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Category of Functors Between Alternative Categories

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The notation and terminology used in this paper are introduced in the following articles: [9], [13], [5], [10], [7], [15], [1], [3], [4], [2], [6], [8], [11], [14], and [12].

1. Preliminaries

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. Observe that every functor from A to B is feasible and id-preserving.

Let A be a transitive non empty category structure with units and let B be a non empty category structure with units. One can check the following observations:

* every functor from A to B which is covariant is also precovariant and comp-preserving,
* every functor from A to B which is precovariant and comp-preserving is also covariant,
* every functor from A to B which is contravariant is also precontravariant and comp-reversing, and
* every functor from A to B which is precontravariant and comp-reversing is also contravariant.

The following proposition is true

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(2)[1] Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, and a be an object of A. Then $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$.

2. Transformations

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. We say that F_1 is transformable to F_2 if and only if:

(Def. 1) For every object a of A holds $\langle F_1(a), F_2(a)\rangle \neq \emptyset$.

Let us note that the predicate F_1 is transformable to F_2 is reflexive.

One can prove the following proposition

(4)[2] Let A, B be transitive non empty category structures with units and F, F_1, F_2 be covariant functors from A to B. Suppose F is transformable to F_1 and F_1 is transformable to F_2. Then F is transformable to F_2.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is transformable to F_2. A many sorted set indexed by the carrier of A is said to be a transformation from F_1 to F_2 if:

(Def. 2) For every object a of A holds it(a) is a morphism from $F_1(a)$ to $F_2(a)$.

Let A, B be transitive non empty category structures with units and let F be a covariant functor from A to B. The functor id_F yielding a transformation from F to F is defined by:

(Def. 3) For every object a of A holds $\mathrm{id}_F(a) = \mathrm{id}_{F(a)}$.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2 and let a be an object of A. The functor $t[a]$ yielding a morphism from $F_1(a)$ to $F_2(a)$ is defined as follows:

(Def. 4) $t[a] = t(a)$.

Let A, B be transitive non empty category structures with units and let F, F_1, F_2 be covariant functors from A to B. Let us assume that F is transformable to F_1 and F_1 is transformable to F_2. Let t_1 be a transformation from F to F_1 and let t_2 be a transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yielding a transformation from F to F_2 is defined by:

(Def. 5) For every object a of A holds $(t_2 \circ t_1)[a] = t_2[a] \cdot t_1[a]$.

We now state four propositions:

[1] The proposition (1) has been removed.
[2] The proposition (3) has been removed.

(5) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is transformable to F_2. Let t_1, t_2 be transformations from F_1 to F_2. If for every object a of A holds $t_1[a] = t_2[a]$, then $t_1 = t_2$.

(6) Let A, B be transitive non empty category structures with units, F be a covariant functor from A to B, and a be an object of A. Then $\mathrm{id}_F[a] = \mathrm{id}_{F(a)}$.

(7) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2. Then $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

(8) Let A, B be categories and F, F_1, F_2, F_3 be covariant functors from A to B. Suppose F is transformable to F_1 and F_1 is transformable to F_2 and F_2 is transformable to F_3. Let t_1 be a transformation from F to F_1, t_2 be a transformation from F_1 to F_2, and t_3 be a transformation from F_2 to F_3. Then $(t_3 \circ t_2) \circ t_1 = t_3 \circ (t_2 \circ t_1)$.

3. Natural Transformations

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. We say that F_1 is naturally transformable to F_2 if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) F_1 is transformable to F_2, and

(ii) there exists a transformation t from F_1 to F_2 such that for all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $t[b] \cdot F_1(f) = F_2(f) \cdot t[a]$.

We now state two propositions:

(9) For all transitive non empty category structures A, B with units holds every covariant functor F from A to B is naturally transformable to F.

(10) Let A, B be categories and F, F_1, F_2 be covariant functors from A to B. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Then F is naturally transformable to F_2.

Let A, B be transitive non empty category structures with units and let F_1, F_2 be covariant functors from A to B. Let us assume that F_1 is naturally transformable to F_2. A transformation from F_1 to F_2 is called a natural transformation from F_1 to F_2 if:

(Def. 7) For all objects a, b of A such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds it $[b] \cdot F_1(f) = F_2(f) \cdot \mathrm{it}[a]$.

Let A, B be transitive non empty category structures with units and let F be a covariant functor from A to B. Then id_F is a natural transformation from F to F.

Let A, B be categories and let F, F_1, F_2 be covariant functors from A to B. Let us assume that F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1 and let t_2 be a natural transformation from F_1 to F_2. The functor $t_2 \circ t_1$ yielding a natural transformation from F to F_2 is defined by:

(Def. 8) $t_2 \circ t_1 = t_2 \circ t_1$.

We now state three propositions:

(11) Let A, B be transitive non empty category structures with units and F_1, F_2 be covariant functors from A to B. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathrm{id}_{(F_2)} \circ t = t$ and $t \circ \mathrm{id}_{(F_1)} = t$.

(12) Let A, B be transitive non empty category structures with units and F, F_1, F_2 be covariant functors from A to B. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2. Let t_1 be a natural transformation from F to F_1, t_2 be a natural transformation from F_1 to F_2, and a be an object of A. Then $(t_2 \circ t_1)[a] = t_2[a] \cdot t_1[a]$.

(13) Let A, B be categories, F, F_1, F_2, F_3 be covariant functors from A to B, t be a natural transformation from F to F_1, and t_1 be a natural transformation from F_1 to F_2. Suppose F is naturally transformable to F_1 and F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3. Let t_3 be a natural transformation from F_2 to F_3. Then $(t_3 \circ t_1) \circ t = t_3 \circ (t_1 \circ t)$.

4. Category of Functors

Let I be a set and let A, B be many sorted sets indexed by I. The functor B^A yields a set and is defined as follows:

(Def. 9)(i) For every set x holds $x \in B^A$ iff x is a many sorted function from A into B if for every set i such that $i \in I$ holds if $B(i) = \emptyset$, then $A(i) = \emptyset$,

(ii) $B^A = \emptyset$, otherwise.

Let A, B be transitive non empty category structures with units. The functor $\mathrm{Funct}(A, B)$ yields a set and is defined as follows:

(Def. 10) For every set x holds $x \in \mathrm{Funct}(A, B)$ iff x is a covariant strict functor from A to B.

Let A, B be categories. The functor B^A yields a strict non empty transitive category structure and is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of $B^A = \mathrm{Funct}(A, B)$,

(ii) for all strict covariant functors F, G from A to B and for every set x holds $x \in$ (the arrows of B^A)(F, G) iff F is naturally transformable to G and x is a natural transformation from F to G, and

(iii) for all strict covariant functors F, G, H from A to B such that F is naturally transformable to G and G is naturally transformable to H and for every natural transformation t_1 from F to G and for every natural transformation t_2 from G to H there exists a function f such that $f =$ (the composition of B^A)(F, G, H) and $f(t_2, t_1) = t_2 \circ t_1$.

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