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Duality in Relation Structures[1]

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The articles [15], [18], [19], [21], [20], [7], [8], [10], [1], [2], [6], [14], [11], [16], [12], [17], [3], [4], [23], [9], [5], [22], and [13] provide the terminology and notation for this paper.

Let L be a relational structure. We introduce $L^{\rm op}$ as a synonym of $L^{\smile}$.

We now state several propositions:

(1) For every relational structure L and for all elements x, y of $L^{\rm op}$ holds $x \leqslant y$ iff $\curvearrowleft x \geqslant \curvearrowleft y$.

(2) Let L be a relational structure, x be an element of L, and y be an element of $L^{\rm op}$. Then

(i) $x \leqslant \curvearrowleft y$ iff $x^{\smile} \geqslant y$, and

(ii) $x \geqslant \curvearrowleft y$ iff $x^{\smile} \leqslant y$.

(3) For every relational structure L holds L is empty iff $L^{\rm op}$ is empty.

(4) For every relational structure L holds L is reflexive iff $L^{\rm op}$ is reflexive.

(5) For every relational structure L holds L is antisymmetric iff $L^{\rm op}$ is antisymmetric.

(6) For every relational structure L holds L is transitive iff $L^{\rm op}$ is transitive.

(7) For every non empty relational structure L holds L is connected iff $L^{\rm op}$ is connected.

Let L be a reflexive relational structure. One can check that $L^{\rm op}$ is reflexive.

Let L be a transitive relational structure. One can check that $L^{\rm op}$ is transitive.

Let L be an antisymmetric relational structure. Note that $L^{\rm op}$ is antisymmetric.

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Let L be a connected non empty relational structure. Observe that $L^{\rm op}$ is connected.

One can prove the following propositions:

(8) Let L be a relational structure, x be an element of L, and X be a set. Then

(i) $x \leqslant X$ iff $x^\smile \geqslant X$, and

(ii) $x \geqslant X$ iff $x^\smile \leqslant X$.

(9) Let L be a relational structure, x be an element of $L^{\rm op}$, and X be a set. Then

(i) $x \leqslant X$ iff $\curvearrowleft x \geqslant X$, and

(ii) $x \geqslant X$ iff $\curvearrowleft x \leqslant X$.

(10) Let L be a relational structure and X be a set. Then sup X exists in L if and only if inf X exists in $L^{\rm op}$.

(11) Let L be a relational structure and X be a set. Then sup X exists in $L^{\rm op}$ if and only if inf X exists in L.

(12) Let L be a non empty relational structure and X be a set. If sup X exists in L or inf X exists in $L^{\rm op}$, then $\bigsqcup_L X = \bigsqcap_{(L^{\rm op})} X$.

(13) Let L be a non empty relational structure and X be a set. If inf X exists in L or sup X exists in $L^{\rm op}$, then $\bigsqcap_L X = \bigsqcup_{(L^{\rm op})} X$.

(14) For all relational structures L_1, L_2 such that the relational structure of L_1 = the relational structure of L_2 and L_1 has g.l.b.'s holds L_2 has g.l.b.'s.

(15) For all relational structures L_1, L_2 such that the relational structure of L_1 = the relational structure of L_2 and L_1 has l.u.b.'s holds L_2 has l.u.b.'s.

(16) For every relational structure L holds L has g.l.b.'s iff $L^{\rm op}$ has l.u.b.'s.

(17) For every non empty relational structure L holds L is complete iff $L^{\rm op}$ is complete.

Let L be a relational structure with g.l.b.'s. Note that $L^{\rm op}$ has l.u.b.'s.

Let L be a relational structure with l.u.b.'s. One can check that $L^{\rm op}$ has g.l.b.'s.

Let L be a complete non empty relational structure. One can check that $L^{\rm op}$ is complete.

The following propositions are true:

(18) Let L be a non empty relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then fininfs(X) = finsups(Y) and finsups(X) = fininfs(Y).

(19) Let L be a relational structure, X be a subset of L, and Y be a subset of $L^{\rm op}$. If $X = Y$, then $\downarrow X = \uparrow Y$ and $\uparrow X = \downarrow Y$.

(20) Let L be a non empty relational structure, x be an element of L, and y be an element of $L^{\rm op}$. If $x = y$, then $\downarrow x = \uparrow y$ and $\uparrow x = \downarrow y$.

(21) For every poset L with g.l.b.'s and for all elements x, y of L holds $x \sqcap y = x^\smile \sqcup y^\smile$.

(22) For every poset L with g.l.b.'s and for all elements x, y of $L^{\rm op}$ holds $\curvearrowleft x \sqcap \curvearrowleft y = x \sqcup y$.

(23) For every poset L with l.u.b.'s and for all elements x, y of L holds $x \sqcup y = x^{\smile} \sqcap y^{\smile}$.

(24) For every poset L with l.u.b.'s and for all elements x, y of $L^{\rm op}$ holds $\curvearrowleft x \sqcup \curvearrowleft y = x \sqcap y$.

(25) For every lattice L holds L is distributive iff $L^{\rm op}$ is distributive.

Let L be a distributive lattice. One can check that $L^{\rm op}$ is distributive.

Next we state a number of propositions:

(26) Let L be a relational structure and x be a set. Then x is a directed subset of L if and only if x is a filtered subset of $L^{\rm op}$.

(27) Let L be a relational structure and x be a set. Then x is a directed subset of $L^{\rm op}$ if and only if x is a filtered subset of L.

(28) Let L be a relational structure and x be a set. Then x is a lower subset of L if and only if x is an upper subset of $L^{\rm op}$.

(29) Let L be a relational structure and x be a set. Then x is a lower subset of $L^{\rm op}$ if and only if x is an upper subset of L.

(30) For every relational structure L holds L is lower-bounded iff $L^{\rm op}$ is upper-bounded.

(31) For every relational structure L holds $L^{\rm op}$ is lower-bounded iff L is upper-bounded.

(32) For every relational structure L holds L is bounded iff $L^{\rm op}$ is bounded.

(33) For every lower-bounded antisymmetric non empty relational structure L holds $(\bot_L)^{\smile} = \top_{L^{\rm op}}$ and $\curvearrowleft(\top_{L^{\rm op}}) = \bot_L$.

(34) For every upper-bounded antisymmetric non empty relational structure L holds $(\top_L)^{\smile} = \bot_{L^{\rm op}}$ and $\curvearrowleft(\bot_{L^{\rm op}}) = \top_L$.

(35) Let L be a bounded lattice and x, y be elements of L. Then y is a complement of x if and only if $y^{\smile}$ is a complement of $x^{\smile}$.

(36) For every bounded lattice L holds L is complemented iff $L^{\rm op}$ is complemented.

Let L be a lower-bounded relational structure. One can verify that $L^{\rm op}$ is upper-bounded.

Let L be an upper-bounded relational structure. Note that $L^{\rm op}$ is lower-bounded.

Let L be a complemented bounded lattice. One can check that $L^{\rm op}$ is complemented.

Next we state the proposition

(37) For every Boolean lattice L and for every element x of L holds $\neg(x^{\smile}) = \neg x$.

Let L be a non empty relational structure. The functor $\neg_L$ yields a map from L into $L^{\rm op}$ and is defined as follows:

(Def. 1) For every element x of L holds $\neg_L(x) = \neg x$.

Let L be a Boolean lattice. Observe that $\neg_L$ is one-to-one.

Let L be a Boolean lattice. One can verify that $\neg_L$ is isomorphic.

The following propositions are true:

(38) For every Boolean lattice L holds L and L^{op} are isomorphic.

(39) Let S, T be non empty relational structures and f be a set. Then
 (i) f is a map from S into T iff f is a map from S^{op} into T,
 (ii) f is a map from S into T iff f is a map from S into T^{op}, and
 (iii) f is a map from S into T iff f is a map from S^{op} into T^{op}.

(40) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from S into T^{op} such that $f = g$. Then
 (i) f is monotone iff g is antitone, and
 (ii) f is antitone iff g is monotone.

(41) Let S, T be non empty relational structures, f be a map from S into T^{op}, and g be a map from S^{op} into T such that $f = g$. Then
 (i) f is monotone iff g is monotone, and
 (ii) f is antitone iff g is antitone.

(42) Let S, T be non empty relational structures, f be a map from S into T, and g be a map from S^{op} into T^{op} such that $f = g$. Then
 (i) f is monotone iff g is monotone, and
 (ii) f is antitone iff g is antitone.

(43) Let S, T be non empty relational structures and f be a set. Then
 (i) f is a connection between S and T iff f is a connection between $S^\smile$ and T,
 (ii) f is a connection between S and T iff f is a connection between S and $T^\smile$, and
 (iii) f is a connection between S and T iff f is a connection between $S^\smile$ and $T^\smile$.

(44) Let S, T be non empty posets, f_1 be a map from S into T, g_1 be a map from T into S, f_2 be a map from $S^\smile$ into $T^\smile$, and g_2 be a map from $T^\smile$ into $S^\smile$. If $f_1 = f_2$ and $g_1 = g_2$, then $\langle f_1, g_1\rangle$ is Galois iff $\langle g_2, f_2\rangle$ is Galois.

(45) Let J be a set, D be a non empty set, K be a many sorted set indexed by J, and F be a set of elements of D double indexed by K. Then $\mathrm{dom}_\kappa F(\kappa) = K$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let j be an element of J, and let k be an element of $K(j)$. Then $F(j)(k)$ is an element of D.

One can prove the following propositions:

(46) Let L be a non empty relational structure, J be a set, K be a many sorted set indexed by J, and x be a set. Then x is a set of elements of L double indexed by K if and only if x is a set of elements of L^{op} double indexed by K.

(47) Let L be a complete lattice, J be a non empty set, K be a non-empty many sorted set indexed by J, and F be a set of elements of L double indexed by K. Then $\mathrm{Sup}(\mathrm{Infs}(F)) \leqslant \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(48) Let L be a complete lattice. Then L is completely-distributive if and only if for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Sup}(\mathrm{Infs}(F)) = \mathrm{Inf}(\mathrm{Sups}(\mathrm{Frege}(F)))$.

(49) Let L be a complete antisymmetric non empty relational structure and F be a function. Then $\bigsqcup_L F = \bigsqcap_{(L^{\mathrm{op}})} F$ and $\bigsqcap_L F = \bigsqcup_{(L^{\mathrm{op}})} F$.

(50) Let L be a complete antisymmetric non empty relational structure and F be a function yielding function. Then $\underline{\bigsqcup}_L F = \overline{\bigsqcap}_{(L^{\mathrm{op}})} F$ and $\overline{\bigsqcap}_L F = \underline{\bigsqcup}_{(L^{\mathrm{op}})} F$.

One can check that every non empty relational structure which is completely-distributive is also complete.

Let us observe that there exists a non empty poset which is completely-distributive, trivial, and strict.

The following proposition is true

(51) For every non empty poset L holds L is completely-distributive iff L^{op} is completely-distributive.

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