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The One-Dimensional Lebesgue Measure

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Summary. The paper is the crowning of a series of articles written in the Mizar language, being a formalization of notions needed for the description of the one-dimensional Lebesgue measure. The formalization of the notion as classical as the Lebesgue measure determines the powers of the PC Mizar system as a tool for the strict, precise notation and verification of the correctness of deductive theories. Following the successive articles [6], [8], [10], [11] constructed so that the final one should include the definition and the basic properties of the Lebesgue measure, we observe one of the paths relatively simple in the sense of the definition, enabling us the formal introduction of this notion. This way, although toilsome, since such is the nature of formal theories, is greatly instructive. It brings home the proper succession of the introduction of the definitions of intermediate notions and points out to those elements of the theory which determine the essence of the complexity of the notion being introduced.

The paper includes the definition of the σ-field of Lebesgue measurable sets, the definition of the Lebesgue measure and the basic set of the theorems describing its properties.

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The terminology and notation used in this paper are introduced in the following articles: [21], [24], [20], [25], [14], [12], [13], [2], [19], [3], [17], [6], [8], [10], [9], [5], [7], [18], [11], [23], [1], [4], [16], [22], and [15].

The following propositions are true:

(1) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $F(n) = 0_{\overline{\mathbb{R}}}$ holds $\sum F = 0_{\overline{\mathbb{R}}}$.

(2) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative and for every natural number n holds $F(n) \leq (\mathrm{Ser}\, F)(n)$.

(3) Let F, G, H be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative and H is non-negative. Suppose that for every natural number n holds $F(n) = G(n) + H(n)$. Let n be a natural number. Then $(\mathrm{Ser}\, F)(n) = (\mathrm{Ser}\, G)(n) + (\mathrm{Ser}\, H)(n)$.

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(4) Let F, G, H be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose that for every natural number n holds $F(n) = G(n) + H(n)$. If G is non-negative and H is non-negative, then $\sum F \leq \sum G + \sum H$.

(5) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and for every natural number n holds $F(n) = G(n)$. Let n be a natural number. Then $(\mathrm{Ser}\, F)(n) = (\mathrm{Ser}\, G)(n)$.

(6) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and for every natural number n holds $F(n) \leq G(n)$. Let n be a natural number. Then $(\mathrm{Ser}\, F)(n) \leq \sum G$.

(7) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative and for every natural number n holds $(\mathrm{Ser}\, F)(n) \leq \sum F$.

Let S be a non empty subset of $\mathbb{N}$, let H be a function from S into $\mathbb{N}$, and let n be an element of S. Then $H(n)$ is a natural number.

Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$, let S be a non empty subset of $\mathbb{N}$, and let H be a function from S into $\mathbb{N}$. The functor $\mathrm{On}(G, H)$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined as follows:

(Def.1) For every element n of $\mathbb{N}$ holds if $n \in S$, then $(\mathrm{On}(G, H))(n) = G(H(n))$ and if $n \notin S$, then $(\mathrm{On}(G, H))(n) = 0_{\overline{\mathbb{R}}}$.

Next we state several propositions:

(8) Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. Then $\mathrm{On}(G, H)$ is non-negative.

(9) Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative. Let n, k be natural numbers. If $n \leq k$, then $(\mathrm{Ser}\, F)(n) \leq (\mathrm{Ser}\, F)(k)$.

(10) Let k be a natural number and let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative. Suppose that for every natural number n such that $n \neq k$ holds $F(n) = 0_{\overline{\mathbb{R}}}$. Then

(i) for every natural number n such that $n < k$ holds $(\mathrm{Ser}\, F)(n) = 0_{\overline{\mathbb{R}}}$, and

(ii) for every natural number n such that $k \leq n$ holds $(\mathrm{Ser}\, F)(n) = F(k)$.

(11) Let G be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. If H is one-to-one and $\mathrm{rng}\, H = \mathbb{N}$, then $\sum \mathrm{On}(G, H) \leq \sum G$.

(12) Let F, G be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose F is non-negative and G is non-negative. Let S be a non empty subset of $\mathbb{N}$ and let H be a function from S into $\mathbb{N}$. Suppose H is one-to-one and $\mathrm{rng}\, H = \mathbb{N}$. Suppose that for every natural number k holds if $k \in S$, then $F(k) = G(H(k))$ and if $k \notin S$, then $F(k) = 0_{\overline{\mathbb{R}}}$. Then $\sum F \leq \sum G$.

Let A be a subset of $\mathbb{R}$. A function from $\mathbb{N}$ into $2^{\mathbb{R}}$ is said to be an interval covering of A if:

(Def.2) $A \subseteq \bigcup \mathrm{rng}\, \mathrm{it}$ and for every natural number n holds $\mathrm{it}(n)$ is an interval.

Let A be a subset of $\mathbb{R}$, let F be an interval covering of A, and let n be a natural number. Then $F(n)$ is an interval.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$. A function from $\mathbb{N}$ into $(2^{\mathbb{R}})^{\mathbb{N}}$ is said to be an interval covering of F if:

(Def.3) For every natural number n holds $\mathrm{it}(n)$ is an interval covering of $F(n)$.

Let A be a subset of $\mathbb{R}$ and let F be an interval covering of A. The functor $(F)\,\mathrm{vol}$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.4) For every natural number n holds $(F)\,\mathrm{vol}(n) = \mathrm{vol}(F(n))$.

The following proposition is true

(13) For every subset A of $\mathbb{R}$ and for every interval covering F of A holds $(F)\,\mathrm{vol}$ is non-negative.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let H be an interval covering of F, and let n be a natural number. Then $H(n)$ is an interval covering of $F(n)$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let G be an interval covering of F. The functor $(G)\,\mathrm{vol}$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}^{\mathbb{N}}$ and is defined by:

(Def.5) For every natural number n holds $(G)\,\mathrm{vol}(n) = (G(n))\,\mathrm{vol}$.

Let A be a subset of $\mathbb{R}$ and let F be an interval covering of A. The functor $\mathrm{vol}(F)$ yields a *Real number* and is defined as follows:

(Def.6) $\mathrm{vol}(F) = \sum((F)\,\mathrm{vol})$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ and let G be an interval covering of F. The functor $\mathrm{vol}(G)$ yielding a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by:

(Def.7) For every natural number n holds $(\mathrm{vol}(G))(n) = \mathrm{vol}(G(n))$.

One can prove the following proposition

(14) Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, and let G be an interval covering of F, and let n be a natural number. Then $0_{\overline{\mathbb{R}}} \leq (\mathrm{vol}(G))(n)$.

Let A be a subset of $\mathbb{R}$. The functor $\mathrm{Svc}(A)$ yielding a non empty subset of $\overline{\mathbb{R}}$ is defined by:

(Def.8) For every *Real number* x holds $x \in \mathrm{Svc}(A)$ iff there exists an interval covering F of A such that $x = \mathrm{vol}(F)$.

Let A be an element of $2^{\mathbb{R}}$. The functor $\mathbb{C}^{A}$ yields an element of $\overline{\mathbb{R}}$ and is defined as follows:

(Def.9) $\mathbb{C}^{A} = \inf \mathrm{Svc}(A)$.

The function OSMeas from $2^{\mathbb{R}}$ into $\overline{\mathbb{R}}$ is defined by:

(Def.10) For every subset A of $\mathbb{R}$ holds $(\mathrm{OSMeas})(A) = \inf \mathrm{Svc}(A)$.

Let F be a function from $\mathbb{N}$ into $\mathbb{N}$ and let n be a natural number. Then $F(n)$ is a natural number.

Let x, y be *Real numbers*. Then $\{x, y\}$ is a subset of $\overline{\mathbb{R}}$.

Let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. The functor $\mathrm{pr1}(H)$ yielding a function from $\mathbb{N}$ into $\mathbb{N}$ is defined by:

(Def.11) For every element n of $\mathbb{N}$ there exists an element s of $\mathbb{N}$ such that $H(n) = \langle \mathrm{pr1}(H)(n), s \rangle$.

Let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. The functor $\mathrm{pr2}(H)$ yielding a function from $\mathbb{N}$ into $\mathbb{N}$ is defined by:

(Def.12) For every element n of $\mathbb{N}$ holds $H(n) = \langle \mathrm{pr1}(H)(n), \mathrm{pr2}(H)(n) \rangle$.

Let F be a function from $\mathbb{N}$ into $2^{\mathbb{R}}$, let G be an interval covering of F, and let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Let us assume that H is one-to-one and $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. The functor $\mathrm{On}(G, H)$ yields an interval covering of $\bigcup \mathrm{rng}\, F$ and is defined by:

(Def.13) For every element n of $\mathbb{N}$ holds $(\mathrm{On}(G, H))(n) = G(\mathrm{pr1}(H)(n))(\mathrm{pr2}(H)(n))$.

Next we state three propositions:

(15) Let H be a function from $\mathbb{N}$ into $[:\mathbb{N}, \mathbb{N}:]$. Suppose H is one-to-one and $\mathrm{rng}\, H = [:\mathbb{N}, \mathbb{N}:]$. Let k be a natural number. Then there exists a natural number m such that for every function F from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering G of F holds $(\mathrm{Ser}((\mathrm{On}(G, H))\, \mathrm{vol}))(k) \leq (\mathrm{Ser}\, \mathrm{vol}(G))(m)$.

(16) For every function F from $\mathbb{N}$ into $2^{\mathbb{R}}$ and for every interval covering G of F holds $\inf \mathrm{Svc}(\bigcup \mathrm{rng}\, F) \leq \sum \mathrm{vol}(G)$.

(17)[1] OSMeas is a Caratheodor's measure on $\mathbb{R}$.

OSMeas is a Caratheodor's measure on $\mathbb{R}$.

The functor L_μ-σFIELD is a σ-field of subsets of $\mathbb{R}$ and is defined by:

(Def.14) L_μ-$\sigma\mathrm{FIELD} = \sigma$-Field(OSMeas).

The σ-measure L_μ on L_μ-σFIELD is defined by:

(Def.15) $L_\mu = \sigma$-Meas(OSMeas).

The following propositions are true:

(18) L_μ is complete on L_μ-σFIELD.

(19) L_μ is a measure on L_μ-σFIELD.

(20) $\emptyset \in L_\mu$-σFIELD and $\mathbb{R} \in L_\mu$-σFIELD.

(21) For every set A such that $A \in L_\mu$-σFIELD holds $\mathbb{R} \setminus A \in L_\mu$-$\sigma$FIELD.

(22) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \cup B \in L_\mu$-σFIELD.

(23) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \cap B \in L_\mu$-σFIELD.

(24) For all sets A, B such that $A \in L_\mu$-σFIELD and $B \in L_\mu$-σFIELD holds $A \setminus B \in L_\mu$-σFIELD.

(25) For every family T of measurable sets of L_μ-σFIELD holds $\bigcap T \in L_\mu$-σFIELD and $\bigcup T \in L_\mu$-σFIELD.

(27)[2] For every denumerable family M of subsets of $\mathbb{R}$ such that $M \subseteq L_\mu$-σFIELD holds $\bigcap M \in L_\mu$-σFIELD.

(28) For all elements A, B of L_μ-σFIELD such that $A \cap B = \emptyset$ holds $L_\mu(A \cup B) = L_\mu(A) + L_\mu(B)$.

(29) For all elements A, B of L_μ-σFIELD such that $A \subseteq B$ holds $L_\mu(A) \leq L_\mu(B)$.

[1]**Editiorial footnote**: The repetition below is caused by the fact that the first sentence is the translation of a Mizar theorem, and the second one – of a Mizar redefinition.

[2]The proposition (26) has been removed.

(30) For all elements A, B of L_μ-σFIELD such that $A \subseteq B$ and $L_\mu(A) < +\infty$ holds $L_\mu(B \setminus A) = L_\mu(B) - L_\mu(A)$.

(31) For all elements A, B of L_μ-σFIELD holds $L_\mu(A \cup B) \leq L_\mu(A) + L_\mu(B)$.

(32) L_μ is non-negative and $L_\mu(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for every sequence F of separated subsets of L_μ-σFIELD holds $\sum(L_\mu \cdot F) = L_\mu(\bigcup \operatorname{rng} F)$.

(33) For every function F from $\mathbb{N}$ into L_μ-σFIELD such that for every element n of $\mathbb{N}$ holds $F(n) \subseteq F(n+1)$ holds $L_\mu(\bigcup \operatorname{rng} F) = \sup \operatorname{rng}(L_\mu \cdot F)$.

(34) Let F be a function from $\mathbb{N}$ into L_μ-σFIELD. Suppose for every element n of $\mathbb{N}$ holds $F(n+1) \subseteq F(n)$ and $L_\mu(F(0)) < +\infty$. Then $L_\mu(\bigcap \operatorname{rng} F) = \inf \operatorname{rng}(L_\mu \cdot F)$.

(35) Let T be a family of measurable sets of L_μ-σFIELD. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. L_μ. Then $\bigcup T$ is a set of measure zero w.r.t. L_μ.

(36) Let T be a family of measurable sets of L_μ-σFIELD. Given a set A such that $A \in T$ and A is a set of measure zero w.r.t. L_μ. Then $\bigcap T$ is a set of measure zero w.r.t. L_μ.

(37) Let T be a family of measurable sets of L_μ-σFIELD. Suppose that for every set A such that $A \in T$ holds A is a set of measure zero w.r.t. L_μ. Then $\bigcap T$ is a set of measure zero w.r.t. L_μ.

(38) Let A be an element of L_μ-σFIELD and let B be a set of measure zero w.r.t. L_μ. If $A \subseteq B$, then A is a set of measure zero w.r.t. L_μ.

(39) Let A, B be sets of measure zero w.r.t. L_μ. Then
(i) $A \cup B$ is a set of measure zero w.r.t. L_μ,
(ii) $A \cap B$ is a set of measure zero w.r.t. L_μ, and
(iii) $A \setminus B$ is a set of measure zero w.r.t. L_μ.

(40) Let A be an element of L_μ-σFIELD and let B be a set of measure zero w.r.t. L_μ. Then $L_\mu(A \cup B) = L_\mu(A)$ and $L_\mu(A \cap B) = 0_{\overline{\mathbb{R}}}$ and $L_\mu(A \setminus B) = L_\mu(A)$.

(41) (i) $\emptyset$ is measurable w.r.t. L_μ,
(ii) $\mathbb{R}$ is measurable w.r.t. L_μ, and
(iii) for all sets A, B such that A is measurable w.r.t. L_μ and B is measurable w.r.t. L_μ holds $\mathbb{R} \setminus A$ is measurable w.r.t. L_μ and $A \cup B$ is measurable w.r.t. L_μ and $A \cap B$ is measurable w.r.t. L_μ.

(42) Let T be a denumerable family of subsets of $\mathbb{R}$. Suppose that for every set A such that $A \in T$ holds A is measurable w.r.t. L_μ. Then $\bigcup T$ is measurable w.r.t. L_μ and $\bigcap T$ is measurable w.r.t. L_μ.

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