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The Steinitz Theorem and the Dimension of a Real Linear Space

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Summary. Finite-dimensional real linear spaces are defined. The dimension of such spaces is the cardinality of a basis. Obviously, each two basis have the same cardinality. We prove the Steinitz theorem and the Exchange Lemma. We also investigate some fundamental facts involving the dimension of real linear spaces.

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The notation and terminology used here are introduced in the following papers: [10], [19], [9], [7], [2], [20], [4], [5], [18], [1], [6], [3], [13], [15], [8], [17], [12], [16], [14], and [11].

1. Prelimiaries

For simplicity, we follow the rules: V denotes a real linear space, W denotes a subspace of V, x denotes a set, n denotes a natural number, v denotes a vector of V, K_1, K_2 denote linear combinations of V, and X denotes a subset of the carrier of V.

We now state a number of propositions:

(1) If X is linearly independent and the support of $K_1 \subseteq X$ and the support of $K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.

(2) Let V be a real linear space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(3) Let L be a linear combination of V and x be a vector of V. Then $x \in$ the support of L if and only if there exists v such that $x = v$ and $L(v) \neq 0$.

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(4) For every finite set X such that $n \leqslant \overline{\overline{X}}$ there exists a finite subset A of X such that $\overline{\overline{A}} = n$.

(5) Let L be a linear combination of V, F, G be finite sequences of elements of the carrier of V, and P be a permutation of $\mathrm{dom}\, F$. If $G = F \cdot P$, then $\sum(L\,F) = \sum(L\,G)$.

(6) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. If the support of L misses $\mathrm{rng}\, F$, then $\sum(L\,F) = 0_V$.

(7) Let F be a finite sequence of elements of the carrier of V. Suppose F is one-to-one. Let L be a linear combination of V. If the support of $L \subseteq \mathrm{rng}\, F$, then $\sum(L\,F) = \sum L$.

(8) Let L be a linear combination of V and F be a finite sequence of elements of the carrier of V. Then there exists a linear combination K of V such that the support of $K = \mathrm{rng}\, F \cap$ the support of L and $L\,F = K\,F$.

(9) Let L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\mathrm{rng}\, F \subseteq$ the carrier of $\mathrm{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(10) Let L be a linear combination of V and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\mathrm{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(11) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L{\restriction}$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(12) Let K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(13) Let L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(14) For every basis I of V and for every vector v of V holds $v \in \mathrm{Lin}(I)$.

(15) Let A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(16) Let A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(17) For every basis A of W there exists a basis B of V such that $A \subseteq B$.

(18) Let A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \mathrm{Lin}(B)$.

(19) Let I be a basis of V and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly-dependent.

(20) For every subset A of V such that $A \subseteq$ the carrier of W holds $\mathrm{Lin}(A)$ is a subspace of W.

(21) For every subset A of V and for every subset B of W such that $A = B$ holds $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

2. The Steinitz Theorem

Next we state two propositions:

(22) Let A, B be finite subsets of V and v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(23) Let A, B be finite subsets of V. Suppose the RLS structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leqslant \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the RLS structure of $V = \mathrm{Lin}(B \cup C)$.

3. Finite Dimensional Vector Spaces

Let V be a real linear space. We say that V is finite dimensional if and only if:

(Def. 1) There exists a finite subset of the carrier of V which is a basis of V.

Let us observe that there exists a real linear space which is strict and finite dimensional.

Let V be a real linear space. Let us observe that V is finite dimensional if and only if:

(Def. 2) There exists a finite subset of V which is a basis of V.

We now state several propositions:

(24) If V is finite dimensional, then every basis of V is finite.

(25) If V is finite dimensional, then for every subset A of V such that A is linearly independent holds A is finite.

(26) If V is finite dimensional, then for all bases A, B of V holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(27) $\mathbf{0}_V$ is finite dimensional.

(28) If V is finite dimensional, then W is finite dimensional.

Let V be a real linear space. One can check that there exists a subspace of V which is finite dimensional and strict.

Let V be a finite dimensional real linear space. Observe that every subspace of V is finite dimensional.

Let V be a finite dimensional real linear space. Note that there exists a subspace of V which is strict.

4. The Dimension of a Vector Space

Let V be a real linear space. Let us assume that V is finite dimensional. The functor $\dim(V)$ yields a natural number and is defined as follows:

(Def. 3) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

We use the following convention: V is a finite dimensional real linear space, W, W_1, W_2 are subspaces of V, and u, v are vectors of V.

Next we state a number of propositions:

(29) $\dim(W) \leqslant \dim(V)$.

(30) For every subset A of V such that A is linearly independent holds $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(31) $\dim(V) = \dim(\Omega_V)$.

(32) $\dim(V) = \dim(W)$ iff $\Omega_V = \Omega_W$.

(33) $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(34) $\dim(V) = 1$ iff there exists v such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(35) $\dim(V) = 2$ iff there exist u, v such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(36) $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(37) $\dim(W_1 \cap W_2) \geqslant (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(38) If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1)+\dim(W_2)$.

(39) $n \leqslant \dim(V)$ iff there exists a strict subspace W of V such that $\dim(W) = n$.

Let V be a finite dimensional real linear space and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def. 4) $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

The following propositions are true:

(40) If $n \leqslant \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(41) If $\dim(V) < n$, then $\mathrm{Sub}_n(V) = \emptyset$.

(42) $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

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