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Lattice of Substitutions

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The articles [8], [6], [5], [7], [1], [9], [2], [4], [11], [3], and [10] provide the terminology and notation for this paper.

1. Preliminaries

In this paper V, C are sets.

Let us consider V, C. The functor SubstitutionSet(V,C) yielding a subset of $\mathrm{Fin}(V \dot{\rightarrow} C)$ is defined as follows:

(Def. 1) $\mathrm{SubstitutionSet}(V,C) = \{A, A$ ranges over elements of $\mathrm{Fin}(V \dot{\rightarrow} C) : \bigwedge_{s,t\,:\,\text{element of } V \dot{\rightarrow} C} (s \in A \wedge t \in A \wedge s \subseteq t \Rightarrow s = t)\}$.

Next we state two propositions:

(1) $\emptyset \in \mathrm{SubstitutionSet}(V,C)$.

(2) $\{\emptyset\} \in \mathrm{SubstitutionSet}(V,C)$.

Let us consider V, C. One can check that SubstitutionSet(V,C) is non empty.

Let us consider V, C and let A, B be elements of SubstitutionSet(V,C). Then $A \cup B$ is an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$.

Let us consider V, C. Note that there exists an element of SubstitutionSet(V,C) which is non empty.

Let us consider V, C. Note that every element of SubstitutionSet(V,C) is finite.

Let us consider V, C and let A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $\Box^{\mathrm{c}}{}_A$ yields an element of SubstitutionSet(V,C) and is defined by:

(Def. 2) $\Box^{\mathrm{c}}{}_A = \{t, t$ ranges over elements of $V \dot{\rightarrow} C : \bigwedge_{s\,:\,\text{element of } V \dot{\rightarrow} C} (s \in A \wedge s \subseteq t \Leftrightarrow s = t)\}$.

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Let us consider V, C and let A be a non empty element of $\mathrm{SubstitutionSet}(V, C)$. Note that every element of A is function-like and relation-like.

Let us consider V, C. One can verify that every element of $V \dot{\rightarrow} C$ is function-like and relation-like.

Let us consider V, C and let A, B be elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $A \frown B$ yields an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$ and is defined as follows:

(Def. 3) $A \frown B = \{s \cup t, s$ ranges over elements of $V \dot{\rightarrow} C, t$ ranges over elements of $V \dot{\rightarrow} C : s \in A \ \wedge \ t \in B \ \wedge \ s \approx t\}$.

In the sequel A, B, D are elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$.

One can prove the following propositions:

(3) $A \frown B = B \frown A$.

(4) If $B = \{\emptyset\}$, then $A \frown B = A$.

(5) For all sets a, b such that $B \in \mathrm{SubstitutionSet}(V, C)$ and $a \in B$ and $b \in B$ and $a \subseteq b$ holds $a = b$.

(6) For every set a such that $a \in \Box^{\mathrm{c}}{}_{B}$ holds $a \in B$ and for every set b such that $b \in B$ and $b \subseteq a$ holds $b = a$.

(7) For every set a such that $a \in B$ and for every set b such that $b \in B$ and $b \subseteq a$ holds $b = a$ holds $a \in \Box^{\mathrm{c}}{}_{B}$.

(8) $\Box^{\mathrm{c}}{}_{A} \subseteq A$.

(9) If $A = \emptyset$, then $\Box^{\mathrm{c}}{}_{A} = \emptyset$.

(10) For every set b such that $b \in B$ there exists a set c such that $c \subseteq b$ and $c \in \Box^{\mathrm{c}}{}_{B}$.

(11) For every element K of $\mathrm{SubstitutionSet}(V, C)$ holds $\Box^{\mathrm{c}}{}_{K} = K$.

(12) $\Box^{\mathrm{c}}{}_{A \cup B} \subseteq \Box^{\mathrm{c}}{}_{A} \cup B$.

(13) $\Box^{\mathrm{c}}{}_{\Box^{\mathrm{c}}{}_{A} \cup B} = \Box^{\mathrm{c}}{}_{A \cup B}$.

(14) If $A \subseteq B$, then $A \frown D \subseteq B \frown D$.

(15) For every set a such that $a \in A \frown B$ there exist sets b, c such that $b \in A$ and $c \in B$ and $a = b \cup c$.

(16) For all elements b, c of $V \dot{\rightarrow} C$ such that $b \in A$ and $c \in B$ and $b \approx c$ holds $b \cup c \in A \frown B$.

(17) $\Box^{\mathrm{c}}{}_{A \frown B} \subseteq (\Box^{\mathrm{c}}{}_{A}) \frown B$.

(18) If $A \subseteq B$, then $D \frown A \subseteq D \frown B$.

(19) $\Box^{\mathrm{c}}{}_{(\Box^{\mathrm{c}}{}_{A}) \frown B} = \Box^{\mathrm{c}}{}_{A \frown B}$.

(20) $\Box^{\mathrm{c}}{}_{A \frown (\Box^{\mathrm{c}}{}_{B})} = \Box^{\mathrm{c}}{}_{A \frown B}$.

(21) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \frown M) = (K \frown L) \frown M$.

(22) For all elements K, L, M of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $K \frown (L \cup M) = K \frown L \cup K \frown M$.

(23) $B \subseteq B \frown B$.

(24) $\Box^{c}{}_{A^\frown A} = \Box^{c}{}_{A}$.

(25) For every element K of SubstitutionSet(V, C) holds $\Box^{c}{}_{K^\frown K} = K$.

2. Definition of the lattice

Let us consider V, C. The functor SubstLatt(V, C) yielding a strict lattice structure is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of SubstLatt(V, C) = SubstitutionSet(V, C), and

(ii) for all elements A, B of SubstitutionSet(V, C) holds (the join operation of SubstLatt(V, C))$(A, B) = \Box^{c}{}_{A \cup B}$ and (the meet operation of SubstLatt(V, C))$(A, B) = \Box^{c}{}_{A^\frown B}$.

Let us consider V, C. One can verify that SubstLatt(V, C) is non empty.

Let us consider V, C. Note that SubstLatt(V, C) is lattice-like.

Let us consider V, C. Observe that SubstLatt(V, C) is distributive and bounded.

One can prove the following two propositions:

(26) $\bot_{\mathrm{SubstLatt}(V,C)} = \emptyset$.

(27) $\top_{\mathrm{SubstLatt}(V,C)} = \{\emptyset\}$.

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