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Dyadic Numbers and T_4 Topological Spaces

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Summary. This article is the first part of a paper proving the fundamental Urysohn's Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into four parts. In the first part, we prove some auxiliary theorems concerning properties of natural numbers and prove two useful schemes about recurrently defined functions; in the second part, we define a special set of rational numbers, which we call dyadic, and prove some of its properties. The next part of the paper contains the definitions of T_1 space and normal space, and we prove related theorems used in later parts of the paper. The final part of this work is developed for proving the theorem about the existence of some special family of subsets of a topological space. This theorem is essential in proving Urysohn's Lemma.

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The notation and terminology used in this paper have been introduced in the following articles: [24], [30], [9], [25], [23], [22], [31], [6], [7], [4], [2], [16], [3], [5], [28], [29], [1], [10], [13], [19], [32], [12], [18], [14], [15], [11], [20], [21], [8], [17], [27], and [26].

1. Preliminaries

The following propositions are true:

(1) $0 \neq \frac{1}{2}$ and $1 \neq \frac{1}{2}$.

(2) $0 < \frac{1}{2}$ and $\frac{1}{2} < 1$.

(3) For every natural number n holds $1 \leq 2^n$.

(4) For every natural number n holds $0 < 2^n$.

In this article we present several logical schemes. The scheme *FuncEx2DChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a non empty set $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

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There exists a function F from $[:\mathcal{A}, \mathcal{B}:]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{P}[x, y, F(\langle x, y\rangle)]$

provided the parameters meet the following requirement:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ there exists an element z of $\mathcal{C}$ such that $\mathcal{P}[x, y, z]$.

The scheme *RecExDN RD* concerns a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a function F from $\mathbb{N}$ into $\mathcal{A}$ such that $F(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, F(n), F(n+1)]$

provided the parameters satisfy the following condition:

- For every natural number n and for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$.

2. Dyadic Numbers

The subset $\mathbb{R}_{<0}$ of $\mathbb{R}$ is defined by:

(Def.1) For every real number x holds $x \in \mathbb{R}_{<0}$ iff $x < 0$.

The subset $\mathbb{R}_{>1}$ of $\mathbb{R}$ is defined by:

(Def.2) For every real number x holds $x \in \mathbb{R}_{>1}$ iff $1 < x$.

Let n be a natural number. The functor $\mathrm{dyadic}(n)$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.3) For every real number x holds $x \in \mathrm{dyadic}(n)$ iff there exists a natural number i such that $0 \leq i$ and $i \leq 2^n$ and $x = \frac{i}{2^n}$.

The subset DYADIC of $\mathbb{R}$ is defined by:

(Def.4) For every real number a holds $a \in \mathrm{DYADIC}$ iff there exists a natural number n such that $a \in \mathrm{dyadic}(n)$.

The subset DOM of $\mathbb{R}$ is defined by:

(Def.5) $\mathrm{DOM} = \mathbb{R}_{<0} \cup \mathrm{DYADIC} \cup \mathbb{R}_{>1}$.

Let T be a topological space, let A be a non empty subset of $\mathbb{R}$, let F be a function from A into $2^{\text{the carrier of } T}$, and let r be an element of A. Then $F(r)$ is a subset of the carrier of T.

One can prove the following three propositions:

(5) For every natural number n and for every real number x such that $x \in \mathrm{dyadic}(n)$ holds $0 \leq x$ and $x \leq 1$.

(6) $\mathrm{dyadic}(0) = \{0, 1\}$.

(7) $\mathrm{dyadic}(1) = \{0, \frac{1}{2}, 1\}$.

Let n be a natural number. Note that $\mathrm{dyadic}(n)$ is non empty.

Next we state the proposition

(8) For every natural number x and for every natural number n holds x^n is a natural number.

Let x, n be natural numbers. Then x^n is a natural number.

The following proposition is true

(9) Let n be a natural number. Then there exists a finite sequence F_1 such that $\operatorname{dom} F_1 = \operatorname{Seg}(2^n + 1)$ and for every natural number i such that $i \in \operatorname{dom} F_1$ holds $F_1(i) = \frac{i-1}{2^n}$.

Let n be a natural number. The functor $\operatorname{dyad}(n)$ yielding a finite sequence is defined by:

(Def.6) $\operatorname{dom} \operatorname{dyad}(n) = \operatorname{Seg}(2^n + 1)$ and for every natural number i such that $i \in \operatorname{dom} \operatorname{dyad}(n)$ holds $(\operatorname{dyad}(n))(i) = \frac{i-1}{2^n}$.

We now state the proposition

(10) For every natural number n holds $\operatorname{dom} \operatorname{dyad}(n) = \operatorname{Seg}(2^n + 1)$ and $\operatorname{rng} \operatorname{dyad}(n) = \operatorname{dyadic}(n)$.

Let us note that DYADIC is non empty.

Let us observe that DOM is non empty.

One can prove the following propositions:

(11) For every natural number n holds $\operatorname{dyadic}(n) \subseteq \operatorname{dyadic}(n + 1)$.

(12) For every natural number n holds $0 \in \operatorname{dyadic}(n)$ and $1 \in \operatorname{dyadic}(n)$.

(13) For every natural number n and for every natural number i such that $0 < i$ and $i \leq 2^n$ holds $\frac{i \cdot 2 - 1}{2^{n+1}} \in \operatorname{dyadic}(n + 1) \setminus \operatorname{dyadic}(n)$.

(14) For every natural number n and for every natural number i such that $0 \leq i$ and $i < 2^n$ holds $\frac{i \cdot 2 + 1}{2^{n+1}} \in \operatorname{dyadic}(n + 1) \setminus \operatorname{dyadic}(n)$.

(15) For every natural number n holds $\frac{1}{2^{n+1}} \in \operatorname{dyadic}(n + 1) \setminus \operatorname{dyadic}(n)$.

Let n be a natural number and let x be an element of $\operatorname{dyadic}(n)$. The functor $\operatorname{axis}(x, n)$ yields a natural number and is defined by:

(Def.7) $x = \frac{\operatorname{axis}(x,n)}{2^n}$.

One can prove the following propositions:

(16) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $x = \frac{\operatorname{axis}(x,n)}{2^n}$ and $0 \leq \operatorname{axis}(x, n)$ and $\operatorname{axis}(x, n) \leq 2^n$.

(17) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $\frac{\operatorname{axis}(x,n)-1}{2^n} < x$ and $x < \frac{\operatorname{axis}(x,n)+1}{2^n}$.

(18) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $\frac{\operatorname{axis}(x,n)-1}{2^n} < \frac{\operatorname{axis}(x,n)+1}{2^n}$.

(19) For every natural number n there exists a natural number k such that $n = k \cdot 2$ or $n = k \cdot 2 + 1$.

(20) Let n be a natural number and let x be an element of $\operatorname{dyadic}(n + 1)$. If $x \notin \operatorname{dyadic}(n)$, then $\frac{\operatorname{axis}(x,n+1)-1}{2^{n+1}} \in \operatorname{dyadic}(n)$ and $\frac{\operatorname{axis}(x,n+1)+1}{2^{n+1}} \in \operatorname{dyadic}(n)$.

(21) For every natural number n and for all elements x_1, x_2 of $\operatorname{dyadic}(n)$ such that $x_1 < x_2$ holds $\operatorname{axis}(x_1, n) < \operatorname{axis}(x_2, n)$.

(22) For every natural number n and for all elements x_1, x_2 of dyadic(n) such that $x_1 < x_2$ holds $x_1 \leq \frac{\mathrm{axis}(x_2,n)-1}{2^n}$ and $\frac{\mathrm{axis}(x_1,n)+1}{2^n} \leq x_2$.

(23) Let n be a natural number and let x_1, x_2 be elements of dyadic$(n+1)$. If $x_1 < x_2$ and $x_1 \notin$ dyadic(n) and $x_2 \notin$ dyadic(n), then $\frac{\mathrm{axis}(x_1,n+1)+1}{2^{n+1}} \leq \frac{\mathrm{axis}(x_2,n+1)-1}{2^{n+1}}$.

3. Normal Spaces

Let T be a topological space and let x be a point of T. A subset of the carrier of T is said to be a neighbourhood of x in T if:

(Def.8) There exists a subset A of the carrier of T such that A is open and $x \in A$ and $A \subseteq$ it.

One can prove the following propositions:

(24) Let T be a topological space and let A be a subset of the carrier of T. Then A is open if and only if for every point x of T such that $x \in A$ there exists a neighbourhood B of x in T such that $B \subseteq A$.

(25) Let T be a topological space, and let A be a subset of the carrier of T, and let x be a point of T. If A is open and $x \in A$, then A is a neighbourhood of x in T.

(26) Let T be a topological space and let A be a subset of the carrier of T. Suppose that for every point x of T such that $x \in A$ holds A is a neighbourhood of x in T. Then A is open.

Let T be a topological space. We say that T is a T_1 space if and only if the condition (Def.9) is satisfied.

(Def.9) Let p, q be points of T. Suppose $p \neq q$. Then there exist subsets W, V of the carrier of T such that W is open and V is open and $p \in W$ and $q \notin W$ and $q \in V$ and $p \notin V$.

Next we state the proposition

(27) For every topological space T holds T is a T_1 space iff for every point p of T holds $\{p\}$ is closed.

Let T be a topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let x be a point of T. Then $F(x)$ is a real number.

The following four propositions are true:

(28) Let T be a topological space. Suppose T is a T_4 space. Let A, B be subsets of the carrier of T. Suppose $A \neq \emptyset$ and A is open and B is open and $\overline{A} \subseteq B$. Then there exists a subset C of the carrier of T such that $C \neq \emptyset$ and C is open and $\overline{A} \subseteq C$ and $\overline{C} \subseteq B$.

(29) Let T be a topological space. Then T is a T_3 space if and only if for every subset A of the carrier of T and for every point p of T such that A is open and $p \in A$ there exists a subset B of the carrier of T such that $p \in B$ and B is open and $\overline{B} \subseteq A$.

(30) Let T be a topological space. Suppose T is a T_4 space and a T_1 space. Let A be a subset of the carrier of T. Suppose A is open and $A \neq \emptyset$. Then there exists a subset B of the carrier of T such that $B \neq \emptyset$ and $\overline{B} \subseteq A$.

(31) Let T be a topological space. Suppose T is a T_4 space. Let A be a subset of the carrier of T. Suppose A is open and $A \neq \emptyset$. Let B be a subset of the carrier of T. Suppose B is closed and $B \neq \emptyset$ and $B \subseteq A$. Then there exists a subset C of the carrier of T such that C is open and $B \subseteq C$ and $\overline{C} \subseteq A$.

4. Some Increasing Family of Sets in Normal Space

Let T be a topological space and let A, B, C be subsets of the carrier of T. We say that C is between A and B if and only if:

(Def.10) $C \neq \emptyset$ and C is open and $\overline{A} \subseteq C$ and $\overline{C} \subseteq B$.

One can prove the following proposition

(32) Let T be a topological space. Suppose T is a T_4 space. Let A, B be subsets of the carrier of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and let G be a function from dyadic(n) into $2^{\text{the carrier of } T}$. Suppose that for all elements r_1, r_2 of dyadic(n) such that $r_1 < r_2$ holds $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$ and $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$. Then there exists a function F from dyadic($n+1$) into $2^{\text{the carrier of } T}$ such that for all elements r_1, r_2, r of dyadic($n+1$) if $r_1 < r_2$, then $F(r_1)$ is open and $F(r_2)$ is open and $\overline{F(r_1)} \subseteq F(r_2)$ and $A \subseteq F(0)$ and $B = \Omega_T \setminus F(1)$ and if $r \in$ dyadic(n), then $F(r) = G(r)$.

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