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Introduction to the Homotopy Theory

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Summary. The paper introduces some preliminary notions concerning the homotopy theory according to [15]: paths and arcwise connected to topological spaces. The basic operations on paths (addition and reversing) are defined. In the last section the predicate: P, Q *are homotopic* is defined. We also showed some properties of the product of two topological spaces needed to prove reflexivity and symmetry of the above predicate.

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The articles [27], [30], [26], [16], [10], [32], [7], [23], [13], [12], [25], [28], [24], [4], [1], [33], [11], [21], [31], [9], [19], [29], [17], [8], [34], [14], [6], [5], [22], [20], [2], [18], and [3] provide the notation and terminology for this paper.

1. Preliminaries

In this paper T, T_1, T_2, S denote non empty topological spaces.

The scheme *FrCard* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$$\overline{\overline{\{\mathcal{F}(w); w \text{ ranges over elements of } \mathcal{A} : w \in \mathcal{B} \wedge \mathcal{P}[w]\}}} \leqslant \overline{\overline{\mathcal{B}}}$$

for all values of the parameters.

The following proposition is true

(1) Let f be a map from T_1 into S and g be a map from T_2 into S. Suppose that

(i) T_1 is a subspace of T,

(ii) T_2 is a subspace of T,

(iii) $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$,

(iv) T_1 is compact,

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(v) T_2 is compact,
(vi) T is a T_2 space,
(vii) f is continuous,
(viii) g is continuous, and
(ix) for every set p such that $p \in \Omega_{(T_1)} \cap \Omega_{(T_2)}$ holds $f(p) = g(p)$.

Then there exists a map h from T into S such that $h = f{+\cdot}g$ and h is continuous.

Let S, T be non empty topological spaces. One can verify that there exists a map from S into T which is continuous.

One can prove the following proposition

(2) For all non empty topological spaces S, T holds every continuous mapping from S into T is a continuous map from S into T.

Let T be a non empty topological structure. Note that id_T is open and continuous.

Let T be a non empty topological structure. Observe that there exists a map from T into T which is continuous and one-to-one.

We now state the proposition

(3) Let S, T be non empty topological spaces and f be a map from S into T. If f is a homeomorphism, then f^{-1} is open.

2. Paths and arcwise connected spaces

Let T be a topological structure and let a, b be points of T. Let us assume that there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. A map from $\mathbb{I}$ into T is said to be a path from a to b if:

(Def. 1) It is continuous and $\mathrm{it}(0) = a$ and $\mathrm{it}(1) = b$.

Next we state the proposition

(4) Let T be a non empty topological space and a be a point of T. Then there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = a$.

Let T be a non empty topological space and let a be a point of T. Note that there exists a path from a to a which is continuous.

Let T be a topological structure. We say that T is arcwise connected if and only if:

(Def. 2) For all points a, b of T there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$.

Let us observe that there exists a topological space which is arcwise connected and non empty.

Let T be an arcwise connected topological structure and let a, b be points of T. Let us note that the path from a to b can be characterized by the following (equivalent) condition:

(Def. 3) It is continuous and it$(0) = a$ and it$(1) = b$.

Let T be an arcwise connected topological structure and let a, b be points of T. Note that every path from a to b is continuous.

Next we state the proposition

(5) For every non empty topological space G_1 such that G_1 is arcwise connected holds G_1 is connected.

Let us mention that every non empty topological space which is arcwise connected is also connected.

3. Basic operations on paths

Let T be a non empty topological space, let a, b, c be points of T, let P be a path from a to b, and let Q be a path from b to c. Let us assume that there exist maps f, g from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$ and g is continuous and $g(0) = b$ and $g(1) = c$. The functor $P + Q$ yielding a path from a to c is defined by the condition (Def. 4).

(Def. 4) Let t be a point of $\mathbb{I}$ and t' be a real number such that $t = t'$. Then

(i) if $0 \leqslant t'$ and $t' \leqslant \frac{1}{2}$, then $(P + Q)(t) = P(2 \cdot t')$, and

(ii) if $\frac{1}{2} \leqslant t'$ and $t' \leqslant 1$, then $(P + Q)(t) = Q(2 \cdot t' - 1)$.

Let T be a non empty topological space and let a be a point of T. Note that there exists a path from a to a which is constant.

One can prove the following two propositions:

(6) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $P = \mathbb{I} \longmapsto a$.

(7) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $P + P = P$.

Let T be a non empty topological space, let a be a point of T, and let P be a constant path from a to a. Observe that $P + P$ is constant.

Let T be a non empty topological space, let a, b be points of T, and let P be a path from a to b. Let us assume that there exists a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. The functor $-P$ yields a path from b to a and is defined as follows:

(Def. 5) For every point t of $\mathbb{I}$ and for every real number t' such that $t = t'$ holds $(-P)(t) = P(1 - t')$.

The following proposition is true

(8) Let T be a non empty topological space, a be a point of T, and P be a constant path from a to a. Then $-P = P$.

Let T be a non empty topological space, let a be a point of T, and let P be a constant path from a to a. One can verify that $-P$ is constant.

4. The product of two topological spaces

One can prove the following proposition

(9) Let X, Y be non empty topological spaces, A be a family of subsets of Y, and f be a map from X into Y. Then $f^{-1}(\bigcup A) = \bigcup(f^{-1}(A))$.

Let S_1, S_2, T_1, T_2 be non empty topological spaces, let f be a map from S_1 into S_2, and let g be a map from T_1 into T_2. Then $[\!: f, g :\!]$ is a map from $[\!: S_1, T_1 :\!]$ into $[\!: S_2, T_2 :\!]$.

Next we state three propositions:

(10) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, g be a continuous map from S_2 into T_2, and P_1, P_2 be subsets of the carrier of $[\!: T_1, T_2 :\!]$. If $P_2 \in \mathrm{BaseAppr}(P_1)$, then $[\!: f, g :\!]^{-1}(P_2)$ is open.

(11) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, g be a continuous map from S_2 into T_2, and P_2 be a subset of the carrier of $[\!: T_1, T_2 :\!]$. If P_2 is open, then $[\!: f, g :\!]^{-1}(P_2)$ is open.

(12) Let S_1, S_2, T_1, T_2 be non empty topological spaces, f be a continuous map from S_1 into T_1, and g be a continuous map from S_2 into T_2. Then $[\!: f, g :\!]$ is continuous.

Let us note that every topological structure which is empty is also T_0.

Let T_1, T_2 be discernible non empty topological spaces. One can check that $[\!: T_1, T_2 :\!]$ is discernible.

We now state two propositions:

(13) For all T_0-spaces T_1, T_2 holds $[\!: T_1, T_2 :\!]$ is a T_0-space.

(14) Let T_1, T_2 be non empty topological spaces. Suppose T_1 is a T_1 space and T_2 is a T_1 space. Then $[\!: T_1, T_2 :\!]$ is a T_1 space.

Let T_1, T_2 be a T_1 space non empty topological spaces. Observe that $[\!: T_1, T_2 :\!]$ is a T_1 space.

Let T_1, T_2 be T_2 non empty topological spaces. Observe that $[\!: T_1, T_2 :\!]$ is T_2.

Let us note that $\mathbb{I}$ is compact and T_2.

Let us mention that $\mathcal{E}^2_{\mathrm{T}}$ is T_2.

Let T be a non empty arcwise connected topological space, let a, b be points of T, and let P, Q be paths from a to b. We say that P, Q are homotopic if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a map f from $[\![\mathbb{I}, \mathbb{I}]\!]$ into T such that

(i) f is continuous, and

(ii) for every point s of $\mathbb{I}$ holds $f(s, 0) = P(s)$ and $f(s, 1) = Q(s)$ and for every point t of $\mathbb{I}$ holds $f(0, t) = a$ and $f(1, t) = b$.

Let us notice that the predicate P, Q are homotopic is reflexive and symmetric.

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