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The Equational Characterization of Continuous Lattices[1]

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Summary. The class of continuous lattices can be characterized by infinitary equations. Therefore, it is closed under the formation of subalgebras and homomorphic images. Following the terminology of [18] we introduce a continuous lattice subframe to be a sublattice closed under the formation of arbitrary infs and directed sups. This notion corresponds with a subalgebra of a continuous lattice in [16].

The class of completely distributive lattices is also introduced in the paper. Such lattices are complete and satisfy the most restrictive type of the general distributivity law. Obviously each completely distributive lattice is a Heyting algebra. It was hard to find the best Mizar implementation of the complete distributivity equational condition (denoted by CD in [16]). The powerful and well developed Many Sorted Theory gives the most convenient way of this formalization. A set double indexed by K, introduced in the paper, corresponds with a family $\{x_{j,k} : j \in J, k \in K(j)\}$. It is defined to be a suitable many sorted function. Two special functors: Sups and Infs as counterparts of Sup and Inf respectively, introduced in [38], are also defined. Originally the equation in Definition 2.4 of [16, p. 58] looks as follows:

$$\bigwedge_{j\in J}\bigvee_{k\in K(j)} x_{j,k} = \bigvee_{f\in M}\bigwedge_{j\in J} x_{j,f(j)},$$

where M is the set of functions defined on J with values $f(j) \in K(j)$.

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The articles [30], [37], [12], [15], [35], [10], [11], [1], [4], [29], [36], [5], [2], [28], [13], [9], [32], [21], [22], [33], [19], [24], [27], [20], [25], [31], [3], [26], [23], [6], [17], [38], [14], [7], [8], and [34] provide the terminology and notation for this paper.

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1. The Continuity of Lattices

In this paper x, y are arbitrary, X denotes a set, and L denotes an up-complete semilattice.

One can prove the following propositions:

(1) L is continuous if and only if for every element x of L holds $\downarrow x$ is an ideal of L and $x \leqslant \sup \downarrow x$ and for every ideal I of L such that $x \leqslant \sup I$ holds $\downarrow x \subseteq I$.

(2) L is continuous if and only if for every element x of L there exists an ideal I of L such that $x \leqslant \sup I$ and for every ideal J of L such that $x \leqslant \sup J$ holds $I \subseteq J$.

(3) For every continuous lower-bounded sup-semilattice L holds $\mathrm{SupMap}(L)$ has a lower adjoint.

(4) For every up-complete lower-bounded lattice L such that $\mathrm{SupMap}(L)$ is upper adjoint holds L is continuous.

(5) For every complete semilattice L such that $\mathrm{SupMap}(L)$ is infs-preserving and sups-preserving holds $\mathrm{SupMap}(L)$ has a lower adjoint.

Let J, D be sets and let K be a many sorted set indexed by J. A set of elements of D double indexed by K is a many sorted function from K into $J \longmapsto D$.

Let J be a set, let K be a many sorted set indexed by J, and let S be a 1-sorted structure. A set of elements of S double indexed by K is a set of elements of the carrier of S double indexed by K.

We now state the proposition

(6) Let J, D be sets, K be a many sorted set indexed by J, F be a set of elements of D double indexed by K, and j be arbitrary. If $j \in J$, then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. Then $F(j)$ is a function from $K(j)$ into D.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, and let j be an element of J. One can check that $\mathrm{rng}\, F(j)$ is non empty.

Let J be a set, let D be a non empty set, and let K be a non-empty many sorted set indexed by J. One can check that every set of elements of D double indexed by K is non-empty.

Next we state four propositions:

(7) For every function yielding function F and for arbitrary f such that $f \in \mathrm{dom}\, \mathrm{Frege}(F)$ holds f is a function.

(8) For every function yielding function F and for every function f such that $f \in \mathrm{dom}\, \mathrm{Frege}(F)$ holds $\mathrm{dom}\, f = \mathrm{dom}\, F$ and $\mathrm{dom}\, F = \mathrm{dom}(\mathrm{Frege}(F))(f)$.

(9) Let F be a function yielding function and f be a function. Suppose $f \in \operatorname{dom} \operatorname{Frege}(F)$. Let i be arbitrary. If $i \in \operatorname{dom} F$, then $f(i) \in \operatorname{dom} F(i)$ and $(\operatorname{Frege}(F))(f)(i) = F(i)(f(i))$ and $F(i)(f(i)) \in \operatorname{rng}(\operatorname{Frege}(F))(f)$.

(10) Let J, D be sets, K be a many sorted set indexed by J, F be a set of elements of D double indexed by K, and f be a function. If $f \in \operatorname{dom} \operatorname{Frege}(F)$, then $(\operatorname{Frege}(F))(f)$ is a function from J into D.

Let f be a non-empty function. Note that $\operatorname{dom}_\kappa f(\kappa)$ is non-empty.

Let J, D be sets, let K be a many sorted set indexed by J, and let F be a set of elements of D double indexed by K. Then $\operatorname{Frege}(F)$ is a set of elements of D double indexed by $\prod(\operatorname{dom}_\kappa F(\kappa)) \longmapsto J$.

Let J, D be non empty sets, let K be a non-empty many sorted set indexed by J, let F be a set of elements of D double indexed by K, let G be a set of elements of D double indexed by $\prod(\operatorname{dom}_\kappa F(\kappa)) \longmapsto J$, and let f be an element of $\prod(\operatorname{dom}_\kappa F(\kappa))$. Then $G(f)$ is a function from J into D.

Let L be a non empty relational structure and let F be a function yielding function. The functor $\underline{\bigsqcup}_L F$ yields a function from $\operatorname{dom} F$ into the carrier of L and is defined as follows:

(Def. 1) For every x such that $x \in \operatorname{dom} F$ holds $(\underline{\bigsqcup}_L F)(x) = \bigsqcup_L F(x)$.

The functor $\overline{\bigsqcap}_L F$ yields a function from $\operatorname{dom} F$ into the carrier of L and is defined by:

(Def. 2) For every x such that $x \in \operatorname{dom} F$ holds $(\overline{\bigsqcap}_L F)(x) = \bigsqcap_L F(x)$.

Let J be a set, let K be a many sorted set indexed by J, let L be a non empty relational structure, and let F be a set of elements of L double indexed by K. We introduce $\operatorname{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\operatorname{Infs}(F)$ as a synonym of $\overline{\bigsqcap}_L F$.

Let I, J be sets, let L be a non empty relational structure, and let F be a set of elements of L double indexed by $I \longmapsto J$. We introduce $\operatorname{Sups}(F)$ as a synonym of $\underline{\bigsqcup}_L F$. We introduce $\operatorname{Infs}(F)$ as a synonym of $\overline{\bigsqcap}_L F$.

The following four propositions are true:

(11) Let L be a non empty relational structure and F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcup_L F(x) = \bigsqcup_L G(x)$, then $\underline{\bigsqcup}_L F = \underline{\bigsqcup}_L G$.

(12) Let L be a non empty relational structure and F, G be function yielding functions. If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $\bigsqcap_L F(x) = \bigsqcap_L G(x)$, then $\overline{\bigsqcap}_L F = \overline{\bigsqcap}_L G$.

(13) Let L be a non empty relational structure and F be a function yielding function. Then

(i) $y \in \operatorname{rng} \underline{\bigsqcup}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcup_L F(x)$, and

(ii) $y \in \operatorname{rng} \overline{\bigsqcap}_L F$ iff there exists x such that $x \in \operatorname{dom} F$ and $y = \bigsqcap_L F(x)$.

(14) Let L be a non empty relational structure, J be a non empty set, K be a many sorted set indexed by J, and F be a set of elements of L double indexed by K. Then

(i) $x \in \operatorname{rng}\operatorname{Sups}(F)$ iff there exists an element j of J such that $x = \operatorname{Sup}(F(j))$, and

(ii) $x \in \operatorname{rng}\operatorname{Infs}(F)$ iff there exists an element j of J such that $x = \operatorname{Inf}(F(j))$.

Let J be a non empty set, let K be a many sorted set indexed by J, let L be a non empty relational structure, and let F be a set of elements of L double indexed by K. Observe that $\operatorname{rng}\operatorname{Sups}(F)$ is non empty and $\operatorname{rng}\operatorname{Infs}(F)$ is non empty.

For simplicity we follow the rules: L is a complete lattice, a, b, c are elements of L, J is a non empty set, and K is a non-empty many sorted set indexed by J.

One can prove the following propositions:

(15) Let F be a function yielding function. If for every function f such that $f \in \operatorname{dom}\operatorname{Frege}(F)$ holds $\bigsqcap_L(\operatorname{Frege}(F))(f) \leqslant a$, then $\operatorname{Sup}(\overline{\bigsqcap}_L \operatorname{Frege}(F)) \leqslant a$.

(16) For every set F of elements of L double indexed by K holds $\operatorname{Inf}(\operatorname{Sups}(F)) \geqslant \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

(17) If L is continuous and for every c such that $c \ll a$ holds $c \leqslant b$, then $a \leqslant b$.

(18) Suppose that for every non empty set J such that $J \in$ the universe of the carrier of L and for every non-empty many sorted set K indexed by J such that for every element j of J holds $K(j) \in$ the universe of the carrier of L and for every set F of elements of L double indexed by K such that for every element j of J holds $\operatorname{rng} F(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$. Then L is continuous.

(19) L is continuous if and only if for all J, K and for every set F of elements of L double indexed by K such that for every element j of J holds $\operatorname{rng} F(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(F)))$.

Let J, K, D be non empty sets and let F be a function from $[\!: J, K :\!]$ into D. Then $\operatorname{curry} F$ is a set of elements of D double indexed by $J \longmapsto K$.

We follow a convention: J, K, D will denote non empty sets, j will denote an element of J, and k will denote an element of K.

One can prove the following four propositions:

(20) For every function F from $[\!: J, K :\!]$ into D holds $\operatorname{dom}\operatorname{curry} F = J$ and $\operatorname{dom}(\operatorname{curry} F)(j) = K$ and $F(\langle j, k\rangle) = (\operatorname{curry} F)(j)(k)$.

(21) L is continuous if and only if for all non empty sets J, K and for every function F from $[\!: J, K :\!]$ into the carrier of L such that for every element j of J holds $\operatorname{rng}(\operatorname{curry} F)(j)$ is directed holds $\operatorname{Inf}(\operatorname{Sups}(\operatorname{curry} F)) = \operatorname{Sup}(\operatorname{Infs}(\operatorname{Frege}(\operatorname{curry} F)))$.

(22) Let F be a function from $[\!: J, K :\!]$ into the carrier of L and X be a subset of L. Suppose $X = \{a, a$ ranges over elements of L: $\bigvee_{f:\ \text{non-empty many sorted set indexed by } J} (f \in (\operatorname{Fin} K)^J \wedge \bigvee_{G:\,\text{set of elements of } L \text{ double indexed by } f} (\bigwedge_{j,x} (x \in f(j) \Rightarrow G(j)(x) =$

$F(\langle j, x\rangle)) \wedge a = \mathrm{Inf}(\mathrm{Sups}(G))))\}$. Then $\mathrm{Inf}(\mathrm{Sups}(\mathrm{curry}\, F)) \geqslant \sup X$.

(23) L is continuous if and only if for all J, K and for every function F from $[: J, K :]$ into the carrier of L and for every subset X of L such that $X = \{a, a$ ranges over elements of L: $\bigvee_{f\,:\ \text{non-empty many sorted set indexed by } J}$ $(f \in (\mathrm{Fin}\, K)^J \wedge \bigvee_{G\,:\,\text{set of elements of } L \text{ double indexed by } f}$ $(\bigwedge_{j,x}$ $(x \in f(j) \Rightarrow G(j)(x) = F(\langle j, x\rangle)) \wedge a = \mathrm{Inf}(\mathrm{Sups}(G))))\}$ holds $\mathrm{Inf}(\mathrm{Sups}(\mathrm{curry}\, F)) = \sup X$.

2. Completely-Distributive Lattices

Let L be a non empty relational structure. We say that L is completely-distributive if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) L is complete, and

(ii) for every non empty set J and for every non-empty many sorted set K indexed by J and for every set F of elements of L double indexed by K holds $\mathrm{Inf}(\mathrm{Sups}(F)) = \mathrm{Sup}(\mathrm{Infs}(\mathrm{Frege}(F)))$.

In the sequel J will denote a non empty set and K will denote a non-empty many sorted set indexed by J.

One can check that every non empty poset which is trivial is also completely-distributive.

One can verify that there exists a lattice which is completely-distributive.

Next we state the proposition

(24) Every completely-distributive lattice is continuous.

Let us observe that every lattice which is completely-distributive is also complete and continuous.

Next we state two propositions:

(25) Let L be a non empty antisymmetric transitive relational structure with g.l.b.'s, x be an element of L, and X, Y be subsets of L. Suppose $\sup X$ exists in L and $\sup Y$ exists in L and $Y = \{x \sqcap y, y$ ranges over elements of L: $y \in X\}$. Then $x \sqcap \sup X \geqslant \sup Y$.

(26) Let L be a completely-distributive lattice, X be a subset of L, and x be an element of L. Then $x \sqcap \sup X = \bigsqcup_L \{x \sqcap y, y$ ranges over elements of L: $y \in X\}$.

Let us note that every lattice which is completely-distributive is also Heyting.

3. Sub–Frames of Continuous Lattices

Let L be a non empty relational structure. A continuous subframe of L is an infs-inheriting directed-sups-inheriting non empty full relational substructure of L.

We now state three propositions:

(27) Let F be a set of elements of L double indexed by K. If for every element j of J holds $\operatorname{rng} F(j)$ is directed, then $\operatorname{rng} \operatorname{Infs}(\operatorname{Frege}(F))$ is directed.

(28) If L is continuous, then every continuous subframe of L is a continuous lattice.

(29) For every non empty poset S such that there exists a map from L into S which is infs-preserving and onto holds S is a complete lattice.

Let J be a set and let y be arbitrary. We introduce $J \Longmapsto y$ as a synonym of $J \longmapsto y$.

Let J be a set and let y be arbitrary. Then $J \longmapsto y$ is a many sorted set indexed by J. We introduce $J \Longmapsto y$ as a synonym of $J \longmapsto y$.

Let A, B, J be sets and let f be a function from A into B. Then $J \Longmapsto f$ is a many sorted function from $J \longmapsto A$ into $J \longmapsto B$.

We now state four propositions:

(30) Let A, B be sets, f be a function from A into B, and g be a function from B into A. If $g \cdot f = \mathrm{id}_A$, then $(J \Longmapsto g) \circ (J \Longmapsto f) = \mathrm{id}_{J \longmapsto A}$.

(31) Let J, A be non empty sets, B be a set, K be a many sorted set indexed by J, F be a set of elements of A double indexed by K, and f be a function from A into B. Then $(J \Longmapsto f) \circ F$ is a set of elements of B double indexed by K.

(32) Let J, A, B be non empty sets, K be a many sorted set indexed by J, F be a set of elements of A double indexed by K, and f be a function from A into B. Then $\mathrm{dom}_\kappa((J \Longmapsto f) \circ F)(\kappa) = \mathrm{dom}_\kappa F(\kappa)$.

(33) Suppose L is continuous. Let S be a non empty poset. Suppose there exists a map from L into S which is infs-preserving, directed-sups-preserving, and onto. Then S is a continuous lattice.

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