FORMALIZED MATHEMATICS
Volume 6, Number 3, 1997
Warsaw University - Białystok

Algebraic Operation on Subsets of Many Sorted Sets

Agnieszka Julia Marasik
Warsaw University
Białystok

MML Identifier: CLOSURE3.

The terminology and notation used in this paper are introduced in the following papers: [14], [17], [13], [12], [18], [3], [4], [1], [6], [5], [15], [16], [2], [11], [9], [7], [8], and [10].

1. Preliminaries

Let S be a non empty 1-sorted structure. One can verify that the 1-sorted structure of S is non empty.

We now state three propositions:

(1) For every non empty set I and for all many sorted sets M, N indexed by I holds $M + \cdot N = N$.

(2) Let I be a set, M, N be many sorted sets indexed by I, and F be a family of many sorted subsets indexed by M. If $N \in F$, then $\bigcap |{:}F{:}| \subseteq N$.

(3) Let S be a non void non empty many sorted signature, M_1 be a strict non-empty algebra over S, and F be a family of many sorted subsets indexed by the sorts of M_1. Suppose $F \subseteq \mathrm{SubSorts}(M_1)$. Let B be a subset of M_1. If $B = \bigcap |{:}F{:}|$, then B is operations closed.

ISSN 1426–2630

2. Relationships between Subsets Families

Let I be a set, let M be a many sorted set indexed by I, let B be a family of many sorted subsets indexed by M, and let A be a family of many sorted subsets indexed by M. We say that A is finer than B if and only if:

(Def. 1) For every set a such that $a \in A$ there exists a set b such that $b \in B$ and $a \subseteq b$.

Let us observe that the predicate A is finer than B is reflexive. We say that B is coarser than A if and only if:

(Def. 2) For every set b such that $b \in B$ there exists a set a such that $a \in A$ and $a \subseteq b$.

Let us notice that the predicate B is coarser than A is reflexive.

We now state two propositions:

(4) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is finer than B and B is finer than C, then A is finer than C.

(5) Let I be a set, M be a many sorted set indexed by I, and A, B, C be families of many sorted subsets indexed by M. If A is coarser than B and B is coarser than C, then A is coarser than C.

Let I be a non empty set and let M be a many sorted set indexed by I. The functor $\mathrm{supp}(M)$ yielding a set is defined by:

(Def. 3) $\mathrm{supp}(M) = \{x, x$ ranges over elements of I: $M(x) \neq \emptyset\}$.

We now state four propositions:

(6) For every non empty set I and for every non-empty many sorted set M indexed by I holds $M = \emptyset_I {+\cdot} M \restriction \mathrm{supp}(M)$.

(7) Let I be a non empty set and M_2, M_3 be non-empty many sorted sets indexed by I. If $\mathrm{supp}(M_2) = \mathrm{supp}(M_3)$ and $M_2 \restriction \mathrm{supp}(M_2) = M_3 \restriction \mathrm{supp}(M_3)$, then $M_2 = M_3$.

(8) Let I be a non empty set, M be a many sorted set indexed by I, and x be an element of I. If $x \notin \mathrm{supp}(M)$, then $M(x) = \emptyset$.

(9) Let I be a non empty set, M be a many sorted set indexed by I, x be an element of $\mathrm{Bool}(M)$, i be an element of I, and y be a set. Suppose $y \in x(i)$. Then there exists an element a of $\mathrm{Bool}(M)$ such that $y \in a(i)$ and a is locally-finite and $\mathrm{supp}(a)$ is finite and $a \subseteq x$.

Let I be a set, let M be a many sorted set indexed by I, and let A be a family of many sorted subsets indexed by M. The functor $\mathrm{MSUnion}(A)$ yielding a many sorted subset indexed by M is defined by:

(Def. 4) For every set i such that $i \in I$ holds $(\mathrm{MSUnion}(A))(i) = \bigcup\{f(i), f$ ranges over elements of $\mathrm{Bool}(M)$: $f \in A\}$.

Let I be a set, let M be a many sorted set indexed by I, and let B be a non empty family of many sorted subsets indexed by M. We see that the element of B is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be an empty family of many sorted subsets indexed by M. One can check that MSUnion(A) is empty yielding.

We now state the proposition

(10) Let I be a set, M be a many sorted set indexed by I, and A be a family of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A) = \bigcup |{:}A{:}|$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cup B$ is a family of many sorted subsets indexed by M.

The following propositions are true:

(11) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A \cup B) = \mathrm{MSUnion}(A) \cup \mathrm{MSUnion}(B)$.

(12) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. If $A \subseteq B$, then $\mathrm{MSUnion}(A) \subseteq \mathrm{MSUnion}(B)$.

Let I be a set, let M be a many sorted set indexed by I, and let A, B be families of many sorted subsets indexed by M. Then $A \cap B$ is a family of many sorted subsets indexed by M.

One can prove the following propositions:

(13) Let I be a set, M be a many sorted set indexed by I, and A, B be families of many sorted subsets indexed by M. Then $\mathrm{MSUnion}(A \cap B) \subseteq \mathrm{MSUnion}(A) \cap \mathrm{MSUnion}(B)$.

(14) Let I be a set, M be a many sorted set indexed by I, and A_1 be a set. Suppose that for every set x such that $x \in A_1$ holds x is a family of many sorted subsets indexed by M. Let A, B be families of many sorted subsets indexed by M. Suppose $B = \{\mathrm{MSUnion}(X), X$ ranges over families of many sorted subsets indexed by M: $X \in A_1\}$ and $A = \bigcup A_1$. Then $\mathrm{MSUnion}(B) = \mathrm{MSUnion}(A)$.

(15) Let I be a non empty set, M, N be many sorted sets indexed by I, and A be a family of many sorted subsets indexed by M. If for every many sorted set x indexed by I holds $x \subseteq N$, then $\mathrm{MSUnion}(A) \subseteq N$.

3. Algebraic Operation on Subsets of Many Sorted Sets

Let I be a non empty set, let M be a many sorted set indexed by I, and let S be a set operation in M. We say that S is algebraic if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let x be an element of $\mathrm{Bool}(M)$. Suppose $x = S(x)$. Then there exists a family A of many sorted subsets indexed by M such that $A = \{S(a), a$ ranges over elements of $\mathrm{Bool}(M)$: a is locally-finite $\wedge$ $\mathrm{supp}(a)$ is finite $\wedge$ $a \subseteq x\}$ and $x = \mathrm{MSUnion}(A)$.

Let I be a non empty set and let M be a many sorted set indexed by I. Note that there exists a set operation in M which is algebraic, reflexive, monotonic, and idempotent.

Let S be a non empty 1-sorted structure and let I_1 be a closure system of S. We say that I_1 is algebraic if and only if:

(Def. 6) $\mathrm{ClOp}(I_1)$ is algebraic.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor $\mathrm{SubAlgCl}(M_1)$ yields a strict closure system structure over S and is defined by:

(Def. 7) The sorts of $\mathrm{SubAlgCl}(M_1)$ = the sorts of M_1 and the family of $\mathrm{SubAlgCl}(M_1) = \mathrm{SubSorts}(M_1)$.

One can prove the following proposition

(16) Let S be a non void non empty many sorted signature and M_1 be a strict non-empty algebra over S. Then $\mathrm{SubSorts}(M_1)$ is an absolutely-multiplicative family of many sorted subsets indexed by the sorts of M_1.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. Note that $\mathrm{SubAlgCl}(M_1)$ is absolutely-multiplicative.

Let S be a non void non empty many sorted signature and let M_1 be a strict non-empty algebra over S. Observe that $\mathrm{SubAlgCl}(M_1)$ is algebraic.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[7] Artur Korniłowicz. Certain facts about families of subsets of many sorted sets. *Formalized Mathematics*, 5(**3**):451–456, 1996.

[8] Artur Korniłowicz. Definitions and basic properties of boolean & union of many sorted sets. *Formalized Mathematics*, 5(**2**):279–281, 1996.
[9] Artur Korniłowicz. On the closure operator and the closure system of many sorted sets. *Formalized Mathematics*, 5(**4**):543–551, 1996.
[10] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[11] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[13] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(**2**):233–236, 1996.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[16] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[17] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received June 23, 1997