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Some Properties of Restrictions of Finite Sequences

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Summary. The aim of the paper is to define some basic notions of restrictions of finite sequences.

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The notation and terminology used in this paper are introduced in the following papers: [12], [15], [11], [14], [9], [2], [16], [5], [6], [3], [13], [1], [4], [7], [10], and [8].

In this paper i, j, k, k_1, k_2, n are natural numbers.

The following propositions are true:

(1) If $i \leq n$, then $(n-i)+1$ is a natural number.

(2) If $i \in \mathrm{Seg}\, n$, then $(n-i)+1 \in \mathrm{Seg}\, n$.

(3) For every function f and for arbitrary x, y such that $f^{-1}\{y\} = \{x\}$ holds $x \in \mathrm{dom}\, f$ and $y \in \mathrm{rng}\, f$ and $f(x) = y$.

(4) For every function f holds f is one-to-one iff for arbitrary x such that $x \in \mathrm{dom}\, f$ holds $f^{-1}\{f(x)\} = \{x\}$.

(5) For every function f and for arbitrary y_1, y_2 such that f is one-to-one and $y_1 \in \mathrm{rng}\, f$ and $y_2 \in \mathrm{rng}\, f$ and $f^{-1}\{y_1\} = f^{-1}\{y_2\}$ holds $y_1 = y_2$.

Let x be arbitrary. Note that $\langle x \rangle$ is non empty.

Let us note that every set which is empty is also trivial.

Let x be arbitrary. Note that $\langle x \rangle$ is trivial. Let y be arbitrary. Observe that $\langle x, y \rangle$ is non trivial.

One can verify that there exists a finite sequence which is one-to-one and non empty.

Next we state three propositions:

(6) For every non empty finite sequence f holds $1 \in \mathrm{dom}\, f$ and $\mathrm{len}\, f \in \mathrm{dom}\, f$.

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(7) For every non empty finite sequence f there exists i such that $i+1 = \operatorname{len} f$.

(8) For arbitrary x and for every finite sequence f holds $\operatorname{len}(\langle x \rangle \frown f) = 1 + \operatorname{len} f$.

The scheme *domSeqLambda* concerns a natural number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a finite sequence p such that $\operatorname{len} p = \mathcal{A}$ and for every k such that $k \in \operatorname{dom} p$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

We now state four propositions:

(9) For every set X such that $X \subseteq \operatorname{Seg} n$ and $1 \leq i$ and $i \leq j$ and $j \leq \operatorname{len} \operatorname{Sgm} X$ and $k_1 = (\operatorname{Sgm} X)(i)$ and $k_2 = (\operatorname{Sgm} X)(j)$ holds $k_1 \leq k_2$.

(10) For every finite sequence f and for arbitrary p, q such that $p \in \operatorname{rng} f$ and $q \in \operatorname{rng} f$ and $p \leftarrow f = q \leftarrow f$ holds $p = q$.

(11) For all finite sequences f, g such that $n+1 \in \operatorname{dom} f$ and $g = f \restriction \operatorname{Seg} n$ holds $f \restriction \operatorname{Seg}(n+1) = g \frown \langle f(n+1) \rangle$.

(12) For every one-to-one finite sequence f such that $i \in \operatorname{dom} f$ holds $f(i) \leftarrow f = i$.

We adopt the following rules: D is a non empty set, p, q are elements of D, and f, g are finite sequences of elements of D.

Let us consider D. One can verify that there exists a finite sequence of elements of D which is one-to-one and non empty.

One can prove the following propositions:

(13) If $\operatorname{dom} f = \operatorname{dom} g$ and for every i such that $i \in \operatorname{dom} f$ holds $\pi_i f = \pi_i g$, then $f = g$.

(14) If $\operatorname{len} f = \operatorname{len} g$ and for every k such that $1 \leq k$ and $k \leq \operatorname{len} f$ holds $\pi_k f = \pi_k g$, then $f = g$.

(15) If $\operatorname{len} f = 1$, then $f = \langle \pi_1 f \rangle$.

(16) $\pi_1(\langle p \rangle \frown f) = p$.

(18)[1] $\operatorname{len}(f \restriction i) \leq \operatorname{len} f$.

(19) $\operatorname{len}(f \restriction i) \leq i$.

(20) $\operatorname{dom}(f \restriction i) \subseteq \operatorname{dom} f$.

(21) $\operatorname{rng}(f \restriction i) \subseteq \operatorname{rng} f$.

Let us consider D, f. Observe that $f \restriction 0$ is empty.

Next we state three propositions:

(22) If $\operatorname{len} f \leq i$, then $f \restriction i = f$.

(23) If f is non empty, then $f \restriction 1 = \langle \pi_1 f \rangle$.

(24) If $i+1 = \operatorname{len} f$, then $f = (f \restriction i) \frown \langle \pi_{\operatorname{len} f} f \rangle$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. One can verify that $f \restriction i$ is one-to-one.

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[1]The proposition (17) has been removed.

The following propositions are true:

(25) If $i \leq \operatorname{len} f$, then $(f \frown g) \upharpoonright i = f \upharpoonright i$.

(26) $(f \frown g) \upharpoonright \operatorname{len} f = f$.

(27) If $p \in \operatorname{rng} f$, then $(f \leftarrow p) \frown \langle p \rangle = f \upharpoonright p \leftrightarrow f$.

(28) $\operatorname{len}(f_{\lfloor i}) \leq \operatorname{len} f$.

(29) If $i \in \operatorname{dom}(f_{\lfloor n})$, then $n + i \in \operatorname{dom} f$.

(30) If $i \in \operatorname{dom}(f_{\lfloor n})$, then $\pi_i f_{\lfloor n} = \pi_{n+i} f$.

(31) $f_{\lfloor 0} = f$.

(32) If f is non empty, then $f = \langle \pi_1 f \rangle \frown (f_{\lfloor 1})$.

(33) If $i + 1 = \operatorname{len} f$, then $f_{\lfloor i} = \langle \pi_{\operatorname{len} f} f \rangle$.

(34) If $j + 1 = i$ and $i \in \operatorname{dom} f$, then $\langle \pi_i f \rangle \frown (f_{\lfloor i}) = f_{\lfloor j}$.

(35) If $\operatorname{len} f \leq i$, then $f_{\lfloor i}$ is empty.

(36) $\operatorname{rng}(f_{\lfloor n}) \subseteq \operatorname{rng} f$.

Let us consider i, D and let f be an one-to-one finite sequence of elements of D. Note that $f_{\lfloor i}$ is one-to-one.

The following propositions are true:

(37) If f is one-to-one, then $\operatorname{rng}(f \upharpoonright n)$ misses $\operatorname{rng}(f_{\lfloor n})$.

(38) If $p \in \operatorname{rng} f$, then $f \rightarrow p = f_{\lfloor p \leftrightarrow f}$.

(39) $(f \frown g)_{\lfloor \operatorname{len} f + i} = g_{\lfloor i}$.

(40) $(f \frown g)_{\lfloor \operatorname{len} f} = g$.

(41) If $p \in \operatorname{rng} f$, then $\pi_{p \leftrightarrow f} f = p$.

(42) If $i \in \operatorname{dom} f$, then $(\pi_i f) \leftrightarrow f \leq i$.

(43) If $p \in \operatorname{rng}(f \upharpoonright i)$, then $p \leftrightarrow (f \upharpoonright i) = p \leftrightarrow f$.

(44) If $i \in \operatorname{dom} f$ and f is one-to-one, then $(\pi_i f) \leftrightarrow f = i$.

Let us consider D, f and let p be arbitrary. The functor $f -: p$ yielding a finite sequence of elements of D is defined as follows:

(Def.1) $f -: p = f \upharpoonright p \leftrightarrow f$.

One can prove the following propositions:

(45) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f -: p) = p \leftrightarrow f$.

(46) If $p \in \operatorname{rng} f$ and $i \in \operatorname{Seg}(p \leftrightarrow f)$, then $\pi_i(f -: p) = \pi_i f$.

(47) If $p \in \operatorname{rng} f$, then $\pi_1(f -: p) = \pi_1 f$.

(48) If $p \in \operatorname{rng} f$, then $\pi_{p \leftrightarrow f}(f -: p) = p$.

(49) If $q \in \operatorname{rng} f$ and $p \in \operatorname{rng} f$ and $q \leftrightarrow f \leq p \leftrightarrow f$, then $q \in \operatorname{rng}(f -: p)$.

(50) If $p \in \operatorname{rng} f$, then $f -: p$ is non empty.

(51) $\operatorname{rng}(f -: p) \subseteq \operatorname{rng} f$.

Let us consider D, p and let f be an one-to-one finite sequence of elements of D. Observe that $f -: p$ is one-to-one.

Let us consider D, f, p. The functor $f :- p$ yielding a finite sequence of elements of D is defined by:

(Def.2) $f :- p = \langle p \rangle \frown (f_{\lfloor p \leftrightarrow f})$.

We now state three propositions:

(52) If $p \in \operatorname{rng} f$, then there exists i such that $i+1 = p \looparrowleft f$ and $f :- p = f_{\lfloor i}$.

(53) If $p \in \operatorname{rng} f$, then $\operatorname{len}(f :- p) = (\operatorname{len} f - p \looparrowleft f) + 1$.

(54) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $j + p \looparrowleft f \in \operatorname{dom} f$.

Let us consider D, p, f. One can check that $f :- p$ is non empty.

Next we state several propositions:

(55) If $p \in \operatorname{rng} f$ and $j + 1 \in \operatorname{dom}(f :- p)$, then $\pi_{j+1}(f :- p) = \pi_{j+p \looparrowleft f} f$.

(56) $\pi_1(f :- p) = p$.

(57) If $p \in \operatorname{rng} f$, then $\pi_{\operatorname{len}(f :- p)}(f :- p) = \pi_{\operatorname{len} f} f$.

(58) If $p \in \operatorname{rng} f$, then $\operatorname{rng}(f :- p) \subseteq \operatorname{rng} f$.

(59) If $p \in \operatorname{rng} f$ and f is one-to-one, then $f :- p$ is one-to-one.

Let f be a finite sequence. The functor $\operatorname{Rev}(f)$ yielding a finite sequence is defined by:

(Def.3) $\operatorname{len} \operatorname{Rev}(f) = \operatorname{len} f$ and for every i such that $i \in \operatorname{dom} \operatorname{Rev}(f)$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

One can prove the following propositions:

(60) For every finite sequence f holds $\operatorname{dom} f = \operatorname{dom} \operatorname{Rev}(f)$ and $\operatorname{rng} f = \operatorname{rng} \operatorname{Rev}(f)$.

(61) For every finite sequence f such that $i \in \operatorname{dom} f$ holds $(\operatorname{Rev}(f))(i) = f((\operatorname{len} f - i) + 1)$.

(62) For every finite sequence f and for all natural numbers i, j such that $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$ holds $j \in \operatorname{dom} \operatorname{Rev}(f)$.

Let f be an empty finite sequence. Observe that $\operatorname{Rev}(f)$ is empty.

Next we state three propositions:

(63) For arbitrary x holds $\operatorname{Rev}(\langle x \rangle) = \langle x \rangle$.

(64) For arbitrary x_1, x_2 holds $\operatorname{Rev}(\langle x_1, x_2 \rangle) = \langle x_2, x_1 \rangle$.

(65) For every non empty finite sequence f holds $f(1) = (\operatorname{Rev}(f))(\operatorname{len} f)$ and $f(\operatorname{len} f) = (\operatorname{Rev}(f))(1)$.

Let f be an one-to-one finite sequence. Note that $\operatorname{Rev}(f)$ is one-to-one.

The following two propositions are true:

(66) For every finite sequence f and for arbitrary x holds $\operatorname{Rev}(f \frown \langle x \rangle) = \langle x \rangle \frown \operatorname{Rev}(f)$.

(67) For all finite sequences f, g holds $\operatorname{Rev}(f \frown g) = (\operatorname{Rev}(g)) \frown \operatorname{Rev}(f)$.

Let us consider D, f. Then $\operatorname{Rev}(f)$ is a finite sequence of elements of D.

We now state two propositions:

(68) If f is non empty, then $\pi_1 f = \pi_{\operatorname{len} f} \operatorname{Rev}(f)$ and $\pi_{\operatorname{len} f} f = \pi_1 \operatorname{Rev}(f)$.

(69) If $i \in \operatorname{dom} f$ and $i + j = \operatorname{len} f + 1$, then $\pi_i f = \pi_j \operatorname{Rev}(f)$.

Let us consider D, f, p, n. The functor $\operatorname{Ins}(f, n, p)$ yielding a finite sequence of elements of D is defined as follows:

(Def.4) $\operatorname{Ins}(f, n, p) = (f \restriction n) \frown \langle p \rangle \frown (f_{\lfloor n})$.

One can prove the following propositions:

(70) $\mathrm{Ins}(f, 0, p) = \langle p \rangle \frown f$.

(71) If $\mathrm{len}\, f \leq n$, then $\mathrm{Ins}(f, n, p) = f \frown \langle p \rangle$.

(72) $\mathrm{len}\, \mathrm{Ins}(f, n, p) = \mathrm{len}\, f + 1$.

(73) $\mathrm{rng}\, \mathrm{Ins}(f, n, p) = \{p\} \cup \mathrm{rng}\, f$.

Let us consider D, f, n, p. Observe that $\mathrm{Ins}(f, n, p)$ is non empty.

The following propositions are true:

(74) $p \in \mathrm{rng}\, \mathrm{Ins}(f, n, p)$.

(75) If $i \in \mathrm{dom}(f \restriction n)$, then $\pi_i \mathrm{Ins}(f, n, p) = \pi_i f$.

(76) If $n \leq \mathrm{len}\, f$, then $\pi_{n+1} \mathrm{Ins}(f, n, p) = p$.

(77) If $n + 1 \leq i$ and $i \leq \mathrm{len}\, f$, then $\pi_{i+1} \mathrm{Ins}(f, n, p) = \pi_i f$.

(78) If $1 \leq n$ and f is non empty, then $\pi_1 \mathrm{Ins}(f, n, p) = \pi_1 f$.

(79) If f is one-to-one and $p \notin \mathrm{rng}\, f$, then $\mathrm{Ins}(f, n, p)$ is one-to-one.

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