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The Cantor Set [1]

Alexander Yu. Shibakov
The Ural University
Ekaterinburg

Andrzej Trybulec
Warsaw University
Białystok

Summary. The aim of the paper is to define some basic notions of the theory of topological spaces like basis and prebasis, and to prove their simple properties. The definition of the Cantor set is given in terms of countable product of $\{0,1\}$ and a collection of its subsets to serve as a prebasis.

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The papers [13], [16], [15], [9], [17], [2], [3], [6], [14], [12], [10], [5], [4], [1], [7], [11], and [8] provide the terminology and notation for this paper.

Let Y be a set and let x be a non empty set. Observe that $Y \longmapsto x$ is non-empty.

Let X be arbitrary and let A be a family of subsets of X. The functor UniCl(A) yields a family of subsets of X and is defined by:

(Def.1) For every subset x of X holds $x \in \mathrm{UniCl}(A)$ iff there exists a family Y of subsets of X such that $Y \subseteq A$ and $x = \bigcup Y$.

Let X be a topological structure. A family of subsets of the carrier of X is called a basis of X if:

(Def.2) It $\subseteq$ the topology of X and the topology of $X \subseteq$ UniCl(it).

We now state three propositions:

(1) For arbitrary X and for every family A of subsets of X holds $A \subseteq \mathrm{UniCl}(A)$.

(2) For every topological structure S holds the topology of S is a basis of S.

(3) For every topological structure S holds the topology of S is open.

Let M be arbitrary and let B be a family of subsets of M. The functor Intersect(B) yielding a subset of M is defined by:

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(Def.3) (i) Intersect$(B) = \bigcap B$ if $B \neq \emptyset$,

(ii) Intersect$(B) = M$, otherwise.

Let X be arbitrary and let A be a family of subsets of X. The functor FinMeetCl(A) yielding a family of subsets of X is defined by the condition (Def.4).

(Def.4) Let x be a subset of X. Then $x \in$ FinMeetCl(A) if and only if there exists a family Y of subsets of X such that $Y \subseteq A$ and Y is finite and $x =$ Intersect(Y).

One can prove the following proposition

(4) For arbitrary X and for every family A of subsets of X holds $A \subseteq$ FinMeetCl(A).

Let T be a topological space. Note that the topology of T is non empty.

The following propositions are true:

(5) For every topological space T holds the topology of $T =$ FinMeetCl(the topology of T).

(6) For every topological space T holds the topology of $T =$ UniCl(the topology of T).

(7) For every topological space T holds the topology of $T =$ UniCl(FinMeetCl(the topology of T)).

(8) For arbitrary X and for every family A of subsets of X holds $X \in$ FinMeetCl(A).

(9) For arbitrary X and for all families A, B of subsets of X such that $A \subseteq B$ holds UniCl$(A) \subseteq$ UniCl(B).

(10) Let X be arbitrary, and let R be a family of subsets of X, and let x be arbitrary. Suppose $x \in X$. Then $x \in$ Intersect(R) if and only if for arbitrary Y such that $Y \in R$ holds $x \in Y$.

(11) For arbitrary X and for all families H, J of subsets of X such that $H \subseteq J$ holds Intersect$(J) \subseteq$ Intersect(H).

(12) Let X be arbitrary, and let R be a non empty family of subsets of 2^X, and let F be a family of subsets of X. If $F = \{$Intersect$(x) : x$ ranges over elements of $R\}$, then Intersect$(F) =$ Intersect$(\bigcup R)$.

Let X, Y be arbitrary, let A be a family of subsets of X, let F be a function from Y into 2^A, and let x be arbitrary. Then $F(x)$ is a family of subsets of X.

We now state four propositions:

(13) For arbitrary X and for every family A of subsets of X holds FinMeetCl$(A) =$ FinMeetCl(FinMeetCl(A)).

(14) Let X be arbitrary, and let A be a family of subsets of X, and let a, b be arbitrary. If $a \in$ FinMeetCl(A) and $b \in$ FinMeetCl(A), then $a \cap b \in$ FinMeetCl(A).

(15) Let X be arbitrary, and let A be a family of subsets of X, and let a, b be arbitrary. If $a \subseteq$ FinMeetCl(A) and $b \subseteq$ FinMeetCl(A), then $a \Cap b \subseteq$ FinMeetCl(A).

(16) For arbitrary X and for all families A, B of subsets of X such that $A \subseteq B$ holds $\mathrm{FinMeetCl}(A) \subseteq \mathrm{FinMeetCl}(B)$.

Let X be arbitrary and let A be a family of subsets of X. Observe that $\mathrm{FinMeetCl}(A)$ is non empty.

One can prove the following proposition

(17) For every non empty set X and for every family A of subsets of X holds $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(A))\rangle$ is topological space-like.

Let X be a topological structure. A family of subsets of the carrier of X is said to be a prebasis of X if:

(Def.5) It $\subseteq$ the topology of X and there exists a basis F of X such that $F \subseteq \mathrm{FinMeetCl}(\mathrm{it})$.

We now state three propositions:

(18) For every non empty set X holds every family of subsets of X is a basis of $\langle X, \mathrm{UniCl}(Y)\rangle$.

(19) Let T_1, T_2 be strict topological spaces and let P be a prebasis of T_1. Suppose the carrier of T_1 = the carrier of T_2 and P is a prebasis of T_2. Then $T_1 = T_2$.

(20) For every non empty set X holds every family of subsets of X is a prebasis of $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(Y))\rangle$.

The strict topological space the Cantor set is defined by the conditions (Def.6).

(Def.6) (i) The carrier of the Cantor set $= \prod(\mathbb{N} \longmapsto \{0,1\})$, and

(ii) there exists a prebasis P of the Cantor set such that for every subset X of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $X \in P$ iff there exist natural numbers N, n such that for every element F of $\prod(\mathbb{N} \longmapsto \{0,1\})$ holds $F \in X$ iff $F(N) = n$.

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