FORMALIZED MATHEMATICS
Volume 5, Number 2, 1996
Warsaw University - Białystok

Subtrees [1]

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Summary. The concepts of root tree, the set of successors of a node in decorated tree and sets of subtrees are introduced.

MML Identifier: `TREES_9`.

The notation and terminology used here are introduced in the following papers: [16], [17], [15], [3], [18], [12], [13], [9], [14], [11], [7], [2], [1], [4], [6], [8], [5], and [10].

1. Root Tree and Successors of Node in Decorated Tree

One can check that every tree which is finite is also finite-order.

The following propositions are true:

(1) For every decorated tree t holds $t \upharpoonright \varepsilon_{\mathbb{N}} = t$.

(2) For every tree t and for all finite sequences p, q of elements of $\mathbb{N}$ such that $p \frown q \in t$ holds $t \upharpoonright (p \frown q) = t \upharpoonright p \upharpoonright q$.

(3) Let t be a decorated tree and let p, q be finite sequences of elements of $\mathbb{N}$. If $p \frown q \in \mathrm{dom}\, t$, then $t \upharpoonright (p \frown q) = t \upharpoonright p \upharpoonright q$.

A decorated tree is root if:

(Def.1) $\mathrm{dom}\,\mathrm{it} =$ the elementary tree of 0.

Let us note that every decorated tree which is root is also finite.

The following three propositions are true:

(4) For every decorated tree t holds t is root iff $\varepsilon \in \mathrm{Leaves}(\mathrm{dom}\, t)$.

(5) For every tree t and for every element p of t holds $t \upharpoonright p =$ the elementary tree of 0 iff $p \in \mathrm{Leaves}(t)$.

[1] This article has been worked out during the visit of the author in Nagano in Summer 1994.

ISSN 1426–2630

(6) For every decorated tree t and for every node p of t holds $t \upharpoonright p$ is root iff $p \in \mathrm{Leaves}(\mathrm{dom}\, t)$.

Let us mention that there exists a decorated tree which is root and there exists a decorated tree which is finite and non root.

Let x be a set. Note that the root tree of x is finite and root.

A tree is finite-branching if:

(Def.2) For every element x of it holds $\mathrm{succ}\, x$ is finite.

Let us mention that every tree which is finite-order is also finite-branching.

Let us note that there exists a tree which is finite.

A decorated tree is finite-order if:

(Def.3) dom it is finite-order.

A decorated tree is finite-branching if:

(Def.4) dom it is finite-branching.

One can check that every decorated tree which is finite is also finite-order and every decorated tree which is finite-order is also finite-branching.

Let us observe that there exists a decorated tree which is finite.

Let t be a finite-order decorated tree. One can verify that $\mathrm{dom}\, t$ is finite-order.

Let t be a finite-branching decorated tree. Note that $\mathrm{dom}\, t$ is finite-branching.

Let t be a finite-branching tree and let p be an element of t. Note that $\mathrm{succ}\, p$ is finite.

The scheme *FinOrdSet* concerns a unary functor $\mathcal{F}$ yielding a set and a finite set $\mathcal{A}$, and states that:

For every natural number n holds $\mathcal{F}(n) \in \mathcal{A}$ iff $n < \mathrm{card}\, \mathcal{A}$

provided the parameters have the following properties:

- For every set x such that $x \in \mathcal{A}$ there exists a natural number n such that $x = \mathcal{F}(n)$,
- For all natural numbers i, j such that $i < j$ and $\mathcal{F}(j) \in \mathcal{A}$ holds $\mathcal{F}(i) \in \mathcal{A}$,
- For all natural numbers i, j such that $\mathcal{F}(i) = \mathcal{F}(j)$ holds $i = j$.

Let X be a set. One can verify that there exists a finite sequence of elements of X which is one-to-one and empty.

The following proposition is true

(7) Let t be a finite-branching tree, and let p be an element of t, and let n be a natural number. Then $p \mathbin{^\frown} \langle n \rangle \in \mathrm{succ}\, p$ if and only if $n < \mathrm{card}\, \mathrm{succ}\, p$.

Let t be a finite-branching tree and let p be an element of t. The functor $\mathrm{Succ}\, p$ yielding an one-to-one finite sequence of elements of t is defined by:

(Def.5) $\mathrm{len}\, \mathrm{Succ}\, p = \mathrm{card}\, \mathrm{succ}\, p$ and $\mathrm{rng}\, \mathrm{Succ}\, p = \mathrm{succ}\, p$ and for every natural number i such that $i < \mathrm{len}\, \mathrm{Succ}\, p$ holds $(\mathrm{Succ}\, p)(i+1) = p \mathbin{^\frown} \langle i \rangle$.

Let t be a finite-branching decorated tree and let p be a finite sequence. Let us assume that $p \in \mathrm{dom}\, t$. The functor $\mathrm{succ}(t, p)$ yielding a finite sequence is defined by:

(Def.6) There exists an element q of $\mathrm{dom}\, t$ such that $q = p$ and $\mathrm{succ}(t, p) = t \cdot \mathrm{Succ}\, q$.

One can prove the following two propositions:

(8) Let t be a finite-branching decorated tree. Then there exists a set x and there exists a decorated tree yielding finite sequence p such that $t = x\text{-tree}(p)$.

(9) For every finite decorated tree t and for every node p of t holds $t \restriction p$ is finite.

Let t be a finite decorated tree and let p be a node of t. Observe that $t \restriction p$ is finite.

The following proposition is true

(10) For every finite tree t and for every element p of t such that $t = t \restriction p$ holds $p = \varepsilon$.

Let D be a non empty set and let S be a non empty subset of $\mathrm{FinTrees}(D)$. One can verify that every element of S is finite.

2. Set of Subtrees of Decorated Tree

Let t be a decorated tree. The functor $\mathrm{Subtrees}(t)$ yielding a constituted of decorated trees non empty set is defined by:

(Def.7) $\mathrm{Subtrees}(t) = \{t \restriction p : p$ ranges over nodes of $t\}$.

Let D be a non empty set and let t be a tree decorated with elements of D. Then $\mathrm{Subtrees}(t)$ is a non empty subset of $\mathrm{Trees}(D)$.

Let D be a non empty set and let t be a finite tree decorated with elements of D. Then $\mathrm{Subtrees}(t)$ is a non empty subset of $\mathrm{FinTrees}(D)$.

Let t be a finite decorated tree. One can verify that every element of $\mathrm{Subtrees}(t)$ is finite.

In the sequel x denotes a set and t, t_1, t_2 denote decorated trees.

One can prove the following propositions:

(11) $x \in \mathrm{Subtrees}(t)$ iff there exists a node n of t such that $x = t \restriction n$.

(12) $t \in \mathrm{Subtrees}(t)$.

(13) If t_1 is finite and $\mathrm{Subtrees}(t_1) = \mathrm{Subtrees}(t_2)$, then $t_1 = t_2$.

(14) For every node n of t holds $\mathrm{Subtrees}(t \restriction n) \subseteq \mathrm{Subtrees}(t)$.

Let t be a decorated tree. The functor $\mathrm{FixedSubtrees}(t)$ yields a non empty subset of $[\!:\mathrm{dom}\, t, \mathrm{Subtrees}(t):\!]$ and is defined as follows:

(Def.8) $\mathrm{FixedSubtrees}(t) = \{\langle p, t \restriction p\rangle : p$ ranges over nodes of $t\}$.

Next we state three propositions:

(15) $x \in \mathrm{FixedSubtrees}(t)$ iff there exists a node n of t such that $x = \langle n, t \restriction n\rangle$.

(16) $\langle \varepsilon, t\rangle \in \mathrm{FixedSubtrees}(t)$.

(17) If $\mathrm{FixedSubtrees}(t_1) = \mathrm{FixedSubtrees}(t_2)$, then $t_1 = t_2$.

Let t be a decorated tree and let C be a set. The functor C-Subtrees(t) yielding a subset of Subtrees(t) is defined as follows:

(Def.9) C-Subtrees$(t) = \{t \restriction p : p$ ranges over nodes of t, $p \notin$ Leaves$(\mathrm{dom}\, t) \vee t(p) \in C\}$.

In the sequel C denotes a set.

We now state two propositions:

(18) $x \in C$-Subtrees(t) iff there exists a node n of t such that $x = t \restriction n$ but $n \notin$ Leaves$(\mathrm{dom}\, t)$ or $t(n) \in C$.

(19) C-Subtrees(t) is empty iff t is root and $t(\varepsilon) \notin C$.

Let t be a finite decorated tree and let C be a set. The functor C-ImmediateSubtrees(t) yields a function from C-Subtrees(t) into $(\text{Subtrees}(t))^*$ and is defined by the condition (Def.10).

(Def.10) Let d be a decorated tree. Suppose $d \in C$-Subtrees(t). Let p be a finite sequence of elements of Subtrees(t). If $p = (C$-ImmediateSubtrees$(t))(d)$, then $d = d(\varepsilon)$-tree(p).

3. Set of Subtrees of Set of Decorated Tree

Let X be a constituted of decorated trees non empty set. The functor Subtrees(X) yielding a constituted of decorated trees non empty set is defined by:

(Def.11) Subtrees$(X) = \{t \restriction p : t$ ranges over elements of X, p ranges over nodes of $t\}$.

Let D be a non empty set and let X be a non empty subset of Trees(D). Then Subtrees(X) is a non empty subset of Trees(D).

Let D be a non empty set and let X be a non empty subset of FinTrees(D). Then Subtrees(X) is a non empty subset of FinTrees(D).

In the sequel X, Y will be non empty constituted of decorated trees sets.

We now state three propositions:

(20) $x \in$ Subtrees(X) iff there exists an element t of X and there exists a node n of t such that $x = t \restriction n$.

(21) If $t \in X$, then $t \in$ Subtrees(X).

(22) If $X \subseteq Y$, then Subtrees$(X) \subseteq$ Subtrees(Y).

Let t be a decorated tree. Observe that $\{t\}$ is non empty and constituted of decorated trees.

Next we state two propositions:

(23) Subtrees$(\{t\}) =$ Subtrees(t).

(24) Subtrees$(X) = \bigcup\{\text{Subtrees}(t) : t$ ranges over elements of $X\}$.

Let X be a constituted of decorated trees non empty set and let C be a set. The functor C-Subtrees(X) yields a subset of Subtrees(X) and is defined as follows:

(Def.12) C-Subtrees$(X) = \{t \restriction p : t$ ranges over elements of X, p ranges over nodes of t, $p \notin \mathrm{Leaves}(\mathrm{dom}\, t) \ \vee \ t(p) \in C\}$.

We now state four propositions:

(25) $x \in C$-Subtrees(X) iff there exists an element t of X and there exists a node n of t such that $x = t \restriction n$ but $n \notin \mathrm{Leaves}(\mathrm{dom}\, t)$ or $t(n) \in C$.

(26) C-Subtrees(X) is empty iff for every element t of X holds t is root and $t(\varepsilon) \notin C$.

(27) C-Subtrees$(\{t\}) = C$-Subtrees(t).

(28) C-Subtrees$(X) = \bigcup\{C$-Subtrees$(t) : t$ ranges over elements of $X\}$.

Let X be a non empty constituted of decorated trees set. Let us assume that every element of X is finite. Let C be a set. The functor C-ImmediateSubtrees(X) yields a function from C-Subtrees(X) into $(\mathrm{Subtrees}(X))^*$ and is defined by the condition (Def.13).

(Def.13) Let d be a decorated tree. Suppose $d \in C$-Subtrees(X). Let p be a finite sequence of elements of Subtrees(X). If $p = (C$-ImmediateSubtrees$(X))(d)$, then $d = d(\varepsilon)$-tree(p).

Let t be a tree. Observe that there exists an element of t which is empty.

We now state four propositions:

(29) For every finite decorated tree t and for every element p of $\mathrm{dom}\, t$ holds $\mathrm{len}\, \mathrm{succ}(t, p) = \mathrm{len}\, \mathrm{Succ}\, p$ and $\mathrm{dom}\, \mathrm{succ}(t, p) = \mathrm{dom}\, \mathrm{Succ}\, p$.

(30) For every finite tree yielding finite sequence p and for every empty element n of $\widehat{p}$ holds $\mathrm{card}\, \mathrm{succ}\, n = \mathrm{len}\, p$.

(31) Let t be a finite decorated tree, and let x be a set, and let p be a decorated tree yielding finite sequence. Suppose $t = x$-tree(p). Let n be an empty element of $\mathrm{dom}\, t$. Then $\mathrm{succ}(t, n) =$ the roots of p.

(32) For every finite decorated tree t and for every node p of t and for every node q of $t \restriction p$ holds $\mathrm{succ}(t, p \frown q) = \mathrm{succ}(t \restriction p, q)$.

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Received November 25, 1994