\mathbb{R}^{\mathbf{1}}$. Suppose $a < b$ and $c < d$ and $P_1 = P$ and f is continuous and one-to-one and P_1 is compact and $f(a) = c$ and $f(b) = d$ and $f^{\circ}P = Q_1$. Then $f(\inf(\Omega_{(P_1)})) = \inf(\Omega_{(Q_1)})$.

(19) Let a, b, c, d be real numbers, f be a map from $[a,\ b]_{\mathrm{T}}$ into $[c,\ d]_{\mathrm{T}}$, P, Q be non empty subsets of $[a,\ b]_{\mathrm{T}}$, and P_1, Q_1 be subsets of $\mathbb{R}^{\mathbf{1}}$. Suppose $a < b$ and $c < d$ and $P_1 = P$ and $Q_1 = Q$ and f is continuous and one-to-one and P_1 is compact and $f(a) = c$ and $f(b) = d$ and $f^{\circ}P = Q$. Then $f(\sup(\Omega_{(P_1)})) = \sup(\Omega_{(Q_1)})$.

(20) For all real numbers a, b such that $a \leqslant b$ holds $\inf[a, b] = a$ and $\sup[a, b] = b$.

(21) Let a, b, c, d, e, f, g, h be real numbers and F be a map from $[a,\ b]_{\mathrm{T}}$ into $[c,\ d]_{\mathrm{T}}$. Suppose $a < b$ and $c < d$ and $e < f$ and $a \leqslant e$ and $f \leqslant b$ and F is a homeomorphism and $F(a) = c$ and $F(b) = d$ and $g = F(e)$ and $h = F(f)$. Then $F^{\circ}[e, f] = [g, h]$.

(22) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. Then there exists a point E_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $E_1 \in P \cap Q$, and

(ii) there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and there exists a real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = E_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $0 \leqslant t$ and $t < s_2$ holds $g(t) \notin Q$.

(23) Let P, Q be subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P meets Q and $P \cap Q$ is closed and P is an arc from p_1 to p_2. Then there exists a point E_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $E_1 \in P \cap Q$, and

(ii) there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ and there exists a real number s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_2) = E_1$ and $0 \leqslant s_2$ and $s_2 \leqslant 1$ and for every real number t such that $1 \geqslant t$ and $t > s_2$ holds $g(t) \notin Q$.

References

[1] Józef Białas and Yatsuka Nakamura. The theorem of Weierstrass. *Formalized Mathematics*, 5(**3**):353–359, 1996.
[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[12] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(**4**):449–454, 1997.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[15] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.
[16] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[17] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.
[18] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.
[19] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[21] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[22] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[23] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[24] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[25] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[26] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[27] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[28] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[29] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received September 10, 1997