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Logical Equivalence of Formulae [1]

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The notation and terminology used here are introduced in the following papers: [11], [9], [10], [8], [1], [12], [4], [2], [7], [5], [3], and [6].

For simplicity we adopt the following rules: p, q, r, s, p_1, q_1 are elements of CQC-WFF, X, Y, Z, X_1, X_2 are subsets of CQC-WFF, h is a formula, and x, y are bound variables.

One can prove the following four propositions:

(1) If $p \in X$, then $X \vdash p$.

(2) If $X \subseteq \mathrm{Cn}\, Y$, then $\mathrm{Cn}\, X \subseteq \mathrm{Cn}\, Y$.

(3) If $X \vdash p$ and $\{p\} \vdash q$, then $X \vdash q$.

(4) If $X \vdash p$ and $X \subseteq Y$, then $Y \vdash p$.

Let p, q be elements of CQC-WFF. The predicate $p \vdash q$ is defined by:

(Def.1) $\{p\} \vdash q$.

We now state two propositions:

(5) $p \vdash p$.

(6) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

Let X, Y be subsets of CQC-WFF. The predicate $X \vdash Y$ is defined as follows:

(Def.2) For every element p of CQC-WFF such that $p \in Y$ holds $X \vdash p$.

We now state several propositions:

(7) $X \vdash Y$ iff $Y \subseteq \mathrm{Cn}\, X$.

(8) $X \vdash X$.

(9) If $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.

(10) $X \vdash \{p\}$ iff $X \vdash p$.

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(11) $\{p\} \vdash \{q\}$ iff $p \vdash q$.

(12) If $X \subseteq Y$, then $Y \vdash X$.

(13) $X \vdash \mathrm{Taut}$.

(14) $\emptyset_{\mathrm{CQC}} \vdash \mathrm{Taut}$.

Let X be a subset of CQC-WFF. The predicate $\vdash X$ is defined by:

(Def.3) For every element p of CQC-WFF such that $p \in X$ holds $\vdash p$.

We now state three propositions:

(15) $\vdash X$ iff $\emptyset_{\mathrm{CQC}} \vdash X$.

(16) $\vdash \mathrm{Taut}$.

(17) $\vdash X$ iff $X \subseteq \mathrm{Taut}$.

Let us consider X, Y. The predicate $X \mapsto\!\dashv Y$ is defined by:

(Def.4) For every p holds $X \vdash p$ iff $Y \vdash p$.

Let us observe that this predicate is reflexive and symmetric.

The following propositions are true:

(18) $X \mapsto\!\dashv Y$ iff $X \vdash Y$ and $Y \vdash X$.

(19) If $X \mapsto\!\dashv Y$ and $Y \mapsto\!\dashv Z$, then $X \mapsto\!\dashv Z$.

(20) $X \mapsto\!\dashv Y$ iff $\mathrm{Cn}\, X = \mathrm{Cn}\, Y$.

(21) $\mathrm{Cn}\, X \cup \mathrm{Cn}\, Y \subseteq \mathrm{Cn}(X \cup Y)$.

(22) $\mathrm{Cn}(X \cup Y) = \mathrm{Cn}(\mathrm{Cn}\, X \cup \mathrm{Cn}\, Y)$.

(23) $X \mapsto\!\dashv \mathrm{Cn}\, X$.

(24) $X \cup Y \mapsto\!\dashv \mathrm{Cn}\, X \cup \mathrm{Cn}\, Y$.

(25) If $X_1 \mapsto\!\dashv X_2$, then $X_1 \cup Y \mapsto\!\dashv X_2 \cup Y$.

(26) If $X_1 \mapsto\!\dashv X_2$ and $X_1 \cup Y \vdash Z$, then $X_2 \cup Y \vdash Z$.

(27) If $X_1 \mapsto\!\dashv X_2$ and $Y \vdash X_1$, then $Y \vdash X_2$.

Let p, q be elements of CQC-WFF. The predicate $p \mapsto\!\dashv q$ is defined by:

(Def.5) $p \vdash q$ and $q \vdash p$.

Let us observe that the predicate defined above is reflexive and symmetric.

We now state a number of propositions:

(28) If $p \mapsto\!\dashv q$ and $q \mapsto\!\dashv r$, then $p \mapsto\!\dashv r$.

(29) $p \mapsto\!\dashv q$ iff $\{p\} \mapsto\!\dashv \{q\}$.

(30) If $p \mapsto\!\dashv q$ and $X \vdash p$, then $X \vdash q$.

(31) $\{p, q\} \mapsto\!\dashv \{p \wedge q\}$.

(32) $p \wedge q \mapsto\!\dashv q \wedge p$.

(33) $X \vdash p \wedge q$ iff $X \vdash p$ and $X \vdash q$.

(34) If $p \mapsto\!\dashv q$ and $r \mapsto\!\dashv s$, then $p \wedge r \mapsto\!\dashv q \wedge s$.

(35) $X \vdash \forall_x p$ iff $X \vdash p$.

(36) $\forall_x p \mapsto\!\dashv p$.

(37) If $p \mapsto\!\dashv q$, then $\forall_x p \mapsto\!\dashv \forall_y q$.

Let p, q be elements of CQC-WFF. We say that p is an universal closure of q if and only if the conditions (Def.6) are satisfied.

(Def.6) (i) p is closed, and

(ii) there exists a natural number n such that $1 \leq n$ and there exists a finite sequence L such that $\operatorname{len} L = n$ and $L(1) = q$ and $L(n) = p$ and for every natural number k such that $1 \leq k$ and $k < n$ there exists a bound variable x and there exists an element r of CQC-WFF such that $r = L(k)$ and $L(k+1) = \forall_x r$.

One can prove the following propositions:

(38) If p is an universal closure of q, then $p \mathrel{\vdash\!\dashv} q$.

(39) If $\vdash p \Rightarrow q$, then $p \vdash q$.

(40) If $X \vdash p \Rightarrow q$, then $X \cup \{p\} \vdash q$.

(41) If p is closed and $p \vdash q$, then $\vdash p \Rightarrow q$.

(42) If p_1 is an universal closure of p, then $X \cup \{p\} \vdash q$ iff $X \vdash p_1 \Rightarrow q$.

(43) If p is closed and $p \vdash q$, then $\neg q \vdash \neg p$.

(44) If p is closed and $X \cup \{p\} \vdash q$, then $X \cup \{\neg q\} \vdash \neg p$.

(45) If p is closed and $\neg p \vdash \neg q$, then $q \vdash p$.

(46) If p is closed and $X \cup \{\neg p\} \vdash \neg q$, then $X \cup \{q\} \vdash p$.

(47) If p is closed and q is closed, then $p \vdash q$ iff $\neg q \vdash \neg p$.

(48) If p_1 is an universal closure of p and q_1 is an universal closure of q, then $p \vdash q$ iff $\neg q_1 \vdash \neg p_1$.

(49) If p_1 is an universal closure of p and q_1 is an universal closure of q, then $p \mathrel{\vdash\!\dashv} q$ iff $\neg p_1 \mathrel{\vdash\!\dashv} \neg q_1$.

Let p, q be elements of CQC-WFF. The predicate $p \equiv q$ is defined by:

(Def.7) $\vdash p \Leftrightarrow q$.

Let us observe that this predicate is reflexive and symmetric.

One can prove the following propositions:

(50) $p \equiv q$ iff $\vdash p \Rightarrow q$ and $\vdash q \Rightarrow p$.

(51) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(52) If $p \equiv q$, then $p \mathrel{\vdash\!\dashv} q$.

(53) $p \equiv q$ iff $\neg p \equiv \neg q$.

(54) If $p \equiv q$ and $r \equiv s$, then $p \wedge r \equiv q \wedge s$.

(55) If $p \equiv q$ and $r \equiv s$, then $p \Rightarrow r \equiv q \Rightarrow s$.

(56) If $p \equiv q$ and $r \equiv s$, then $p \vee r \equiv q \vee s$.

(57) If $p \equiv q$ and $r \equiv s$, then $p \Leftrightarrow r \equiv q \Leftrightarrow s$.

(58) If $p \equiv q$, then $\forall_x p \equiv \forall_x q$.

(59) If $p \equiv q$, then $\exists_x p \equiv \exists_x q$.

(60) For all sets X, Y, Z such that $Y \cap Z = \emptyset$ holds $(X \setminus Y) \cup Z = (X \cup Z) \setminus Y$.

(61) Let k be a natural number, and let l be a list of variables of the length k, and let a be a free variable, and let x be a bound variable. Then $\mathrm{snb}(l) \subseteq \mathrm{snb}(l[a \longmapsto x])$.

(62) Let k be a natural number, and let l be a list of variables of the length k, and let a be a free variable, and let x be a bound variable. Then $\mathrm{snb}(l[a \longmapsto x]) \subseteq \mathrm{snb}(l) \cup \{x\}$.

(63) For every h holds $\mathrm{snb}(h) \subseteq \mathrm{snb}(h(x))$.

(64) For every h holds $\mathrm{snb}(h(x)) \subseteq \mathrm{snb}(h) \cup \{x\}$.

(65) If $p = h(x)$ and $x \neq y$ and $y \notin \mathrm{snb}(h)$, then $y \notin \mathrm{snb}(p)$.

(66) If $p = h(x)$ and $q = h(y)$ and $x \notin \mathrm{snb}(h)$ and $y \notin \mathrm{snb}(h)$, then $\forall_x p \equiv \forall_y q$.

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