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The Subformula Tree of a Formula of the First Order Language

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Summary. A continuation of [12]. The notions of list of immediate constituents of a formula and subformula tree of a formula are introduced. The some propositions related to these notions are proved.

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The terminology and notation used in this paper are introduced in the following articles: [15], [18], [3], [11], [19], [9], [10], [13], [8], [17], [1], [4], [6], [5], [7], [14], [2], and [16].

1. Preliminaries

The following propositions are true:

(1) For all real numbers x, y, z such that $x \leq y$ and $y < z$ holds $x < z$.

(2) For all natural numbers m, k holds $m + 1 \leq k$ iff $m < k$.

(3) For every finite sequence r holds $r = r \restriction \mathrm{Seg}\,\mathrm{len}\, r$.

(4) For every natural number n and for every finite sequence r there exists a finite sequence q such that $q = r \restriction \mathrm{Seg}\, n$ and $q \preceq r$.

(5) For all finite sequences p, q, r such that $q \preceq r$ holds $p \mathbin{^\frown} q \preceq p \mathbin{^\frown} r$.

(6) Let D be a non empty set, and let r be a finite sequence of elements of D, and let r_1, r_2 be finite sequences, and let k be a natural number. Suppose $k + 1 \leq \mathrm{len}\, r$ and $r_1 = r \restriction \mathrm{Seg}(k + 1)$ and $r_2 = r \restriction \mathrm{Seg}\, k$. Then there exists an element x of D such that $r_1 = r_2 \mathbin{^\frown} \langle x \rangle$.

(7) Let D be a non empty set, and let r be a finite sequence of elements of D, and let r_1 be a finite sequence. If $1 \leq \mathrm{len}\, r$ and $r_1 = r \restriction \mathrm{Seg}\, 1$, then there exists an element x of D such that $r_1 = \langle x \rangle$.

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Let D be a non empty set and let T be a tree decorated with elements of D. Observe that every element of $\operatorname{dom} T$ is function-like and relation-like.

Let D be a non empty set and let T be a tree decorated with elements of D. One can verify that every element of $\operatorname{dom} T$ is finite sequence-like.

Let D be a non empty set. One can check that there exists a tree decorated with elements of D which is finite.

In the sequel T will be a decorated tree and p will be a finite sequence of elements of $\mathbb{N}$.

Next we state the proposition

(8) If $p \in \operatorname{dom} T$, then $T(p) = (T \restriction p)(\varepsilon)$.

In the sequel T is a finite-branching decorated tree, t is an element of $\operatorname{dom} T$, x is a finite sequence, and n is a natural number.

The following propositions are true:

(9) $\operatorname{succ}(T, t) = T \cdot \operatorname{Succ} t$.

(10) $\operatorname{dom}(T \cdot \operatorname{Succ} t) = \operatorname{dom} \operatorname{Succ} t$.

(11) $\operatorname{dom} \operatorname{succ}(T, t) = \operatorname{dom} \operatorname{Succ} t$.

(12) $t \frown \langle n \rangle \in \operatorname{dom} T$ iff $n + 1 \in \operatorname{dom} \operatorname{Succ} t$.

(13) For all T, x, n such that $x \frown \langle n \rangle \in \operatorname{dom} T$ holds $T(x \frown \langle n \rangle) = (\operatorname{succ}(T, x))(n + 1)$.

In the sequel x, x' will be elements of $\operatorname{dom} T$ and y' will be arbitrary.

One can prove the following two propositions:

(14) If $x' \in \operatorname{succ} x$, then $T(x') \in \operatorname{rng} \operatorname{succ}(T, x)$.

(15) If $y' \in \operatorname{rng} \operatorname{succ}(T, x)$, then there exists x' such that $y' = T(x')$ and $x' \in \operatorname{succ} x$.

In the sequel n, k, m will denote natural numbers.

The scheme *ExDecTrees* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a finite sequence of elements of $\mathcal{A}$, and states that:

> There exists a finite-branching tree T decorated with elements of $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\operatorname{dom} T$ and for every element w of $\mathcal{A}$ such that $w = T(t)$ holds $\operatorname{succ}(T, t) = \mathcal{F}(w)$

for all values of the parameters.

The following propositions are true:

(16) For every tree T and for every element t of T holds $\operatorname{Seg}_{\preceq}(t)$ is a finite chain of T.

(17) For every tree T holds T-level$(0) = \{\varepsilon\}$.

(18) For every tree T holds T-level$(n + 1) = \bigcup\{\operatorname{succ} w : w$ ranges over elements of T, $\operatorname{len} w = n\}$.

(19) For every finite-branching tree T and for every natural number n holds T-level(n) is finite.

(20) For every finite-branching tree T holds T is finite iff there exists a natural number n such that T-level$(n) = \emptyset$.

(21) For every finite-branching tree T such that T is not finite holds there exists chain of T which is not finite.

(22) For every finite-branching tree T such that T is not finite holds there exists branch of T which is not finite.

(23) Let T be a tree, and let C be a chain of T, and let t be an element of T. If $t \in C$ and C is not finite, then there exists an element t' of T such that $t' \in C$ and $t \prec t'$.

(24) Let T be a tree, and let B be a branch of T, and let t be an element of T. Suppose $t \in B$ and B is not finite. Then there exists an element t' of T such that $t' \in B$ and $t' \in \operatorname{succ} t$.

(25) Let f be a function from $\mathbb{N}$ into $\mathbb{N}$. Suppose that for every n holds $f(n+1)$ **qua** natural number $\leq f(n)$ **qua** natural number. Then there exists m such that for every n such that $m \leq n$ holds $f(n) = f(m)$.

The scheme *FinDecTree* concerns a non empty set $\mathcal{A}$, a finite-branching tree $\mathcal{B}$ decorated with elements of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a natural number, and states that:

$\mathcal{B}$ is finite

provided the parameters meet the following requirement:

- For all elements t, t' of $\operatorname{dom} \mathcal{B}$ and for every element d of $\mathcal{A}$ such that $t' \in \operatorname{succ} t$ and $d = \mathcal{B}(t')$ holds $\mathcal{F}(d) < \mathcal{F}(\mathcal{B}(t))$.

In the sequel D will denote a non empty set and T will denote a tree decorated with elements of D.

Next we state two propositions:

(26) For arbitrary y such that $y \in \operatorname{rng} T$ holds y is an element of D.

(27) For arbitrary x such that $x \in \operatorname{dom} T$ holds $T(x)$ is an element of D.

2. Subformula tree

In the sequel F, G, H will denote elements of WFF.

One can prove the following propositions:

(28) If F is a subformula of G, then $\operatorname{len}({}^{@}F) \leq \operatorname{len}({}^{@}G)$.

(29) If F is a subformula of G and $\operatorname{len}({}^{@}F) = \operatorname{len}({}^{@}G)$, then $F = G$.

Let p be an element of WFF. The list of immediate constituents of p yields a finite sequence of elements of WFF and is defined by:

(Def.1) (i) The list of immediate constituents of $p = \varepsilon_{\mathrm{WFF}}$ if $p =$ VERUM or p is atomic,

(ii) the list of immediate constituents of $p = \langle \operatorname{Arg}(p) \rangle$ if p is negative,

(iii) the list of immediate constituents of $p = \langle \operatorname{LeftArg}(p), \operatorname{RightArg}(p) \rangle$ if p is conjunctive,

(iv) the list of immediate constituents of $p = \langle \operatorname{Scope}(p) \rangle$, otherwise.

Next we state two propositions:

(30) Suppose $k \in \operatorname{dom}$(the list of immediate constituents of F) and $G =$ (the list of immediate constituents of F)(k). Then G is an immediate constituent of F.

(31) rng (the list of immediate constituents of F) $= \{G : G$ ranges over elements of WFF, G is an immediate constituent of $F\}$.

Let p be an element of WFF. The tree of subformulae of p yields a finite tree decorated with elements of WFF and is defined by the conditions (Def.2).

(Def.2) (i) (The tree of subformulae of p)$(\varepsilon) = p$, and

(ii) for every element x of dom (the tree of subformulae of p) holds succ(the tree of subformulae of p, x) $=$ the list of immediate constituents of (the tree of subformulae of p)(x).

In the sequel t, t' will be elements of dom (the tree of subformulae of F).

One can prove the following propositions:

(32) (The tree of subformulae of F)$(\varepsilon) = F$.

(33) succ(the tree of subformulae of F, t) $=$ the list of immediate constituents of (the tree of subformulae of F)(t).

(34) $F \in$ rng (the tree of subformulae of F).

(35) Suppose $t \frown \langle n \rangle \in$ dom (the tree of subformulae of F). Then there exists G such that

(i) $G =$ (the tree of subformulae of F)$(t \frown \langle n \rangle)$, and

(ii) G is an immediate constituent of (the tree of subformulae of F)(t).

(36) The following statements are equivalent

(i) H is an immediate constituent of (the tree of subformulae of F)(t),

(ii) there exists n such that $t \frown \langle n \rangle \in$ dom (the tree of subformulae of F) and $H =$ (the tree of subformulae of F)$(t \frown \langle n \rangle)$.

(37) Suppose $G \in$ rng (the tree of subformulae of F) and H is an immediate constituent of G. Then $H \in$ rng (the tree of subformulae of F).

(38) If $G \in$ rng (the tree of subformulae of F) and H is a subformula of G, then $H \in$ rng (the tree of subformulae of F).

(39) $G \in$ rng (the tree of subformulae of F) iff G is a subformula of F.

(40) rng (the tree of subformulae of F) $=$ Subformulae F.

(41) Suppose $t' \in$ succ t. Then (the tree of subformulae of F)(t') is an immediate constituent of (the tree of subformulae of F)(t).

(42) If $t \preceq t'$, then (the tree of subformulae of F)(t') is a subformula of (the tree of subformulae of F)(t).

(43) If $t \prec t'$, then len($^{@}$(the tree of subformulae of F)(t')) $<$ len($^{@}$(the tree of subformulae of F)(t)).

(44) If $t \prec t'$, then (the tree of subformulae of F)$(t') \neq$ (the tree of subformulae of F)(t).

(45) If $t \prec t'$, then (the tree of subformulae of F)(t') is a proper subformula of (the tree of subformulae of F)(t).

(46) (The tree of subformulae of F)$(t) = F$ iff $t = \varepsilon$.

(47) Suppose $t \neq t'$ and (the tree of subformulae of F)(t) = (the tree of subformulae of F)(t'). Then t and t' are not comparable.

Let F, G be elements of WFF. The F-entry points in subformula tree of G yields an antichain of prefixes of dom (the tree of subformulae of F) and is defined by the condition (Def.3).

(Def.3) Let t be an element of dom (the tree of subformulae of F). Then $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of F)$(t) = G$.

We now state several propositions:

(48) $t \in$ the F-entry points in subformula tree of G iff (the tree of subformulae of F)$(t) = G$.

(49) The F-entry points in subformula tree of $G = \{t : t$ ranges over elements of dom (the tree of subformulae of F), (the tree of subformulae of F)$(t) = G\}$.

(50) G is a subformula of F iff the F-entry points in subformula tree of $G \neq \emptyset$.

(51) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is negative. Then (the tree of subformulae of F)(t') = Arg((the tree of subformulae of F)(t)) and $m = 0$.

(52) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is conjunctive. Then

(i) (the tree of subformulae of F)(t') = LeftArg((the tree of subformulae of F)(t)) and $m = 0$, or

(ii) (the tree of subformulae of F)(t') = RightArg((the tree of subformulae of F)(t)) and $m = 1$.

(53) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is universal. Then (the tree of subformulae of F)(t') = Scope((the tree of subformulae of F)(t)) and $m = 0$.

(54) Suppose (the tree of subformulae of F)(t) is negative. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F), and

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = Arg((the tree of subformulae of F)(t)).

(55) Suppose (the tree of subformulae of F)(t) is conjunctive. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F),

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = LeftArg((the tree of subformulae of F)(t)),

(iii) $t \frown \langle 1 \rangle \in$ dom (the tree of subformulae of F), and

(iv) (the tree of subformulae of F)$(t \frown \langle 1 \rangle)$ = RightArg((the tree of subformulae of F)(t)).

(56) Suppose (the tree of subformulae of F)(t) is universal. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F), and

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = Scope((the tree of subformulae of F)(t)).

In the sequel t will be an element of dom (the tree of subformulae of F) and s will be an element of dom (the tree of subformulae of G).

Next we state the proposition

(57) Suppose $t \in$ the F-entry points in subformula tree of G and $s \in$ the G-entry points in subformula tree of H. Then $t ^\frown s \in$ the F-entry points in subformula tree of H.

In the sequel t will be an element of dom (the tree of subformulae of F) and s will be a finite sequence.

Next we state several propositions:

(58) Suppose $t \in$ the F-entry points in subformula tree of G and $t ^\frown s \in$ the F-entry points in subformula tree of H. Then $s \in$ the G-entry points in subformula tree of H.

(59) Given F, G, H. Then $\{t ^\frown s : t$ ranges over elements of dom (the tree of subformulae of F), s ranges over elements of dom (the tree of subformulae of G), $t \in$ the F-entry points in subformula tree of $G \ \wedge \ s \in$ the G-entry points in subformula tree of $H\} \subseteq$ the F-entry points in subformula tree of H.

(60) (The tree of subformulae of F) $\restriction t =$ the tree of subformulae of (the tree of subformulae of F)(t).

(61) $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of F) $\restriction t =$ the tree of subformulae of G.

(62) The F-entry points in subformula tree of $G = \{t : t$ ranges over elements of dom (the tree of subformulae of F), (the tree of subformulae of F) $\restriction t =$ the tree of subformulae of $G\}$.

In the sequel C is a chain of dom (the tree of subformulae of F).

Next we state the proposition

(63) Given F, G, H, C. Suppose that

(i) $G \in \{$(the tree of subformulae of F)(t) : t ranges over elements of dom (the tree of subformulae of F), $t \in C\}$, and

(ii) $H \in \{$(the tree of subformulae of F)(t) : t ranges over elements of dom (the tree of subformulae of F), $t \in C\}$.

Then G is a subformula of H or H is a subformula of G.

Let F be an element of WFF. An element of WFF is said to be a subformula of F if:

(Def.4) It is a subformula of F.

Let F be an element of WFF and let G be a subformula of F. An element of dom (the tree of subformulae of F) is said to be an entry point in subformula tree of G if:

(Def.5) (The tree of subformulae of F)(it) $= G$.

In the sequel G will denote a subformula of F.

Next we state the proposition

(64) t is an entry point in subformula tree of G iff (the tree of subformulae of F)(t) $= G$.

In the sequel t, t' are entry points in subformula tree of G.

The following proposition is true

(65) If $t \neq t'$, then t and t' are not comparable.

Let F be an element of WFF and let G be a subformula of F. The entry points in subformula tree of G yields a non empty antichain of prefixes of dom (the tree of subformulae of F) and is defined as follows:

(Def.6) The entry points in subformula tree of $G =$ the F-entry points in subformula tree of G.

We now state three propositions:

(66) The entry points in subformula tree of $G =$ the F-entry points in subformula tree of G.

(67) $t \in$ the entry points in subformula tree of G.

(68) The entry points in subformula tree of $G = \{t : t$ ranges over entry points in subformula tree of G, $t = t\}$.

In the sequel G_1, G_2 will denote subformulae of F, t_1 will denote an entry point in subformula tree of G_1, and s will denote an element of dom (the tree of subformulae of G_1).

We now state the proposition

(69) If $s \in$ the G_1-entry points in subformula tree of G_2, then $t_1 \frown s$ is an entry point in subformula tree of G_2.

In the sequel s will be a finite sequence.

Next we state three propositions:

(70) If $t_1 \frown s$ is an entry point in subformula tree of G_2, then $s \in$ the G_1-entry points in subformula tree of G_2.

(71) Given F, G_1, G_2. Then $\{t \frown s : t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom (the tree of subformulae of G_1), $s \in$ the G_1-entry points in subformula tree of $G_2\} = \{t \frown s : t$ ranges over elements of dom (the tree of subformulae of F), s ranges over elements of dom (the tree of subformulae of G_1), $t \in$ the F-entry points in subformula tree of $G_1 \ \wedge \ s \in$ the G_1-entry points in subformula tree of $G_2\}$.

(72) Given F, G_1, G_2. Then $\{t \frown s : t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom (the tree of subformulae of G_1), $s \in$ the G_1-entry points in subformula tree of $G_2\} \subseteq$ the entry points in subformula tree of G_2.

In the sequel G_1, G_2 will denote subformulae of F, t_1 will denote an entry point in subformula tree of G_1, and t_2 will denote an entry point in subformula tree of G_2.

The following two propositions are true:

(73) If there exist t_1, t_2 such that $t_1 \preceq t_2$, then G_2 is a subformula of G_1.

(74) If G_2 is a subformula of G_1, then for every t_1 there exists t_2 such that $t_1 \preceq t_2$.

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