FORMALIZED MATHEMATICS
Volume 6, Number **1**, 1997
Warsaw University - Białystok

The "Way-Below" Relation [1]

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the paper the "way-below" relation, in symbols $x \ll y$, is introduced. Some authors prefer the term "relatively compact" or "way inside", since in the poset of open sets of a topology it is natural to read $U \ll V$ as "U is relatively compact in V". A compact element of a poset (or an element isolated from below) is defined to be way below itself. So, the compactness in the poset of open sets of a topology is equivalent to the compactness in that topology.

The article includes definitions, facts and examples 1.1–1.8 presented in [15, pp. 38–42].

MML Identifier: WAYBEL_3.

The terminology and notation used in this paper have been introduced in the following articles: [5], [25], [29], [30], [31], [20], [14], [23], [8], [28], [10], [11], [22], [24], [6], [19], [7], [26], [33], [27], [21], [32], [13], [12], [9], [4], [2], [1], [16], [3], [17], and [18].

1. The "Way-Below" Relation

Let L be a non empty reflexive relational structure and let x, y be elements of L. We say that x is way below y if and only if:

(Def. 1) For every non empty directed subset D of L such that $y \leq \sup D$ there exists an element d of L such that $d \in D$ and $x \leq d$.

We introduce $x \ll y$ and $y \gg x$ as synonyms of x is way below y.

Let L be a non empty reflexive relational structure and let x be an element of L. We say that x is compact if and only if:

(Def. 2) x is way below x.

[1] This work has been partially supported by Office of Naval Research Grant N00014-95-1-1336.

ISSN 1426–2630

We introduce x is isolated from below as a synonym of x is compact.

Next we state several propositions:

(1) Let L be a non empty reflexive antisymmetric relational structure and let x, y be elements of L. If $x \ll y$, then $x \leq y$.

(2) Let L be a non empty reflexive transitive relational structure and let u, x, y, z be elements of L. If $u \leq x$ and $x \ll y$ and $y \leq z$, then $u \ll z$.

(3) Let L be a non empty poset. Suppose L is inf-complete or has l.u.b.'s. Let x, y, z be elements of L. If $x \ll z$ and $y \ll z$, then sup $\{x, y\}$ exists in L and $x \sqcup y \ll z$.

(4) Let L be a lower-bounded antisymmetric reflexive non empty relational structure and let x be an element of L. Then $\bot_L \ll x$.

(5) For every non empty poset L and for all elements x, y, z of L such that $x \ll y$ and $y \ll z$ holds $x \ll z$.

(6) Let L be a non empty reflexive antisymmetric relational structure and let x, y be elements of L. If $x \ll y$ and $x \gg y$, then $x = y$.

Let L be a non empty reflexive relational structure and let x be an element of L. The functor $\twoheaddownarrow x$ yields a subset of L and is defined as follows:

(Def. 3) $\twoheaddownarrow x = \{y : y$ ranges over elements of L, $y \ll x\}$.

The functor $\twoheaduparrow x$ yielding a subset of L is defined by:

(Def. 4) $\twoheaduparrow x = \{y : y$ ranges over elements of L, $y \gg x\}$.

We now state several propositions:

(7) For every non empty reflexive relational structure L and for all elements x, y of L holds $x \in \twoheaddownarrow y$ iff $x \ll y$.

(8) For every non empty reflexive relational structure L and for all elements x, y of L holds $x \in \twoheaduparrow y$ iff $x \gg y$.

(9) For every non empty reflexive antisymmetric relational structure L and for every element x of L holds $x \geq \twoheaddownarrow x$.

(10) For every non empty reflexive antisymmetric relational structure L and for every element x of L holds $x \leq \twoheaduparrow x$.

(11) Let L be a non empty reflexive antisymmetric relational structure and let x be an element of L. Then $\twoheaddownarrow x \subseteq \downarrow x$ and $\twoheaduparrow x \subseteq \uparrow x$.

(12) Let L be a non empty reflexive transitive relational structure and let x, y be elements of L. If $x \leq y$, then $\twoheaddownarrow x \subseteq \twoheaddownarrow y$ and $\twoheaduparrow y \subseteq \twoheaduparrow x$.

Let L be a lower-bounded non empty reflexive antisymmetric relational structure and let x be an element of L. Note that $\twoheaddownarrow x$ is non empty.

Let L be a non empty reflexive transitive relational structure and let x be an element of L. Note that $\twoheaddownarrow x$ is lower and $\twoheaduparrow x$ is upper.

Let L be a sup-semilattice and let x be an element of L. One can verify that $\twoheaddownarrow x$ is directed.

Let L be an inf-complete non empty poset and let x be an element of L. Note that $\twoheaddownarrow x$ is directed.

Let L be a connected non empty relational structure. One can check that every subset of L is directed and filtered.

Let us note that every non empty chain which is up-complete and lower-bounded is also complete.

One can verify that there exists a non empty chain which is complete.

We now state several propositions:

(13) For every up-complete non empty chain L and for all elements x, y of L such that $x < y$ holds $x \ll y$.

(14) Let L be a non empty reflexive antisymmetric relational structure and let x, y be elements of L. If x is not compact and $x \ll y$, then $x < y$.

(15) For every non empty lower-bounded reflexive antisymmetric relational structure L holds $\perp_L$ is compact.

(16) For every up-complete non empty poset L and for every non empty finite directed subset D of L holds $\sup D \in D$.

(17) For every up-complete non empty poset L such that L is finite holds every element of L is isolated from below.

2. The Way-Below Relation in Other Terms

The scheme *SSubsetEx* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a subset X of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

We now state several propositions:

(18) Let L be a complete lattice and let x, y be elements of L. Suppose $x \ll y$. Let X be a subset of L. If $y \leq \sup X$, then there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$.

(19) Let L be a complete lattice and let x, y be elements of L. Suppose that for every subset X of L such that $y \leq \sup X$ there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$. Then $x \ll y$.

(20) Let L be a non empty reflexive transitive relational structure and let x, y be elements of L. If $x \ll y$, then for every ideal I of L such that $y \leq \sup I$ holds $x \in I$.

(21) Let L be an up-complete non empty poset and let x, y be elements of L. If for every ideal I of L such that $y \leq \sup I$ holds $x \in I$, then $x \ll y$.

(22) Let L be a lower-bounded lattice. Suppose L is meet-continuous. Let x, y be elements of L. Then $x \ll y$ if and only if for every ideal I of L such that $y = \sup I$ holds $x \in I$.

(23) Let L be a complete lattice. Then every element of L is compact if and only if for every non empty subset X of L there exists an element x of

L such that $x \in X$ and for every element y of L such that $y \in X$ holds $x \nless y$.

3. Continuous Lattices

Let L be a non empty reflexive relational structure. We say that L satisfies axiom of approximation if and only if:

(Def. 5) For every element x of L holds $x = \sup \downarrow\!\!\downarrow x$.

Let us note that every non empty reflexive relational structure which is trivial satisfies axiom of approximation.

Let L be a non empty reflexive relational structure. We say that L is continuous if and only if:

(Def. 6) For every element x of L holds $\downarrow\!\!\downarrow x$ is non empty and directed and L is up-complete and satisfies axiom of approximation.

One can check that every non empty reflexive relational structure which is continuous is also up-complete and satisfies axiom of approximation and every lower-bounded sup-semilattice which is up-complete and satisfies axiom of approximation is also continuous.

Let us note that there exists a lattice which is continuous, complete, and strict.

Let L be a continuous non empty reflexive relational structure and let x be an element of L. One can verify that $\downarrow\!\!\downarrow x$ is non empty and directed.

Next we state two propositions:

(24) Let L be an up-complete semilattice. Suppose that for every element x of L holds $\downarrow\!\!\downarrow x$ is non empty and directed. Then L satisfies axiom of approximation if and only if for all elements x, y of L such that $x \not\leq y$ there exists an element u of L such that $u \ll x$ and $u \not\leq y$.

(25) For every continuous lattice L and for all elements x, y of L holds $x \leq y$ iff $\downarrow\!\!\downarrow x \subseteq \downarrow\!\!\downarrow y$.

One can verify that every non empty chain which is complete satisfies axiom of approximation.

The following proposition is true

(26) For every complete lattice L such that every element of L is compact holds L satisfies axiom of approximation.

4. The Way-Below Relation in Direct Powers

Let f be a binary relation. We say that f is nonempty if and only if:

(Def. 7) For every 1-sorted structure S such that $S \in \operatorname{rng} f$ holds S is non empty.

We say that f is reflexive-yielding if and only if:

(Def. 8) For every relational structure S such that $S \in \operatorname{rng} f$ holds S is reflexive.

Let I be a set. Observe that there exists a many sorted set indexed by I which is relational structure yielding, nonempty, and reflexive-yielding.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. Observe that $\prod J$ is non empty.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty relational structure.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. Note that every element of $\prod J$ is function-like and relation-like.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let i be an element of I, and let X be a subset of $\prod J$. Then $\pi_i X$ is a subset of $J(i)$.

Next we state two propositions:

(27) Let I be a non empty set, and let J be a relational structure yielding nonempty many sorted set indexed by I, and let x be a function. Then x is an element of $\prod J$ if and only if $\operatorname{dom} x = I$ and for every element i of I holds $x(i)$ is an element of $J(i)$.

(28) Let I be a non empty set, and let J be a relational structure yielding nonempty many sorted set indexed by I, and let x, y be elements of $\prod J$. Then $x \leq y$ if and only if for every element i of I holds $x(i) \leq y(i)$.

Let I be a non empty set and let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Note that $\prod J$ is reflexive. Let i be an element of I. Then $J(i)$ is a non empty reflexive relational structure.

Let I be a non empty set, let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

One can prove the following propositions:

(29) Let I be a non empty set and let J be a relational structure yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is transitive, then $\prod J$ is transitive.

(30) Let I be a non empty set and let J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is antisymmetric. Then $\prod J$ is antisymmetric.

(31) Let I be a non empty set and let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Then $\prod J$ is a complete lattice.

(32) Let I be a non empty set and let J be a relational structure yielding

nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let X be a subset of $\prod J$ and let i be an element of I. Then $(\sup X)(i) = \sup \pi_i X$.

(33) Let I be a non empty set and let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let x, y be elements of $\prod J$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) for every element i of I holds $x(i) \ll y(i)$, and

(ii) there exists a finite subset K of I such that for every element i of I such that $i \notin K$ holds $x(i) = \perp_{J(i)}$.

5. The Way-Below Relation in Topological Spaces

One can prove the following four propositions:

(34) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose x is way below y. Let F be a family of subsets of T. If F is open and $y \subseteq \bigcup F$, then there exists a finite subset G of F such that $x \subseteq \bigcup G$.

(35) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose that for every family F of subsets of T such that F is open and $y \subseteq \bigcup F$ there exists a finite subset G of F such that $x \subseteq \bigcup G$. Then x is way below y.

(36) Let T be a non empty topological space, and let x be an element of $\langle$the topology of T, $\subseteq\rangle$, and let X be a subset of T. If $x = X$, then x is compact iff X is compact.

(37) Let T be a non empty topological space and let x be an element of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x =$ the carrier of T. Then x is compact if and only if T is compact.

Let T be a non empty topological space. We say that T is locally-compact if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x be a point of T and let X be a subset of T. Suppose $x \in X$ and X is open. Then there exists a subset Y of T such that $x \in \operatorname{Int} Y$ and $Y \subseteq X$ and Y is compact.

Let us observe that every non empty topological space which is compact and T_2 is also T_3 , T_4 , and locally-compact.

We now state the proposition

(38) For every set x holds $\{x\}_{\rm top}$ is T_2.

One can verify that there exists a non empty topological space which is compact and T_2 .

One can prove the following two propositions:

(39) Let T be a non empty topological space and let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact, then $x \ll y$.

(40) Let T be a non empty topological space. Suppose T is locally-compact. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact.

Let T be a topological structure and let X be a subset of the carrier of T. Then $\overline{X}$ is a subset of T.

The following three propositions are true:

(41) Let T be a non empty topological space. Suppose T is locally-compact and a T_2 space. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $Z = x$ and $\overline{Z} \subseteq y$ and $\overline{Z}$ is compact.

(42) Let X be a non empty topological space. Suppose X is a T_3 space and $\langle$the topology of X, $\subseteq\rangle$ is continuous. Then X is locally-compact.

(43) For every non empty topological space T such that T is locally-compact holds $\langle$the topology of T, $\subseteq\rangle$ is continuous.

References

[1] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.

[2] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.

[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.

[4] Grzegorz Bancerek. Filters - Part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.

[5] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[6] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[7] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.

[8] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[9] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[10] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[11] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[12] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[13] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[14] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[15] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[16] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.

[17] Artur Korniłowicz. Definitions and properties of the join and meet of subsets. *Formalized Mathematics*, 6(**1**):153–158, 1997.

[18] Artur Korniłowicz. Meet – continuous lattices. *Formalized Mathematics*, 6(**1**):159–167, 1997.
[19] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.
[20] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[21] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[22] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[23] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[24] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[27] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[29] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[30] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[31] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[32] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.
[33] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received October 11, 1996