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Vertex Sequences Induced by Chains [1]

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Summary. In the three preliminary sections to the article we define two operations on finite sequences which seem to be of general interest. The first is the *cut* operation that extracts a contiguous chunk of a finite sequence from a position to a position. The second operation is a glueing catenation that given two finite sequences catenates them with removal of the first element of the second sequence. The main topic of the article is to define an operation which for a given chain in a graph returns the sequence of vertices through which the chain passes. We define the exact conditions when such an operation is uniquely definable. This is done with the help of the so called two-valued alternating finite sequences. We also prove theorems about the existence of simple chains which are subchains of a given chain. In order to do this we define the notion of a finite subsequence of a typed finite sequence.

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The articles [16], [20], [9], [21], [6], [7], [4], [5], [19], [15], [8], [3], [1], [14], [10], [11], [2], [18], [17], [12], and [13] provide the notation and terminology for this paper.

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1. Preliminaries

We adopt the following convention: p, q are finite sequences, X, Y are sets, and i, k, l, m, n, r are natural numbers.

The scheme *FinSegRng* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i) : \mathcal{A} \leq i \ \wedge \ i \leq \mathcal{B} \ \wedge \ \mathcal{P}[i]\}$ is finite

for all values of the parameters.

One can prove the following propositions:

(1) $m+1 \leq k$ and $k \leq n$ iff there exists a natural number i such that $m \leq i$ and $i < n$ and $k = i+1$.

(2) If $q = p \restriction \mathrm{Seg}\, n$, then $\mathrm{len}\, q \leq \mathrm{len}\, p$ and for every i such that $1 \leq i$ and $i \leq \mathrm{len}\, q$ holds $p(i) = q(i)$.

(3) If $X \subseteq \mathrm{Seg}\, k$ and $Y \subseteq \mathrm{dom}\, \mathrm{Sgm}\, X$, then $\mathrm{Sgm}\, X \cdot \mathrm{Sgm}\, Y = \mathrm{Sgm}\, \mathrm{rng}(\mathrm{Sgm}\, X \restriction Y)$.

(4) For all natural numbers m, n holds $\overline{\overline{\{k : m \leq k \ \wedge \ k \leq m+n\}}} = n+1$.

(5) For every l such that $1 \leq l$ and $l \leq n$ holds $(\mathrm{Sgm}\{k_1 : k_1$ ranges over natural numbers, $m+1 \leq k_1 \ \wedge \ k_1 \leq m+n\})(l) = m+l$.

2. The cut operation for finite sequences

Let p be a finite sequence and let m, n be natural numbers. The functor $\langle p(m), \ldots, p(n)\rangle$ yields a finite sequence and is defined by:

(Def.1) (i) $\mathrm{len}\langle p(m), \ldots, p(n)\rangle + m = n+1$ and for every natural number i such that $i < \mathrm{len}\langle p(m), \ldots, p(n)\rangle$ holds $\langle p(m), \ldots, p(n)\rangle(i+1) = p(m+i)$ if $1 \leq m$ and $m \leq n+1$ and $n \leq \mathrm{len}\, p$,

(ii) $\langle p(m), \ldots, p(n)\rangle = \varepsilon$, otherwise.

We now state several propositions:

(6) If $1 \leq m$ and $m \leq \mathrm{len}\, p$, then $\langle p(m), \ldots, p(m)\rangle = \langle p(m)\rangle$.

(7) $\langle p(1), \ldots, p(\mathrm{len}\, p)\rangle = p$.

(8) If $m \leq n$ and $n \leq r$ and $r \leq \mathrm{len}\, p$, then $\langle p(m+1), \ldots, p(n)\rangle \frown \langle p(n+1), \ldots, p(r)\rangle = \langle p(m+1), \ldots, p(r)\rangle$.

(9) If $1 \leq m$ and $m \leq \mathrm{len}\, p$, then $\langle p(1), \ldots, p(m)\rangle \frown \langle p(m+1), \ldots, p(\mathrm{len}\, p)\rangle = p$.

(10) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, p$, then $\langle p(1), \ldots, p(m)\rangle \frown \langle p(m+1), \ldots, p(n)\rangle \frown \langle p(n+1), \ldots, p(\mathrm{len}\, p)\rangle = p$.

(11) $\mathrm{rng}\langle p(m), \ldots, p(n)\rangle \subseteq \mathrm{rng}\, p$.

Let D be a set, let p be a finite sequence of elements of D, and let m, n be natural numbers. Then $\langle p(m), \ldots, p(n)\rangle$ is a finite sequence of elements of D.

Next we state the proposition

(12) If $p \neq \varepsilon$ and $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} p$, then $\langle p(m), \ldots, p(n)\rangle(1) = p(m)$ and $\langle p(m), \ldots, p(n)\rangle(\operatorname{len}\langle p(m), \ldots, p(n)\rangle) = p(n)$.

3. The glueing catenation of finite sequences

Let p, q be finite sequences. The functor $p \frown\!\!\frown q$ yielding a finite sequence is defined as follows:

(Def.2) $p \frown\!\!\frown q = p \mathbin{^\frown} \langle q(2), \ldots, q(\operatorname{len} q)\rangle$.

Next we state several propositions:

(13) If $q \neq \varepsilon$, then $\operatorname{len}(p \frown\!\!\frown q) + 1 = \operatorname{len} p + \operatorname{len} q$.

(14) If $1 \leq k$ and $k \leq \operatorname{len} p$, then $(p \frown\!\!\frown q)(k) = p(k)$.

(15) If $1 \leq k$ and $k < \operatorname{len} q$, then $(p \frown\!\!\frown q)(\operatorname{len} p + k) = q(k+1)$.

(16) If $1 < \operatorname{len} q$, then $(p \frown\!\!\frown q)(\operatorname{len}(p \frown\!\!\frown q)) = q(\operatorname{len} q)$.

(17) $\operatorname{rng}(p \frown\!\!\frown q) \subseteq \operatorname{rng} p \cup \operatorname{rng} q$.

Let D be a set and let p, q be finite sequences of elements of D. Then $p \frown\!\!\frown q$ is a finite sequence of elements of D.

Next we state the proposition

(18) If $p \neq \varepsilon$ and $q \neq \varepsilon$ and $p(\operatorname{len} p) = q(1)$, then $\operatorname{rng}(p \frown\!\!\frown q) = \operatorname{rng} p \cup \operatorname{rng} q$.

4. Two valued alternating finite sequences

A finite sequence is two-valued if:

(Def.3) $\operatorname{card}\operatorname{rng}\mathrm{it} = 2$.

The following proposition is true

(19) p is two-valued iff $\operatorname{len} p > 1$ and there exist arbitrary x, y such that $x \neq y$ and $\operatorname{rng} p = \{x, y\}$.

A finite sequence is alternating if:

(Def.4) For every natural number i such that $1 \leq i$ and $i + 1 \leq \operatorname{len}\mathrm{it}$ holds $\mathrm{it}(i) \neq \mathrm{it}(i+1)$.

One can check that there exists a finite sequence which is two-valued and alternating.

In the sequel a, a_1, a_2 are two-valued alternating finite sequences.

One can prove the following propositions:

(20) If $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$ and $a_1(1) = a_2(1)$, then $a_1 = a_2$.

(21) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every i such that $1 \leq i$ and $i \leq \operatorname{len} a_1$ holds $a_1(i) \neq a_2(i)$.

(22) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every a such that $\operatorname{len} a = \operatorname{len} a_1$ and $\operatorname{rng} a = \operatorname{rng} a_1$ holds $a = a_1$ or $a = a_2$.

(23) If $X \neq Y$ and $n > 1$, then there exists a_1 such that $\operatorname{rng} a_1 = \{X, Y\}$ and $\operatorname{len} a_1 = n$ and $a_1(1) = X$.

5. Finite subsequence of finite sequences

Let us consider X and let f_1 be a finite sequence of elements of X. A finite subsequence is called a FinSubsequence of f_1 if:

(Def.5) It $\subseteq f_1$.

In the sequel s_1 will denote a finite subsequence.

The following propositions are true:

(24) If s_1 is a finite sequence, then $\operatorname{Seq} s_1 = s_1$.

(25) If $\operatorname{rng} p \subseteq \operatorname{dom} s_1$, then $s_1 \cdot p$ is a finite sequence.

(26) Let f be a finite subsequence and let g, h, f_2, f_3, f_4 be finite sequences. If $\operatorname{rng} g \subseteq \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} f$ and $f_2 = f \cdot g$ and $f_3 = f \cdot h$ and $f_4 = f \cdot (g \frown h)$, then $f_4 = f_2 \frown f_3$.

We follow the rules: f_1, f_5, f_6 will be finite sequences of elements of X and f_7, f_8 will be FinSubsequence of f_1.

We now state four propositions:

(27) $\operatorname{dom} f_7 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} f_7 \subseteq \operatorname{rng} f_1$.

(28) f_1 is a FinSubsequence of f_1.

(29) $f_7 \restriction Y$ is a FinSubsequence of f_1.

(30) For every FinSubsequence f_9 of f_5 such that $\operatorname{Seq} f_7 = f_5$ and $\operatorname{Seq} f_9 = f_6$ and $f_8 = f_7 \restriction \operatorname{rng}(\operatorname{Sgm} \operatorname{dom} f_7 \restriction \operatorname{dom} f_9)$ holds $\operatorname{Seq} f_8 = f_6$.

6. Vertex sequences induced by chains

In the sequel G is a graph.

Let us consider G. One can verify that the vertices of G is non empty.

We follow the rules: v, v_1, v_2, v_3, v_4 will denote elements of the vertices of G and e will be arbitrary.

We now state two propositions:

(31) If e joins v_1 with v_2, then e joins v_2 with v_1.

(32) If e joins v_1 with v_2 and e joins v_3 with v_4, then $v_1 = v_3$ and $v_2 = v_4$ or $v_1 = v_4$ and $v_2 = v_3$.

Let us consider G. We see that the chain of G is a finite sequence of elements of the edges of G.

Let us consider G. A path of G is a path-like chain of G.

We follow the rules: v_5, v_6, v_7 will denote finite sequences of elements of the vertices of G and c, c_1, c_2 will denote chains of G.

The following proposition is true

(33) ε is a chain of G.

Let us consider G. One can check that there exists a chain of G which is empty.

Let us consider G, X. The functor (G)-VSet(X) yields a set and is defined as follows:

(Def.6) $(G)\text{-VSet}(X) = \{v : \bigvee_{e\,:\,\text{element of the edges of } G}\ e \in X \ \wedge\ (v = (\text{the source of } G)(e) \ \vee\ v = (\text{the target of } G)(e))\}$.

Let us consider G, v_5 and let c be a finite sequence. We say that v_5 is vertex sequence of c if and only if:

(Def.7) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} c$ holds $c(n)$ joins $\pi_n v_5$ with $\pi_{n+1} v_5$.

One can prove the following four propositions:

(34) If $c \neq \varepsilon$ and v_5 is vertex sequence of c, then (G)-VSet$(\operatorname{rng} c) = \operatorname{rng} v_5$.

(35) $\langle v \rangle$ is vertex sequence of ε.

(36) There exists v_5 which is vertex sequence of c.

(37) Suppose $c \neq \varepsilon$ and v_6 is vertex sequence of c and v_7 is vertex sequence of c and $v_6 \neq v_7$. Then $v_6(1) \neq v_7(1)$ and for every v_5 such that v_5 is vertex sequence of c holds $v_5 = v_6$ or $v_5 = v_7$.

Let us consider G and let c be a finite sequence. We say that c alternates vertices in G if and only if:

(Def.8) $\operatorname{len} c \geq 1$ and $\overline{\overline{(G)\text{-VSet}(\operatorname{rng} c)}} = 2$ and for every n such that $n \in \operatorname{dom} c$ holds (the source of G)$(c(n)) \neq$ (the target of G)$(c(n))$.

One can prove the following propositions:

(38) If c alternates vertices in G and v_5 is vertex sequence of c, then for every k such that $k \in \operatorname{dom} c$ holds $v_5(k) \neq v_5(k+1)$.

(39) Suppose c alternates vertices in G and v_5 is vertex sequence of c. Then $\operatorname{rng} v_5 = \{(\text{the source of } G)(c(1)), (\text{the target of } G)(c(1))\}$.

(40) Suppose c alternates vertices in G and v_5 is vertex sequence of c. Then v_5 is a two-valued alternating finite sequence.

(41) Suppose c alternates vertices in G. Then there exist v_6, v_7 such that

(i) $v_6 \neq v_7$,

(ii) v_6 is vertex sequence of c,

(iii) v_7 is vertex sequence of c, and

(iv) for every v_5 such that v_5 is vertex sequence of c holds $v_5 = v_6$ or $v_5 = v_7$.

(42) Suppose v_5 is vertex sequence of c. Then $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \varepsilon$ and c does not alternate vertices in G if and only if for every v_6 such that v_6 is vertex sequence of c holds $v_6 = v_5$.

Let us consider G, c. Let us assume that $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \varepsilon$ and c does not alternate vertices in G. The functor vertex-seq(c) yielding a finite sequence of elements of the vertices of G is defined as follows:

(Def.9) vertex-seq(c) is vertex sequence of c.

We now state several propositions:

(43) If v_5 is vertex sequence of c and $c_1 = c \restriction \mathrm{Seg}\, n$ and $v_6 = v_5 \restriction \mathrm{Seg}(n+1)$, then v_6 is vertex sequence of c_1.

(44) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then q is a chain of G.

(45) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and v_5 is vertex sequence of c and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$, then v_6 is vertex sequence of c_1.

(46) If v_6 is vertex sequence of c_1 and v_7 is vertex sequence of c_2 and $v_6(\mathrm{len}\, v_6) = v_7(1)$, then $c_1 \frown c_2$ is a chain of G.

(47) Suppose v_6 is vertex sequence of c_1 and v_7 is vertex sequence of c_2 and $v_6(\mathrm{len}\, v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\!\!\frown v_7$. Then v_5 is vertex sequence of c.

7. Vertex sequences induced by simple chains, paths and ordered chains

Let us consider G. A chain of G is simple if it satisfies the condition (Def.10).

(Def.10) There exists v_5 such that v_5 is vertex sequence of it and for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \mathrm{len}\, v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \mathrm{len}\, v_5$.

Let us consider G. One can check that there exists a chain of G which is simple.

In the sequel s_2 denotes a simple chain of G.

Next we state several propositions:

(49)[2] $s_2 \restriction \mathrm{Seg}\, n$ is a simple chain of G.

(50) If $2 < \mathrm{len}\, s_2$ and v_6 is vertex sequence of s_2 and v_7 is vertex sequence of s_2, then $v_6 = v_7$.

(51) If v_5 is vertex sequence of s_2, then for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \mathrm{len}\, v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \mathrm{len}\, v_5$.

(52) Suppose c is not a simple chain of G and v_5 is vertex sequence of c. Then there exists a FinSubsequence f_{10} of c and there exists a FinSubsequence f_{11} of v_5 and there exist c_1, v_6 such that $\mathrm{len}\, c_1 < \mathrm{len}\, c$ and v_6 is vertex sequence of c_1 and $\mathrm{len}\, v_6 < \mathrm{len}\, v_5$ and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$ and $\mathrm{Seq}\, f_{10} = c_1$ and $\mathrm{Seq}\, f_{11} = v_6$.

(53) Suppose v_5 is vertex sequence of c. Then there exists a FinSubsequence f_{10} of c and there exists a FinSubsequence f_{11} of v_5 and there exist s_2, v_6 such that $\mathrm{Seq}\, f_{10} = s_2$ and $\mathrm{Seq}\, f_{11} = v_6$ and v_6 is vertex sequence of s_2 and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$.

[2] The proposition (48) has been removed.

Let us consider G. One can check that every chain of G which is empty is also path-like.

We now state the proposition

(54) If p is a path of G, then $p \upharpoonright \mathrm{Seg}\, n$ is a path of G.

Let us consider G. One can verify that there exists a path of G which is simple.

We now state two propositions:

(55) If $2 < \mathrm{len}\, s_2$, then s_2 is a path of G.

(56) s_2 is a path of G iff $\mathrm{len}\, s_2 = 0$ or $\mathrm{len}\, s_2 = 1$ or $s_2(1) \neq s_2(2)$.

Let us consider G. Observe that every chain of G which is empty is also oriented.

Let us consider G and let o_1 be an oriented chain of G. Let us assume that $o_1 \neq \varepsilon$. The functor vertex-seq(o_1) yields a finite sequence of elements of the vertices of G and is defined as follows:

(Def.11) vertex-seq(o_1) is vertex sequence of o_1 and (vertex-seq(o_1))(1) = (the source of G)($o_1(1)$).

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On the Lattice of Subspaces of a Vector Space

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The terminology and notation used here are introduced in the following articles: [18], [11], [5], [17], [6], [20], [14], [15], [13], [1], [16], [10], [19], [3], [4], [2], [12], [9], [7], and [8].

In this paper F denotes a field and V_1 denotes a strict vector space over F.

Let us consider F, V_1. The functor $\mathbb{L}_{(V_1)}$ yields a strict bounded lattice and is defined as follows:

(Def.1) $\mathbb{L}_{(V_1)} = \langle \mathrm{Subspaces}\, V_1, \mathrm{SubJoin}\, V_1, \mathrm{SubMeet}\, V_1 \rangle$.

Let us consider F, V_1. Family of subspaces of V_1 is defined as follows:

(Def.2) For arbitrary x such that $x \in$ it holds x is a subspace of V_1.

Let us consider F, V_1. Note that there exists a family of subspaces of V_1 which is non empty.

Let us consider F, V_1. Then Subspaces V_1 is a non empty family of subspaces of V_1. Let X be a non empty family of subspaces of V_1. We see that the element of X is a subspace of V_1.

Let us consider F, V_1 and let x be an element of Subspaces V_1. The functor $\overline{x}$ yielding a subset of the carrier of V_1 is defined as follows:

(Def.3) There exists a subspace X of V_1 such that $x = X$ and $\overline{x} =$ the carrier of X.

Let us consider F, V_1. The functor $\overline{V_1}$ yielding a function from Subspaces V_1 into $2^{\text{the carrier of } V_1}$ is defined by:

(Def.4) For every element h of Subspaces V_1 and for every subspace H of V_1 such that $h = H$ holds $\overline{V_1}(h) =$ the carrier of H.

We now state two propositions:

(1) For every strict vector space V_1 over F and for every non empty subset H of Subspaces V_1 holds $\overline{V_1}^{\circ} H$ is non empty.

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(2) For every strict vector space V_1 over F and for every strict subspace H of V_1 holds $0_{(V_1)} \in \overline{V_1}(H)$.

Let us consider F, V_1 and let G be a non empty subset of Subspaces V_1. The functor $\bigcap G$ yielding a strict subspace of V_1 is defined by:

(Def.5) The carrier of $\bigcap G = \bigcap(\overline{V_1}^{\circ} G)$.

Next we state several propositions:

(3) Subspaces $V_1 =$ the carrier of $\mathbb{L}_{(V_1)}$.

(4) The meet operation of $\mathbb{L}_{(V_1)} = \mathrm{SubMeet}\, V_1$.

(5) The join operation of $\mathbb{L}_{(V_1)} = \mathrm{SubJoin}\, V_1$.

(6) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be strict subspaces of V_1. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(7) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcup q = H_1 + H_2$.

(8) Let V_1 be a strict vector space over F, and let p, q be elements of the carrier of $\mathbb{L}_{(V_1)}$, and let H_1, H_2 be subspaces of V_1. If $p = H_1$ and $q = H_2$, then $p \sqcap q = H_1 \cap H_2$.

Let us observe that a non empty lattice structure is complete if it satisfies the condition (Def.6).

(Def.6) Let X be a subset of the carrier of it. Then there exists an element a of the carrier of it such that $a \sqsubseteq X$ and for every element b of the carrier of it such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

The following propositions are true:

(9) For every V_1 holds $\mathbb{L}_{(V_1)}$ is complete.

(10) Let x be arbitrary, and let V_1 be a strict vector space over F, and let S be a subset of the carrier of V_1. If S is non empty and linearly closed, then if $x \in \mathrm{Lin}(S)$, then $x \in S$.

Let F be a field, let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. The functor FuncLatt(f) yields a function from the carrier of $\mathbb{L}_A$ into the carrier of $\mathbb{L}_B$ and is defined by the condition (Def.7).

(Def.7) Let S be a strict subspace of A and let B_0 be a subset of the carrier of B. If $B_0 = f^{\circ}(\text{the carrier of } S)$, then $(\mathrm{FuncLatt}(f))(S) = \mathrm{Lin}(B_0)$.

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is called a lower homomorphism between L_1 and L_2 if:

(Def.8) For all elements a, b of the carrier of L_1 holds $\mathrm{it}(a \sqcap b) = \mathrm{it}(a) \sqcap \mathrm{it}(b)$.

Let L_1, L_2 be lattices. A function from the carrier of L_1 into the carrier of L_2 is called an upper homomorphism between L_1 and L_2 if:

(Def.9) For all elements a, b of the carrier of L_1 holds $\mathrm{it}(a \sqcup b) = \mathrm{it}(a) \sqcup \mathrm{it}(b)$.

One can prove the following propositions:

(11) Let L_1, L_2 be lattices and let f be a function from the carrier of L_1 into the carrier of L_2. Then f is a homomorphism from L_1 to L_2 if and only if f is an upper homomorphism between L_1 and L_2 and a lower homomorphism between L_1 and L_2.

(12) Let F be a field, and let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. If f is linear, then $\mathrm{FuncLatt}(f)$ is an upper homomorphism between $\mathbb{L}_A$ and $\mathbb{L}_B$.

(13) Let F be a field, and let A, B be strict vector spaces over F, and let f be a function from the carrier of A into the carrier of B. Suppose f is one-to-one and linear. Then $\mathrm{FuncLatt}(f)$ is a homomorphism from $\mathbb{L}_A$ to $\mathbb{L}_B$.

(14) Let A, B be strict vector spaces over F and let f be a function from the carrier of A into the carrier of B. If f is linear and one-to-one, then $\mathrm{FuncLatt}(f)$ is one-to-one.

(15) Let A be a strict vector space over F and let f be a function from the carrier of A into the carrier of A. If $f = \mathrm{id}_{(\text{the carrier of } A)}$, then $\mathrm{FuncLatt}(f) = \mathrm{id}_{(\text{the carrier of } \mathbb{L}_A)}$.

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On the Lattice of Subgroups of a Group

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The articles [15], [3], [16], [8], [4], [2], [17], [13], [7], [10], [12], [9], [11], [14], [1], [6], and [5] provide the terminology and notation for this paper.

The following propositions are true:

(1) Let G be a group and let H_1, H_2 be subgroups of G. Then the carrier of $H_1 \cap H_2 =$ (the carrier of H_1) $\cap$ (the carrier of H_2).

(2) For every group G and for arbitrary h holds $h \in \mathrm{SubGr}\, G$ iff there exists a strict subgroup H of G such that $h = H$.

(3) Let G be a group, and let A be a subset of the carrier of G, and let H be a strict subgroup of G. If $A =$ the carrier of H, then $\mathrm{gr}(A) = H$.

(4) Let G be a group, and let H_1, H_2 be subgroups of G, and let A be a subset of the carrier of G. If $A =$ (the carrier of H_1) $\cup$ (the carrier of H_2), then $H_1 \sqcup H_2 = \mathrm{gr}(A)$.

(5) Let G be a group, and let H_1, H_2 be subgroups of G, and let g be an element of the carrier of G. If $g \in H_1$ or $g \in H_2$, then $g \in H_1 \sqcup H_2$.

(6) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1 be a subgroup of G_1. Then there exists a strict subgroup H_2 of G_2 such that the carrier of $H_2 = f^\circ$(the carrier of H_1).

(7) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_2 be a subgroup of G_2. Then there exists a strict subgroup H_1 of G_1 such that the carrier of $H_1 = f^{-1}$ (the carrier of H_2).

(8) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_1. Suppose the carrier of $H_1 \subseteq$ the carrier of H_2. Then f°(the carrier of H_1) $\subseteq f^\circ$(the carrier of H_2).

(9) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_2. Suppose the carrier of $H_1 \subseteq$ the carrier of H_2. Then f^{-1} (the carrier of H_1) $\subseteq f^{-1}$ (the carrier of H_2).

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(10) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_1, and let H_3, H_4 be subgroups of G_2. Suppose the carrier of $H_3 = f^\circ$(the carrier of H_1) and the carrier of $H_4 = f^\circ$(the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(11) Let G_1, G_2 be groups, and let f be a homomorphism from G_1 to G_2, and let H_1, H_2 be subgroups of G_2, and let H_3, H_4 be subgroups of G_1. Suppose the carrier of $H_3 = f^{-1}$ (the carrier of H_1) and the carrier of $H_4 = f^{-1}$ (the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(12) Let G_1, G_2 be groups, and let f be a function from the carrier of G_1 into the carrier of G_2, and let A be a subset of the carrier of G_1. Then $f^\circ A \subseteq f^\circ$(the carrier of $\mathrm{gr}(A)$).

(13) Let G_1, G_2 be groups, and let H_1, H_2 be subgroups of G_1, and let f be a function from the carrier of G_1 into the carrier of G_2, and let A be a subset of the carrier of G_1. Suppose $A =$ (the carrier of H_1) $\cup$ (the carrier of H_2). Then f°(the carrier of $H_1 \sqcup H_2$) $= f^\circ$(the carrier of $\mathrm{gr}(A)$).

(14) For every group G and for every subset A of the carrier of G such that $A = \{1_G\}$ holds $\mathrm{gr}(A) = \{\mathbf{1}\}_G$.

(15) For all non empty sets X, Y and for all subsets A_1, A_2 of Y and for every function f from X into Y holds $f^{-1}(A_1 \cup A_2) = f^{-1} A_1 \cup f^{-1} A_2$.

(16) For all non empty sets X, Y and for all subsets A_1, A_2 of X and for every function f from X into Y holds $f^\circ(A_1 \cup A_2) = f^\circ A_1 \cup f^\circ A_2$.

Let G be a group. The functor $\overline{G}$ yields a function from $\mathrm{SubGr}\, G$ into $2^{\text{the carrier of } G}$ and is defined as follows:

(Def.1) For every element h of $\mathrm{SubGr}\, G$ and for every subgroup H of G such that $h = H$ holds $\overline{G}(h) =$ the carrier of H.

Next we state several propositions:

(17) Let G be a group, and let h be an element of $\mathrm{SubGr}\, G$, and let H be a subgroup of G. If $h = H$, then $\overline{G}(h) =$ the carrier of H.

(18) Let G be a group, and let H be a strict subgroup of G, and let x be an element of the carrier of G. Then $x \in \overline{G}(H)$ if and only if $x \in H$.

(19) For every group G and for every strict subgroup H of G holds $1_G \in \overline{G}(H)$.

(20) For every group G and for every strict subgroup H of G holds $\overline{G}(H) \neq \emptyset$.

(21) Let G be a group, and let H be a strict subgroup of G, and let g_1, g_2 be elements of the carrier of G. If $g_1 \in \overline{G}(H)$ and $g_2 \in \overline{G}(H)$, then $g_1 \cdot g_2 \in \overline{G}(H)$.

(22) Let G be a group, and let H be a strict subgroup of G, and let g be an element of the carrier of G. If $g \in \overline{G}(H)$, then $g^{-1} \in \overline{G}(H)$.

(23) For every group G and for all strict subgroups H_1, H_2 of G holds the carrier of $H_1 \cap H_2 = \overline{G}(H_1) \cap \overline{G}(H_2)$.

(24) For every group G and for all strict subgroups H_1, H_2 of G holds $\overline{G}(H_1 \cap H_2) = \overline{G}(H_1) \cap \overline{G}(H_2)$.

Let G be a group and let F be a non empty subset of $\mathrm{SubGr}\, G$. The functor $\bigcap F$ yielding a strict subgroup of G is defined by:

(Def.2) The carrier of $\bigcap F = \bigcap(\overline{G}^{\circ} F)$.

Next we state several propositions:

(25) For every group G and for every non empty subset F of $\mathrm{SubGr}\, G$ such that $\{\mathbf{1}\}_G \in F$ holds $\bigcap F = \{\mathbf{1}\}_G$.

(26) For every group G and for every element h of $\mathrm{SubGr}\, G$ and for every non empty subset F of $\mathrm{SubGr}\, G$ such that $F = \{h\}$ holds $\bigcap F = h$.

(27) Let G be a group, and let H_1, H_2 be subgroups of G, and let h_1, h_2 be elements of the carrier of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcup h_2 = H_1 \sqcup H_2$.

(28) Let G be a group, and let H_1, H_2 be subgroups of G, and let h_1, h_2 be elements of the carrier of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcap h_2 = H_1 \cap H_2$.

(29) Let G be a group, and let p be an element of the carrier of $\mathbb{L}_G$, and let H be a subgroup of G. If $p = H$, then H is a strict subgroup of G.

(30) Let G be a group, and let H_1, H_2 be subgroups of G, and let p, q be elements of the carrier of $\mathbb{L}_G$. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(31) Let G be a group, and let H_1, H_2 be subgroups of G, and let p, q be elements of the carrier of $\mathbb{L}_G$. If $p = H_1$ and $q = H_2$, then $p \sqsubseteq q$ iff H_1 is a subgroup of H_2.

(32) For every group G holds $\mathbb{L}_G$ is complete.

Let G_1, G_2 be groups and let f be a function from the carrier of G_1 into the carrier of G_2. The functor $\mathrm{FuncLatt}(f)$ yielding a function from the carrier of $\mathbb{L}_{(G_1)}$ into the carrier of $\mathbb{L}_{(G_2)}$ is defined by the condition (Def.3).

(Def.3) Let H be a strict subgroup of G_1 and let A be a subset of the carrier of G_2. If $A = f^{\circ}(\text{the carrier of } H)$, then $(\mathrm{FuncLatt}(f))(H) = \mathrm{gr}(A)$.

One can prove the following propositions:

(33) Let G be a group and let f be a function from the carrier of G into the carrier of G. If $f = \mathrm{id}_{(\text{the carrier of } G)}$, then $\mathrm{FuncLatt}(f) = \mathrm{id}_{(\text{the carrier of } \mathbb{L}_G)}$.

(34) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 such that f is one-to-one holds $\mathrm{FuncLatt}(f)$ is one-to-one.

(35) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 holds $(\mathrm{FuncLatt}(f))(\{\mathbf{1}\}_{(G_1)}) = \{\mathbf{1}\}_{(G_2)}$.

(36) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. Suppose f is one-to-one. Then $\mathrm{FuncLatt}(f)$ is a lower homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(37) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. Then FuncLatt(f) is an upper homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(38) Let G_1, G_2 be groups and let f be a homomorphism from G_1 to G_2. If f is one-to-one, then FuncLatt(f) is a homomorphism from $\mathbb{L}_{(G_1)}$ to $\mathbb{L}_{(G_2)}$.

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On the Lattice of Subalgebras of a Universal Algebra

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The papers [13], [16], [15], [12], [17], [6], [7], [2], [8], [5], [4], [18], [1], [10], [11], [9], [14], and [3] provide the terminology and notation for this paper.

In this paper U_0 is a universal algebra, H is a non empty subset of the carrier of U_0, and o is an operation of U_0.

Let us consider U_0. Family of subalgebras of U_0 is defined by:

(Def.1) For arbitrary U_1 such that $U_1 \in$ it holds U_1 is a subalgebra of U_0.

Let us consider U_0. One can check that there exists a family of subalgebras of U_0 which is non empty.

Let us consider U_0. Then Subalgebras(U_0) is a non empty family of subalgebras of U_0. Let U_2 be a non empty family of subalgebras of U_0. We see that the element of U_2 is a subalgebra of U_0.

Let us consider U_0. Then $\bigsqcup_{(U_0)}$ is a binary operation on Subalgebras(U_0). Then $\bigsqcap_{(U_0)}$ is a binary operation on Subalgebras(U_0).

Let us consider U_0 and let u be an element of Subalgebras(U_0). The functor $\overline{u}$ yielding a subset of the carrier of U_0 is defined as follows:

(Def.2) There exists a subalgebra U_1 of U_0 such that $u = U_1$ and $\overline{u} =$ the carrier of U_1.

Let us consider U_0. The functor Carr(U_0) yields a function from Subalgebras(U_0) into $2^{\text{the carrier of } U_0}$ and is defined by:

(Def.3) For every element u of Subalgebras(U_0) holds $(\mathrm{Carr}(U_0))(u) = \overline{u}$.

We now state several propositions:

(1) For arbitrary u holds $u \in$ Subalgebras(U_0) iff there exists a strict subalgebra U_1 of U_0 such that $u = U_1$.

(2) Let H be a non empty subset of the carrier of U_0 and given o. If arity $o = 0$, then H is closed on o iff $o(\varepsilon) \in H$.

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(3) For every subalgebra U_1 of U_0 holds the carrier of $U_1 \subseteq$ the carrier of U_0.

(4) For every non empty subset H of the carrier of U_0 and for every o such that H is closed on o and arity $o = 0$ holds $o_H = o$.

(5) If U_0 has constants, then Constants$(U_0) = \{o(\varepsilon) : o$ ranges over operation of U_0, arity $o = 0\}$.

(6) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds Constants$(U_0) =$ Constants(U_1).

Let U_0 be a universal algebra with constants. Note that every subalgebra of U_0 has constants.

The following proposition is true

(7) For every universal algebra U_0 with constants and for all subalgebras U_1, U_3 of U_0 holds Constants$(U_1) =$ Constants(U_3).

Let us consider U_0. Then Carr(U_0) is a function from Subalgebras(U_0) into $2^{\text{the carrier of } U_0}$ and it can be characterized by the condition:

(Def.4) For every element u of Subalgebras(U_0) and for every subalgebra U_1 of U_0 such that $u = U_1$ holds $(\text{Carr}(U_0))(u) =$ the carrier of U_1.

One can prove the following propositions:

(8) For every strict subalgebra H of U_0 and for every element u of U_0 holds $u \in (\text{Carr}(U_0))(H)$ iff $u \in H$.

(9) For every non empty subset H of Subalgebras(U_0) holds $(\text{Carr}(U_0))^\circ H$ is non empty.

(10) For every universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds Constants$(U_0) \subseteq (\text{Carr}(U_0))(U_1)$.

(11) Let U_0 be a universal algebra with constants, and let U_1 be a subalgebra of U_0, and let a be arbitrary. If a is an element of Constants(U_0), then $a \in$ the carrier of U_1.

(12) Let U_0 be a universal algebra with constants and let H be a non empty subset of Subalgebras(U_0). Then $\bigcap((\text{Carr}(U_0))^\circ H)$ is a non empty subset of the carrier of U_0.

(13) For every universal algebra U_0 with constants holds the carrier of the lattice of subalgebras of $U_0 =$ Subalgebras(U_0).

(14) Let U_0 be a universal algebra with constants, and let H be a non empty subset of Subalgebras(U_0), and let S be a non empty subset of the carrier of U_0. If $S = \bigcap((\text{Carr}(U_0))^\circ H)$, then S is operations closed.

Let U_0 be a strict universal algebra with constants and let H be a non empty subset of Subalgebras(U_0). The functor $\bigcap H$ yielding a strict subalgebra of U_0 is defined as follows:

(Def.5) The carrier of $\bigcap H = \bigcap((\text{Carr}(U_0))^\circ H)$.

One can prove the following propositions:

(15) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcup l_2 = U_1 \bigsqcup U_3$.

(16) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcap l_2 = U_1 \cap U_3$.

(17) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. Suppose $l_1 = U_1$ and $l_2 = U_3$. Then $l_1 \sqsubseteq l_2$ if and only if the carrier of $U_1 \subseteq$ the carrier of U_3.

(18) Let U_0 be a universal algebra with constants, and let l_1, l_2 be elements of the carrier of the lattice of subalgebras of U_0, and let U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqsubseteq l_2$ iff U_1 is a subalgebra of U_3.

(19) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is bounded.

(20) For every strict universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(21) For every strict universal algebra U_0 with constants holds $\perp_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(22) Let U_0 be a strict universal algebra with constants, and let U_1 be a subalgebra of U_0, and let H be a subset of the carrier of U_0. If $H =$ the carrier of U_0, then $\mathrm{Gen}^{\mathrm{UA}}(H) \bigsqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(23) Let U_0 be a strict universal algebra with constants and let H be a subset of the carrier of U_0. Suppose $H =$ the carrier of U_0. Then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(24) For every strict universal algebra U_0 with constants holds $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

(25) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is complete.

Let U_4, U_5 be universal algebras with constants and let F be a function from the carrier of U_4 into the carrier of U_5. The functor $\mathrm{FuncLatt}(F)$ yielding a function from the carrier of the lattice of subalgebras of U_4 into the carrier of the lattice of subalgebras of U_5 is defined by the condition (Def.6).

(Def.6) Let U_1 be a strict subalgebra of U_4 and let H be a subset of the carrier of U_5. If $H = F^{\circ}$(the carrier of U_1), then $(\mathrm{FuncLatt}(F))(U_1) = \mathrm{Gen}^{\mathrm{UA}}(H)$.

We now state the proposition

(26) Let U_0 be a strict universal algebra with constants and let F be a function from the carrier of U_0 into the carrier of U_0. Suppose $F = \mathrm{id}_{(\text{the carrier of } U_0)}$. Then $\mathrm{FuncLatt}(F) = \mathrm{id}_{(\text{the carrier of the lattice of subalgebras of } U_0)}$.

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On the Decomposition of Finite Sequences

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The notation and terminology used here are introduced in the following papers: [10], [12], [9], [7], [1], [13], [4], [2], [11], [8], [5], [3], and [6].

1. Preliminaries

We introduce degenerated as a synonym of trivial.

Let us observe that every set which is non trivial is also non empty.

In the sequel x, y, z will be arbitrary.

Let us consider x, y. Observe that $\langle x, y \rangle$ is non trivial.

Let us consider x, y, z. Note that $\langle x, y, z \rangle$ is non trivial.

Let f be a non empty finite sequence. One can check that $\mathrm{Rev}(f)$ is non empty.

2. Decomposing a finite sequence

For simplicity we adopt the following rules: f_1, f_2, f_3 will denote finite sequences, p, p_1, p_2, p_3 will be arbitrary, f will denote a finite sequence, and i, k will denote natural numbers.

Next we state a number of propositions:

(3)[1] For every set X and for every i such that $X \subseteq \mathrm{Seg}\, i$ and $1 \in X$ holds $(\mathrm{Sgm}\, X)(1) = 1$.

(4) For every finite sequence f such that $k \in \mathrm{dom}\, f$ and for every i such that $1 \leq i$ and $i < k$ holds $f(i) \neq f(k)$ holds $f(k) \leftarrow\!\!\shortmid f = k$.

[1]The propositions (1) and (2) have been removed.

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(5) $\langle p_1, p_2 \rangle \upharpoonright \mathrm{Seg}\, 1 = \langle p_1 \rangle$.
(6) $\langle p_1, p_2, p_3 \rangle \upharpoonright \mathrm{Seg}\, 1 = \langle p_1 \rangle$.
(7) $\langle p_1, p_2, p_3 \rangle \upharpoonright \mathrm{Seg}\, 2 = \langle p_1, p_2 \rangle$.
(8) If $p \in \mathrm{rng}\, f_1$, then $p \looparrowleft (f_1 \frown f_2) = p \looparrowleft f_1$.
(9) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $p \looparrowleft (f_1 \frown f_2) = \mathrm{len}\, f_1 + p \looparrowleft f_2$.
(10) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \rightarrow p = (f_1 \rightarrow p) \frown f_2$.
(11) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \rightarrow p = f_2 \rightarrow p$.
(12) $f_1 \subseteq f_1 \frown f_2$.
(13) For every set A such that $A \subseteq \mathrm{dom}\, f_1$ holds $(f_1 \frown f_2) \upharpoonright A = f_1 \upharpoonright A$.
(14) If $p \in \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \leftarrow p = f_1 \leftarrow p$.

Let us consider f_1, i. Observe that $f_1 \upharpoonright \mathrm{Seg}\, i$ is finite sequence-like.

The following propositions are true:

(15) If $f_1 \subseteq f_2$, then $f_3 \frown f_1 \subseteq f_3 \frown f_2$.
(16) $(f_1 \frown f_2) \upharpoonright \mathrm{Seg}(\mathrm{len}\, f_1 + i) = f_1 \frown (f_2 \upharpoonright \mathrm{Seg}\, i)$.
(17) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \frown f_2 \leftarrow p = f_1 \frown (f_2 \leftarrow p)$.
(18) For every finite sequence f and for arbitrary p, q such that $p \in \mathrm{rng}\, f$ and $q \in \mathrm{rng}\, f$ and $p \looparrowleft f = q \looparrowleft f$ holds $p = q$.
(19) If $f_1 \frown f_2$ yields p just once, then $p \in \mathrm{rng}\, f_1 \mathbin{\dot{-}} \mathrm{rng}\, f_2$.
(20) If $f_1 \frown f_2$ yields p just once and $p \in \mathrm{rng}\, f_1$, then f_1 yields p just once.
(21) If $\mathrm{rng}\, f$ is non trivial, then f is non trivial.
(22) $p \looparrowleft \langle p \rangle = 1$.
(23) $p_1 \looparrowleft \langle p_1, p_2 \rangle = 1$.
(24) If $p_1 \neq p_2$, then $p_2 \looparrowleft \langle p_1, p_2 \rangle = 2$.
(25) $p_1 \looparrowleft \langle p_1, p_2, p_3 \rangle = 1$.
(26) If $p_1 \neq p_2$, then $p_2 \looparrowleft \langle p_1, p_2, p_3 \rangle = 2$.
(27) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $p_3 \looparrowleft \langle p_1, p_2, p_3 \rangle = 3$.
(28) For every finite sequence f holds $\mathrm{Rev}(\langle p \rangle \frown f) = (\mathrm{Rev}(f)) \frown \langle p \rangle$.
(29) For every finite sequence f holds $\mathrm{Rev}(\mathrm{Rev}(f)) = f$.
(30) If $x \neq y$, then $\langle x, y \rangle \leftarrow y = \langle x \rangle$.
(31) If $x \neq y$, then $\langle x, y, z \rangle \leftarrow y = \langle x \rangle$.
(32) If $x \neq z$ and $y \neq z$, then $\langle x, y, z \rangle \leftarrow z = \langle x, y \rangle$.
(33) $\langle x, y \rangle \rightarrow x = \langle y \rangle$.
(34) If $x \neq y$, then $\langle x, y, z \rangle \rightarrow y = \langle z \rangle$.
(35) $\langle x, y, z \rangle \rightarrow x = \langle y, z \rangle$.
(36) $\langle z \rangle \rightarrow z = \varepsilon$ and $\langle z \rangle \leftarrow z = \varepsilon$.
(37) If $x \neq y$, then $\langle x, y \rangle \rightarrow y = \varepsilon$.
(38) If $x \neq z$ and $y \neq z$, then $\langle x, y, z \rangle \rightarrow z = \varepsilon$.
(39) If $x \in \mathrm{rng}\, f$ and $y \in \mathrm{rng}(f \leftarrow x)$, then $(f \leftarrow x) \leftarrow y = f \leftarrow y$.
(40) If $x \notin \mathrm{rng}\, f_1$, then $x \looparrowleft (f_1 \frown \langle x \rangle \frown f_2) = \mathrm{len}\, f_1 + 1$.

(41) If f yields x just once, then $x \looparrowleft f + x \looparrowleft \mathrm{Rev}(f) = \mathrm{len}\, f + 1$.

(42) If f yields x just once, then $\mathrm{Rev}(f \leftarrow x) = \mathrm{Rev}(f) \rightarrow x$.

(43) If f yields x just once, then $\mathrm{Rev}(f)$ yields x just once.

3. Finite sequences with elements from a non empty set

We adopt the following convention: D will denote a non empty set, p, p_1, p_2, p_3 will denote elements of D, and f, f_1, f_2 will denote finite sequences of elements of D.

One can prove the following propositions:

(44) If $p \in \mathrm{rng}\, f$, then $f \mathbin{-:} p = (f \leftarrow p) \mathbin{^\frown} \langle p \rangle$.

(45) If $p \in \mathrm{rng}\, f$, then $f \mathbin{:-} p = \langle p \rangle \mathbin{^\frown} (f \rightarrow p)$.

(46) If $f \neq \varepsilon$, then $\pi_1 f \in \mathrm{rng}\, f$.

(47) If $f \neq \varepsilon$, then $(\pi_1 f) \looparrowleft f = 1$.

(48) If $f \neq \varepsilon$ and $\pi_1 f = p$, then $f \mathbin{-:} p = \langle p \rangle$ and $f \mathbin{:-} p = f$.

(49) $(\langle p_1 \rangle \mathbin{^\frown} f)_{\restriction 1} = f$.

(50) $\langle p_1, p_2 \rangle_{\restriction 1} = \langle p_2 \rangle$.

(51) $\langle p_1, p_2, p_3 \rangle_{\restriction 1} = \langle p_2, p_3 \rangle$.

(52) If $k \in \mathrm{dom}\, f$ and for every i such that $1 \leq i$ and $i < k$ holds $\pi_i f \neq \pi_k f$, then $(\pi_k f) \looparrowleft f = k$.

(53) If $p_1 \neq p_2$, then $\langle p_1, p_2 \rangle \mathbin{-:} p_2 = \langle p_1, p_2 \rangle$.

(54) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle \mathbin{-:} p_2 = \langle p_1, p_2 \rangle$.

(55) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle \mathbin{-:} p_3 = \langle p_1, p_2, p_3 \rangle$.

(56) $\langle p \rangle \mathbin{:-} p = \langle p \rangle$ and $\langle p \rangle \mathbin{-:} p = \langle p \rangle$.

(57) If $p_1 \neq p_2$, then $\langle p_1, p_2 \rangle \mathbin{:-} p_2 = \langle p_2 \rangle$.

(58) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle \mathbin{:-} p_2 = \langle p_2, p_3 \rangle$.

(59) If $p_1 \neq p_3$ and $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle \mathbin{:-} p_3 = \langle p_3 \rangle$.

(60) If $x \in \mathrm{rng}\, f$ and $p \in \mathrm{rng}\, f$ and $x \looparrowleft f \leq p \looparrowleft f$, then $x \in \mathrm{rng}(f \mathbin{-:} p)$.

(61) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $p \looparrowleft f = k + p \looparrowleft (f_{\restriction k})$.

(62) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $p \in \mathrm{rng}(f_{\restriction k})$.

(63) If $k < i$ and $i \in \mathrm{dom}\, f$, then $\pi_i f \in \mathrm{rng}(f_{\restriction k})$.

(64) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f > k$, then $f_{\restriction k} \mathbin{-:} p = (f \mathbin{-:} p)_{\restriction k}$.

(65) If $p \in \mathrm{rng}\, f$ and $p \looparrowleft f \neq 1$, then $f_{\restriction 1} \mathbin{-:} p = (f \mathbin{-:} p)_{\restriction 1}$.

(66) $p \in \mathrm{rng}(f \mathbin{:-} p)$.

(67) If $x \in \mathrm{rng}\, f$ and $p \in \mathrm{rng}\, f$ and $x \looparrowleft f \geq p \looparrowleft f$, then $x \in \mathrm{rng}(f \mathbin{:-} p)$.

(68) If $p \in \mathrm{rng}\, f$ and $k \leq \mathrm{len}\, f$ and $k \geq p \looparrowleft f$, then $\pi_k f \in \mathrm{rng}(f \mathbin{:-} p)$.

(69) If $p \in \mathrm{rng}\, f_1$, then $f_1 \mathbin{^\frown} f_2 \mathbin{:-} p = (f_1 \mathbin{:-} p) \mathbin{^\frown} f_2$.

(70) If $p \in \mathrm{rng}\, f_2 \setminus \mathrm{rng}\, f_1$, then $f_1 \mathbin{^\frown} f_2 \mathbin{:-} p = f_2 \mathbin{:-} p$.

(71) If $p \in \mathrm{rng}\, f_1$, then $f_1 \mathbin{^\frown} f_2 \mathbin{-:} p = f_1 \mathbin{-:} p$.

(72) If $p \in \operatorname{rng} f_2 \setminus \operatorname{rng} f_1$, then $f_1 \frown f_2 -: p = f_1 \frown (f_2 -: p)$.
(73) $f :- p :- p = f :- p$.
(74) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f -: p_1)$, then $f \rightarrow p_2 = (f \rightarrow p_1) \rightarrow p_2$.
(75) If $p \in \operatorname{rng} f$, then $\operatorname{rng} f = \operatorname{rng}(f -: p) \cup \operatorname{rng}(f :- p)$.
(76) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f -: p_1)$, then $f :- p_1 :- p_2 = f :- p_2$.
(77) If $p \in \operatorname{rng} f$, then $p \looparrowleft (f -: p) = p \looparrowleft f$.
(78) $f \restriction i \restriction i = f \restriction i$.
(79) If $p \in \operatorname{rng} f$, then $f -: p -: p = f -: p$.
(80) If $p_1 \in \operatorname{rng} f$ and $p_2 \in \operatorname{rng}(f -: p_1)$, then $f -: p_1 -: p_2 = f -: p_2$.
(81) If $p \in \operatorname{rng} f$, then $(f -: p) \frown ((f :- p)_{\lfloor 1}) = f$.
(82) If $f_1 \neq \varepsilon$, then $(f_1 \frown f_2)_{\lfloor 1} = ((f_1)_{\lfloor 1}) \frown f_2$.
(83) If $p_2 \in \operatorname{rng} f$ and $p_2 \looparrowleft f \neq 1$, then $p_2 \in \operatorname{rng}(f_{\lfloor 1})$.
(84) If $p \in \operatorname{rng} f$, then $p \looparrowleft (f :- p) = 1$.
(86)[2] $(\varepsilon_D)_{\lfloor k} = \varepsilon_D$.
(87) $f_{\lfloor i+k} = (f_{\lfloor i})_{\lfloor k}$.
(88) If $p \in \operatorname{rng} f$ and $p \looparrowleft f > k$, then $f_{\lfloor k} :- p = f :- p$.
(89) If $p \in \operatorname{rng} f$ and $p \looparrowleft f \neq 1$, then $f_{\lfloor 1} :- p = f :- p$.
(90) If $i + k = \operatorname{len} f$, then $\operatorname{Rev}(f_{\lfloor k}) = \operatorname{Rev}(f) \restriction i$.
(91) If $i + k = \operatorname{len} f$, then $\operatorname{Rev}(f \restriction k) = (\operatorname{Rev}(f))_{\lfloor i}$.
(92) If f yields p just once, then $\operatorname{Rev}(f \rightarrow p) = \operatorname{Rev}(f) \leftarrow p$.
(93) If f yields p just once, then $\operatorname{Rev}(f :- p) = \operatorname{Rev}(f) -: p$.
(94) If f yields p just once, then $\operatorname{Rev}(f -: p) = \operatorname{Rev}(f) :- p$.

4. Rotating a finite sequence

Let D be a non empty set. A finite sequence of elements of D is circular if:

(Def.1) $\pi_1 \mathrm{it} = \pi_{\operatorname{len} \mathrm{it}} \mathrm{it}$.

Let us consider D, f, p. The functor $f^p_{\circlearrowleft}$ yielding a finite sequence of elements of D is defined by:

(Def.2) (i) $f^p_{\circlearrowleft} = (f :- p) \frown ((f -: p)_{\lfloor 1})$ if $p \in \operatorname{rng} f$,
(ii) $f^p_{\circlearrowleft} = f$, otherwise.

Let us consider D, let f be a non empty finite sequence of elements of D, and let p be an element of D. One can verify that $f^p_{\circlearrowleft}$ is non empty.

Let us consider D. Observe that there exists a finite sequence of elements of D which is circular non empty and trivial and there exists a finite sequence of elements of D which is circular non empty and non trivial.

The following proposition is true

(95) $f^{\pi_1 f}_{\circlearrowleft} = f$.

[2]The proposition (85) has been removed.

Let us consider D, p and let f be a circular non empty finite sequence of elements of D. Observe that $f^{p}_{\circlearrowleft}$ is circular.

We now state a number of propositions:

(96) If f is circular and $p \in \operatorname{rng} f$, then $\operatorname{rng}(f^{p}_{\circlearrowleft}) = \operatorname{rng} f$.

(97) If $p \in \operatorname{rng} f$, then $p \in \operatorname{rng}(f^{p}_{\circlearrowleft})$.

(98) If $p \in \operatorname{rng} f$, then $\pi_1 f^{p}_{\circlearrowleft} = p$.

(99) $(f^{p}_{\circlearrowleft})^{p}_{\circlearrowleft} = f^{p}_{\circlearrowleft}$.

(100) $\langle p \rangle^{p}_{\circlearrowleft} = \langle p \rangle$.

(101) $\langle p_1, p_2 \rangle^{p_1}_{\circlearrowleft} = \langle p_1, p_2 \rangle$.

(102) $\langle p_1, p_2 \rangle^{p_2}_{\circlearrowleft} = \langle p_2, p_2 \rangle$.

(103) $\langle p_1, p_2, p_3 \rangle^{p_1}_{\circlearrowleft} = \langle p_1, p_2, p_3 \rangle$.

(104) If $p_1 \neq p_2$, then $\langle p_1, p_2, p_3 \rangle^{p_2}_{\circlearrowleft} = \langle p_2, p_3, p_2 \rangle$.

(105) If $p_2 \neq p_3$, then $\langle p_1, p_2, p_3 \rangle^{p_3}_{\circlearrowleft} = \langle p_3, p_2, p_3 \rangle$.

(106) For every circular non trivial finite sequence f of elements of D holds $\operatorname{rng}(f_{\lfloor 1}) = \operatorname{rng} f$.

(107) $\operatorname{rng}(f_{\lfloor 1}) \subseteq \operatorname{rng}(f^{p}_{\circlearrowleft})$.

(108) If $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f -: p_1)$, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(109) If $p_2 \looparrowleft f \neq 1$ and $p_2 \in \operatorname{rng} f \setminus \operatorname{rng}(f :- p_1)$, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(110) If $p_2 \in \operatorname{rng}(f_{\lfloor 1})$ and f yields p_2 just once, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(111) If f is circular and f yields p_2 just once, then $(f^{p_1}_{\circlearrowleft})^{p_2}_{\circlearrowleft} = f^{p_2}_{\circlearrowleft}$.

(112) If f is circular and f yields p just once, then $\operatorname{Rev}(f^{p}_{\circlearrowleft}) = (\operatorname{Rev}(f))^{p}_{\circlearrowleft}$.

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Decomposing a Go-Board into Cells

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The articles [20], [23], [6], [22], [9], [2], [14], [17], [18], [24], [1], [5], [3], [4], [21], [10], [11], [16], [15], [7], [8], [12], [13], and [19] provide the terminology and notation for this paper.

For simplicity we follow a convention: q will be a point of $\mathcal{E}^2_{\rm T}$, i, i_1, i_2, j, j_1, j_2, k will be natural numbers, r, s will be real numbers, and G will be a Go-board.

We now state the proposition

(1) Let M be a tabular finite sequence and given i, j. If $\langle i, j\rangle \in$ the indices of M, then $1 \leq i$ and $i \leq \operatorname{len} M$ and $1 \leq j$ and $j \leq \operatorname{width} M$.

Let us consider G, i. The functor $\operatorname{vstrip}(G,i)$ yielding a subset of the carrier of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def.1) (i) $\operatorname{vstrip}(G,i) = \{[r,s] : (G_{i,1})_{\mathbf{1}} \leq r \ \wedge \ r \leq (G_{i+1,1})_{\mathbf{1}}\}$ if $1 \leq i$ and $i < \operatorname{len} G$,

(ii) $\operatorname{vstrip}(G,i) = \{[r,s] : (G_{i,1})_{\mathbf{1}} \leq r\}$ if $i \geq \operatorname{len} G$,

(iii) $\operatorname{vstrip}(G,i) = \{[r,s] : r \leq (G_{i+1,1})_{\mathbf{1}}\}$, otherwise.

The functor $\operatorname{hstrip}(G,i)$ yields a subset of the carrier of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def.2) (i) $\operatorname{hstrip}(G,i) = \{[r,s] : (G_{1,i})_{\mathbf{2}} \leq s \ \wedge \ s \leq (G_{1,i+1})_{\mathbf{2}}\}$ if $1 \leq i$ and $i < \operatorname{width} G$,

(ii) $\operatorname{hstrip}(G,i) = \{[r,s] : (G_{1,i})_{\mathbf{2}} \leq s\}$ if $i \geq \operatorname{width} G$,

(iii) $\operatorname{hstrip}(G,i) = \{[r,s] : s \leq (G_{1,i+1})_{\mathbf{2}}\}$, otherwise.

We now state a number of propositions:

(2) If $1 \leq j$ and $j \leq \operatorname{width} G$ and $1 \leq i$ and $i \leq \operatorname{len} G$, then $(G_{i,j})_{\mathbf{2}} = (G_{1,j})_{\mathbf{2}}$.

(3) If $1 \leq j$ and $j \leq \operatorname{width} G$ and $1 \leq i$ and $i \leq \operatorname{len} G$, then $(G_{i,j})_{\mathbf{1}} = (G_{i,1})_{\mathbf{1}}$.

(4) If $1 \leq j$ and $j \leq \operatorname{width} G$ and $1 \leq i_1$ and $i_1 < i_2$ and $i_2 \leq \operatorname{len} G$, then $(G_{i_1,j})_{\mathbf{1}} < (G_{i_2,j})_{\mathbf{1}}$.

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(5) If $1 \leq j_1$ and $j_1 < j_2$ and $j_2 \leq \operatorname{width} G$ and $1 \leq i$ and $i \leq \operatorname{len} G$, then $(G_{i,j_1})_{\mathbf{2}} < (G_{i,j_2})_{\mathbf{2}}$.

(6) If $1 \leq j$ and $j < \operatorname{width} G$ and $1 \leq i$ and $i \leq \operatorname{len} G$, then $\operatorname{hstrip}(G, j) = \{[r, s] : (G_{i,j})_{\mathbf{2}} \leq s \ \wedge \ s \leq (G_{i,j+1})_{\mathbf{2}}\}$.

(7) If $1 \leq i$ and $i \leq \operatorname{len} G$, then $\operatorname{hstrip}(G, \operatorname{width} G) = \{[r, s] : (G_{i,\operatorname{width} G})_{\mathbf{2}} \leq s\}$.

(8) If $1 \leq i$ and $i \leq \operatorname{len} G$, then $\operatorname{hstrip}(G, 0) = \{[r, s] : s \leq (G_{i,1})_{\mathbf{2}}\}$.

(9) If $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$, then $\operatorname{vstrip}(G, i) = \{[r, s] : (G_{i,j})_{\mathbf{1}} \leq r \ \wedge \ r \leq (G_{i+1,j})_{\mathbf{1}}\}$.

(10) If $1 \leq j$ and $j \leq \operatorname{width} G$, then $\operatorname{vstrip}(G, \operatorname{len} G) = \{[r, s] : (G_{\operatorname{len} G,j})_{\mathbf{1}} \leq r\}$.

(11) If $1 \leq j$ and $j \leq \operatorname{width} G$, then $\operatorname{vstrip}(G, 0) = \{[r, s] : r \leq (G_{1,j})_{\mathbf{1}}\}$.

Let G be a Go-board and let us consider i, j. The functor $\operatorname{cell}(G, i, j)$ yields a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def.3) $\operatorname{cell}(G, i, j) = \operatorname{vstrip}(G, i) \cap \operatorname{hstrip}(G, j)$.

A finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is s.c.c. if:

(Def.4) For all i, j such that $i + 1 < j$ but $i > 1$ and $j < \operatorname{len} \mathrm{it}$ or $j + 1 < \operatorname{len} \mathrm{it}$ holds $\mathcal{L}(\mathrm{it}, i) \cap \mathcal{L}(\mathrm{it}, j) = \emptyset$.

A non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is standard if:

(Def.5) It is a sequence which elements belong to the Go-board of it.

One can verify that there exists a non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is non constant special unfolded circular s.c.c. and standard.

We now state two propositions:

(12) Let f be a standard non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $k \in \operatorname{dom} f$. Then there exist i, j such that $\langle i, j \rangle \in$ the indices of the Go-board of f and $\pi_k f = (\text{the Go-board of } f)_{i,j}$.

(13) Let f be a standard non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Suppose $n \in \operatorname{dom} f$ and $n + 1 \in \operatorname{dom} f$. Let m, k, i, j be natural numbers. Suppose that
 (i) $\langle m, k \rangle \in$ the indices of the Go-board of f,
 (ii) $\langle i, j \rangle \in$ the indices of the Go-board of f,
 (iii) $\pi_n f = (\text{the Go-board of } f)_{m,k}$, and
 (iv) $\pi_{n+1} f = (\text{the Go-board of } f)_{i,j}$.
 Then $|m - i| + |k - j| = 1$.

A special circular sequence is a special unfolded circular s.c.c. non empty finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$.

In the sequel f is a standard special circular sequence.

Let us consider f, k. Let us assume that $1 \leq k$ and $k+1 \leq \operatorname{len} f$. The functor $\operatorname{rightcell}(f, k)$ yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def.6).

(Def.6) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose that
 (i) $\langle i_1, j_1 \rangle \in$ the indices of the Go-board of f,

(ii) $\langle i_2, j_2\rangle \in$ the indices of the Go-board of f,
(iii) $\pi_k f =$ (the Go-board of $f)_{i_1,j_1}$, and
(iv) $\pi_{k+1} f =$ (the Go-board of $f)_{i_2,j_2}$.
Then
(v) $i_1 = i_2$ and $j_1 + 1 = j_2$ and rightcell$(f, k) =$ cell(the Go-board of f, i_1, j_1), or
(vi) $i_1 + 1 = i_2$ and $j_1 = j_2$ and rightcell$(f, k) =$ cell(the Go-board of f, $i_1, j_1 -' 1$), or
(vii) $i_1 = i_2 + 1$ and $j_1 = j_2$ and rightcell$(f, k) =$ cell(the Go-board of f, i_2, j_2), or
(viii) $i_1 = i_2$ and $j_1 = j_2 + 1$ and rightcell$(f, k) =$ cell(the Go-board of f, $i_1 -' 1, j_2$).

The functor leftcell(f, k) yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def.7).

(Def.7) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose that
(i) $\langle i_1, j_1\rangle \in$ the indices of the Go-board of f,
(ii) $\langle i_2, j_2\rangle \in$ the indices of the Go-board of f,
(iii) $\pi_k f =$ (the Go-board of $f)_{i_1,j_1}$, and
(iv) $\pi_{k+1} f =$ (the Go-board of $f)_{i_2,j_2}$.
Then
(v) $i_1 = i_2$ and $j_1 + 1 = j_2$ and leftcell$(f, k) =$ cell(the Go-board of f, $i_1 -' 1, j_1$), or
(vi) $i_1 + 1 = i_2$ and $j_1 = j_2$ and leftcell$(f, k) =$ cell(the Go-board of f, i_1, j_1), or
(vii) $i_1 = i_2 + 1$ and $j_1 = j_2$ and leftcell$(f, k) =$ cell(the Go-board of f, $i_2, j_2 -' 1$), or
(viii) $i_1 = i_2$ and $j_1 = j_2 + 1$ and leftcell$(f, k) =$ cell(the Go-board of f, i_1, j_2).

Next we state a number of propositions:

(14) If $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(G_{i+1,j}, G_{i+1,j+1}) \subseteq$ vstrip(G, i).

(15) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i,j+1}) \subseteq$ vstrip(G, i).

(16) If $j < \operatorname{width} G$ and $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(G_{i,j+1}, G_{i+1,j+1}) \subseteq$ hstrip(G, j).

(17) If $1 \leq j$ and $j \leq \operatorname{width} G$ and $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \subseteq$ hstrip(G, j).

(18) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j + 1 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i,j+1}) \subseteq$ hstrip(G, j).

(19) If $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(G_{i+1,j}, G_{i+1,j+1}) \subseteq$ cell(G, i, j).

(20) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i,j+1}) \subseteq$ cell(G, i, j).

(21) If $1 \leq j$ and $j \leq \operatorname{width} G$ and $1 \leq i$ and $i+1 \leq \operatorname{len} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \subseteq \operatorname{vstrip}(G, i)$.

(22) If $j < \operatorname{width} G$ and $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(G_{i,j+1}, G_{i+1,j+1}) \subseteq \operatorname{cell}(G, i, j)$.

(23) If $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \subseteq \operatorname{cell}(G, i, j)$.

(24) If $i+1 \leq \operatorname{len} G$, then $\operatorname{vstrip}(G, i) \cap \operatorname{vstrip}(G, i+1) = \{q : q_{\mathbf{1}} = (G_{i+1,1})_{\mathbf{1}}\}$.

(25) If $j+1 \leq \operatorname{width} G$, then $\operatorname{hstrip}(G, j) \cap \operatorname{hstrip}(G, j+1) = \{q : q_{\mathbf{2}} = (G_{1,j+1})_{\mathbf{2}}\}$.

(26) For every Go-board G such that $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$ holds $\operatorname{cell}(G, i, j) \cap \operatorname{cell}(G, i+1, j) = \mathcal{L}(G_{i+1,j}, G_{i+1,j+1})$.

(27) For every Go-board G such that $j < \operatorname{width} G$ and $1 \leq i$ and $i < \operatorname{len} G$ holds $\operatorname{cell}(G, i, j) \cap \operatorname{cell}(G, i, j+1) = \mathcal{L}(G_{i,j+1}, G_{i+1,j+1})$.

(28) Suppose that

(i) $1 \leq k$,
(ii) $k+1 \leq \operatorname{len} f$,
(iii) $\langle i+1, j \rangle \in$ the indices of the Go-board of f,
(iv) $\langle i+1, j+1 \rangle \in$ the indices of the Go-board of f,
(v) $\pi_k f =$ (the Go-board of $f)_{i+1,j}$, and
(vi) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$.

Then $\operatorname{leftcell}(f, k) = \operatorname{cell}($the Go-board of f, $i, j)$ and $\operatorname{rightcell}(f, k) = \operatorname{cell}($the Go-board of f, $i+1, j)$.

(29) Suppose that

(i) $1 \leq k$,
(ii) $k+1 \leq \operatorname{len} f$,
(iii) $\langle i, j+1 \rangle \in$ the indices of the Go-board of f,
(iv) $\langle i+1, j+1 \rangle \in$ the indices of the Go-board of f,
(v) $\pi_k f =$ (the Go-board of $f)_{i,j+1}$, and
(vi) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$.

Then $\operatorname{leftcell}(f, k) = \operatorname{cell}($the Go-board of f, $i, j+1)$ and $\operatorname{rightcell}(f, k) = \operatorname{cell}($the Go-board of f, $i, j)$.

(30) Suppose that

(i) $1 \leq k$,
(ii) $k+1 \leq \operatorname{len} f$,
(iii) $\langle i, j+1 \rangle \in$ the indices of the Go-board of f,
(iv) $\langle i+1, j+1 \rangle \in$ the indices of the Go-board of f,
(v) $\pi_k f =$ (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_{k+1} f =$ (the Go-board of $f)_{i,j+1}$.

Then $\operatorname{leftcell}(f, k) = \operatorname{cell}($the Go-board of f, $i, j)$ and $\operatorname{rightcell}(f, k) = \operatorname{cell}($the Go-board of f, $i, j+1)$.

(31) Suppose that

(i) $1 \leq k$,
(ii) $k+1 \leq \operatorname{len} f$,

(iii) $\langle i+1,\, j+1\rangle \in$ the indices of the Go-board of f,
(iv) $\langle i+1,\, j\rangle \in$ the indices of the Go-board of f,
(v) $\pi_k f = (\text{the Go-board of } f)_{i+1,j+1}$, and
(vi) $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$.

Then leftcell$(f,k) =$ cell(the Go-board of f, $i+1, j$) and rightcell$(f,k) =$ cell(the Go-board of f, i, j).

(32) If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then $\operatorname{leftcell}(f,k) \cap \operatorname{rightcell}(f,k) = \mathcal{L}(f,k)$.

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Indexed Category

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Summary. The concept of indexing of a category (a part of indexed category, see [18]) is introduced as a pair formed by a many sorted category and a many sorted functor. The indexing of a category C against to [18] is not a functor but it can be treated as a functor from C into some categorial category (see [1]). The goal of the article is to work out the notation necessary to define institutions (see [13]).

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The articles [23], [25], [11], [24], [26], [4], [5], [19], [9], [7], [22], [20], [21], [15], [16], [14], [3], [6], [12], [8], [2], [10], [17], and [1] provide the notation and terminology for this paper.

1. Category-yielding Functions

Let A be a non empty set. One can check that there exists a many sorted set indexed by A which is non empty yielding.

Let A be a non empty set. One can verify that every many sorted set indexed by A which is non-empty is also non empty yielding.

Let C be a categorial category and let f be a morphism of C. Then $f_{\mathbf{2}}$ is a functor from $f_{\mathbf{1},\mathbf{1}}$ to $f_{\mathbf{1},\mathbf{2}}$.

We now state two propositions:

(1) For every categorial category C and for all morphisms f, g of C such that $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f = \langle\langle \operatorname{dom} f, \operatorname{cod} g\rangle, g_{\mathbf{2}} \cdot f_{\mathbf{2}}\rangle$.

(2) Let C be a category, and let D, E be categorial categories, and let F be a functor from C to D, and let G be a functor from C to E. If $F = G$, then $\operatorname{Obj} F = \operatorname{Obj} G$.

A function is category-yielding if:

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(Def.1) For arbitrary x such that $x \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(x)$ is a category.

Let us note that there exists a function which is category-yielding.

Let X be a set. Observe that there exists a many sorted set indexed by X which is category-yielding.

Let A be a set. A many sorted category indexed by A is a category-yielding many sorted set indexed by A.

Let C be a category. A many sorted set indexed by C is a many sorted set indexed by the objects of C. A many sorted category indexed by C is a many sorted category indexed by the objects of C.

Let X be a set and let x be a category. One can verify that $X \longmapsto x$ is category-yielding.

Let X be a set and let x be a function. One can check that $X \longmapsto x$ is function yielding.

Let X be a non empty set. One can check that every many sorted set indexed by X is non empty.

Let f be a non empty function. One can check that $\operatorname{rng} f$ is non empty.

Let f be a category-yielding function. Observe that $\operatorname{rng} f$ is categorial.

Let X be a non empty set, let f be a many sorted category indexed by X, and let x be an element of X. Then $f(x)$ is a category.

Let B be a set, let A be a non empty set, let f be a function from B into A, and let g be a many sorted category indexed by A. Observe that $g \cdot f$ is category-yielding.

Let F be a category-yielding function. The functor $\mathrm{Objs}(F)$ yields a non-empty function and is defined by the conditions (Def.2).

(Def.2) (i) $\operatorname{dom} \mathrm{Objs}(F) = \operatorname{dom} F$, and

(ii) for every set x such that $x \in \operatorname{dom} F$ and for every category C such that $C = F(x)$ holds $(\mathrm{Objs}(F))(x) =$ the objects of C.

The functor $\mathrm{Mphs}(F)$ yields a non-empty function and is defined by the conditions (Def.3).

(Def.3) (i) $\operatorname{dom} \mathrm{Mphs}(F) = \operatorname{dom} F$, and

(ii) for every set x such that $x \in \operatorname{dom} F$ and for every category C such that $C = F(x)$ holds $(\mathrm{Mphs}(F))(x) =$ the morphisms of C.

Let A be a non empty set and let F be a many sorted category indexed by A. Then $\mathrm{Objs}(F)$ is a non-empty many sorted set indexed by A. Then $\mathrm{Mphs}(F)$ is a non-empty many sorted set indexed by A.

The following proposition is true

(3) For every set X and for every category C holds $\mathrm{Objs}(X \longmapsto C) = X \longmapsto$ the objects of C and $\mathrm{Mphs}(X \longmapsto C) = X \longmapsto$ the morphisms of C.

2. Pairs of Many Sorted Sets

Let A, B be sets. Pair of many sorted sets indexed by A and B is defined by:

(Def.4) There exists a many sorted set f indexed by A and there exists a many sorted set g indexed by B such that it $= \langle f, g \rangle$.

Let A, B be sets, let f be a many sorted set indexed by A, and let g be a many sorted set indexed by B. Then $\langle f, g \rangle$ is a pair of many sorted sets indexed by A and B.

Let A, B be sets and let X be a pair of many sorted sets indexed by A and B. Then $X_{\mathbf{1}}$ is a many sorted set indexed by A. Then $X_{\mathbf{2}}$ is a many sorted set indexed by B.

Let A, B be sets. A pair of many sorted sets indexed by A and B is category-yielding on first if:

(Def.5) $\mathrm{it}_{\mathbf{1}}$ is category-yielding.

A pair of many sorted sets indexed by A and B is function-yielding on second if:

(Def.6) $\mathrm{it}_{\mathbf{2}}$ is function yielding.

Let A, B be sets. One can check that there exists a pair of many sorted sets indexed by A and B which is category-yielding on first and function-yielding on second.

Let A, B be sets and let X be a category-yielding on first pair of many sorted sets indexed by A and B. Then $X_{\mathbf{1}}$ is a many sorted category indexed by A.

Let A, B be sets and let X be a function-yielding on second pair of many sorted sets indexed by A and B. Then $X_{\mathbf{2}}$ is a many sorted function of B.

Let f be a function yielding function. One can check that $\mathrm{rng}\, f$ is functional.

Let A, B be sets, let f be a many sorted category indexed by A, and let g be a many sorted function of B. Then $\langle f, g \rangle$ is a category-yielding on first function-yielding on second pair of many sorted sets indexed by A and B.

Let A be a non empty set and let F, G be many sorted categories indexed by A. A many sorted function of A is called a many sorted functor from F to G if:

(Def.7) For every element a of A holds $\mathrm{it}(a)$ is a functor from $F(a)$ to $G(a)$.

The scheme *LambdaMSFr* deals with a non empty set $\mathcal{A}$, many sorted categories $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a many sorted functor F from $\mathcal{B}$ to $\mathcal{C}$ such that for every element a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$

provided the parameters meet the following requirement:

- For every element a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is a functor from $\mathcal{B}(a)$ to $\mathcal{C}(a)$.

Let A be a non empty set, let F, G be many sorted categories indexed by A, let f be a many sorted functor from F to G, and let a be an element of A. Then $f(a)$ is a functor from $F(a)$ to $G(a)$.

3. Indexing

Let A, B be non empty sets and let F, G be functions from B into A. A category-yielding on first pair of many sorted sets indexed by A and B is said to be an indexing of F and G if:

(Def.8) $\mathrm{it}_{\mathbf{2}}$ is a many sorted functor from $\mathrm{it}_{\mathbf{1}} \cdot F$ to $\mathrm{it}_{\mathbf{1}} \cdot G$.

Next we state two propositions:

(4) Let A, B be non empty sets, and let F, G be functions from B into A, and let I be an indexing of F and G, and let m be an element of B. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(F(m))$ to $I_{\mathbf{1}}(G(m))$.

(5) Let C be a category, and let I be an indexing of the dom-map of C and the cod-map of C, and let m be a morphism of C. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(\operatorname{dom} m)$ to $I_{\mathbf{1}}(\operatorname{cod} m)$.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. Then $I_{\mathbf{2}}$ is a many sorted functor from $I_{\mathbf{1}} \cdot F$ to $I_{\mathbf{1}} \cdot G$.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. A categorial category is called a target category of I if it satisfies the conditions (Def.9).

(Def.9) (i) For every element a of A holds $I_{\mathbf{1}}(a)$ is an object of it, and

(ii) for every element b of B holds $\langle\langle I_{\mathbf{1}}(F(b)), I_{\mathbf{1}}(G(b))\rangle, I_{\mathbf{2}}(b)\rangle$ is a morphism of it.

Let A, B be non empty sets, let F, G be functions from B into A, and let I be an indexing of F and G. One can verify that there exists a target category of I which is full and strict.

Let A, B be non empty sets, let F, G be functions from B into A, let c be a partial function from $[\!:B, B:\!]$ to B, and let i be a function from A into B. Let us assume that there exists a category C such that $C = \langle A, B, F, G, c, i\rangle$. An indexing of F and G is called an indexing of F, G, c and i if it satisfies the conditions (Def.10).

(Def.10) (i) For every element a of A holds $\mathrm{it}_{\mathbf{2}}(i(a)) = \mathrm{id}_{\mathrm{it}_{\mathbf{1}}(a)}$, and

(ii) for all elements m_1, m_2 of B such that $F(m_2) = G(m_1)$ holds $\mathrm{it}_{\mathbf{2}}(c(\langle m_2, m_1\rangle)) = \mathrm{it}_{\mathbf{2}}(m_2) \cdot \mathrm{it}_{\mathbf{2}}(m_1)$.

Let C be a category. An indexing of C is an indexing of the dom-map of C, the cod-map of C, the composition of C and the id-map of C. A coindexing of C is an indexing of the cod-map of C, the dom-map of C, $\curvearrowleft$(the composition of C) and the id-map of C.

One can prove the following propositions:

(6) Let C be a category and let I be an indexing of the dom-map of C and the cod-map of C. Then I is an indexing of C if and only if the following conditions are satisfied:

(i) for every object a of C holds $I_{\mathbf{2}}(\mathrm{id}_a) = \mathrm{id}_{I_{\mathbf{1}}(a)}$, and

(ii) for all morphisms m_1, m_2 of C such that $\operatorname{dom} m_2 = \operatorname{cod} m_1$ holds $I_{\mathbf{2}}(m_2 \cdot m_1) = I_{\mathbf{2}}(m_2) \cdot I_{\mathbf{2}}(m_1)$.

(7) Let C be a category and let I be an indexing of the cod-map of C and the dom-map of C. Then I is a coindexing of C if and only if the following conditions are satisfied:

(i) for every object a of C holds $I_{\mathbf{2}}(\mathrm{id}_a) = \mathrm{id}_{I_{\mathbf{1}}(a)}$, and

(ii) for all morphisms m_1, m_2 of C such that $\operatorname{dom} m_2 = \operatorname{cod} m_1$ holds $I_{\mathbf{2}}(m_2 \cdot m_1) = I_{\mathbf{2}}(m_1) \cdot I_{\mathbf{2}}(m_2)$.

(8) For every category C and for every set x holds x is a coindexing of C iff x is an indexing of C^{op}.

(9) Let C be a category, and let I be an indexing of C, and let c_1, c_2 be objects of C. Suppose $\hom(c_1, c_2)$ is non empty. Let m be a morphism from c_1 to c_2. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(c_1)$ to $I_{\mathbf{1}}(c_2)$.

(10) Let C be a category, and let I be a coindexing of C, and let c_1, c_2 be objects of C. Suppose $\hom(c_1, c_2)$ is non empty. Let m be a morphism from c_1 to c_2. Then $I_{\mathbf{2}}(m)$ is a functor from $I_{\mathbf{1}}(c_2)$ to $I_{\mathbf{1}}(c_1)$.

Let C be a category, let I be an indexing of C, and let T be a target category of I. The functor I-functor(C, T) yielding a functor from C to T is defined as follows:

(Def.11) For every morphism f of C holds $(I\text{-functor}(C,T))(f) = \langle\langle I_{\mathbf{1}}(\operatorname{dom} f), I_{\mathbf{1}}(\operatorname{cod} f)\rangle, I_{\mathbf{2}}(f)\rangle$.

We now state three propositions:

(11) Let C be a category, and let I be an indexing of C, and let T_1, T_2 be target categories of I. Then $I\text{-functor}(C, T_1) = I\text{-functor}(C, T_2)$ and $\mathrm{Obj}(I\text{-functor}(C, T_1)) = \mathrm{Obj}(I\text{-functor}(C, T_2))$.

(12) For every category C and for every indexing I of C and for every target category T of I holds $\mathrm{Obj}(I\text{-functor}(C, T)) = I_{\mathbf{1}}$.

(13) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let x be an object of C. Then $(I\text{-functor}(C, T))(x) = I_{\mathbf{1}}(x)$.

Let C be a category and let I be an indexing of C. The functor $\operatorname{rng} I$ yielding a strict target category of I is defined by:

(Def.12) For every target category T of I holds $\operatorname{rng} I = \mathrm{Im}(I\text{-functor}(C, T))$.

Next we state the proposition

(14) Let C be a category, and let I be an indexing of C, and let D be a categorial category. Then $\operatorname{rng} I$ is a subcategory of D if and only if D is a target category of I.

Let C be a category, let I be an indexing of C, and let m be a morphism of C. The functor $I(m)$ yielding a functor from $I_{\mathbf{1}}(\operatorname{dom} m)$ to $I_{\mathbf{1}}(\operatorname{cod} m)$ is defined by:

(Def.13) $I(m) = I_{\mathbf{2}}(m)$.

Let C be a category, let I be a coindexing of C, and let m be a morphism of C. The functor $I(m)$ yielding a functor from $I_{\mathbf{1}}(\operatorname{cod} m)$ to $I_{\mathbf{1}}(\operatorname{dom} m)$ is defined as follows:

(Def.14) $I(m) = I_{\mathbf{2}}(m)$.

The following proposition is true

(15) Let C, D be categories. Then
(i) $\langle$(the objects of C) $\longmapsto (D)$, (the morphisms of C) $\longmapsto \mathrm{id}_D\rangle$ is an indexing of C, and
(ii) $\langle$(the objects of C) $\longmapsto (D)$, (the morphisms of C) $\longmapsto \mathrm{id}_D\rangle$ is a coindexing of C.

4. Indexing vs Functors

Let A be a set and let B be a non empty set. We see that the function from A into B is a many sorted set indexed by A.

Let C, D be categories and let F be a function from the morphisms of C into the morphisms of D. Then $\operatorname{Obj} F$ is a function from the objects of C into the objects of D.

Let C be a category, let D be a categorial category, and let F be a functor from C to D. Note that $\operatorname{Obj} F$ is category-yielding.

Let C be a category, let D be a categorial category, and let F be a functor from C to D. Then $\mathrm{pr2}(F)$ is a many sorted functor from $\operatorname{Obj} F \cdot$ (the dom-map of C) to $\operatorname{Obj} F \cdot$ (the cod-map of C).

Next we state the proposition

(16) Let C be a category, and let D be a categorial category, and let F be a functor from C to D. Then $\langle \operatorname{Obj} F, \mathrm{pr2}(F)\rangle$ is an indexing of C.

Let C be a category, let D be a categorial category, and let F be a functor from C to D. The functor F-indexing of C yields an indexing of C and is defined by:

(Def.15) F-indexing of $C = \langle \operatorname{Obj} F, \mathrm{pr2}(F)\rangle$.

One can prove the following propositions:

(17) Let C be a category, and let D be a categorial category, and let F be a functor from C to D. Then D is a target category of F-indexing of C.

(18) Let C be a category, and let D be a categorial category, and let F be a functor from C to D, and let T be a target category of F-indexing of C. Then $F = F$-indexing of C-functor(C, T).

(19) Let C be a category, and let D, E be categorial categories, and let F be a functor from C to D, and let G be a functor from C to E. If $F = G$, then F-indexing of $C = G$-indexing of C.

(20) For every category C and for every indexing I of C and for every target category T of I holds $\mathrm{pr2}(I\text{-functor}(C, T)) = I_{\mathbf{2}}$.

(21) For every category C and for every indexing I of C and for every target category T of I holds $(I\text{-functor}(C,T))$-indexing of $C = I$.

5. Composing Indexings and Functors

Let C, D, E be categories, let F be a functor from C to D, and let I be an indexing of E. Let us assume that $\operatorname{Im} F$ is a subcategory of E. The functor $I \cdot F$ yielding an indexing of C is defined by:

(Def.16) For every functor F' from C to E such that $F' = F$ holds $I \cdot F = ((I\text{-functor}(E, \operatorname{rng} I)) \cdot F')$-indexing of C.

Next we state several propositions:

(22) Let C, D_1, D_2, E be categories, and let I be an indexing of E, and let F be a functor from C to D_1, and let G be a functor from C to D_2. Suppose $\operatorname{Im} F$ is a subcategory of E and $\operatorname{Im} G$ is a subcategory of E and $F = G$. Then $I \cdot F = I \cdot G$.

(23) Let C, D be categories, and let F be a functor from C to D, and let I be an indexing of D, and let T be a target category of I. Then $I \cdot F = ((I\text{-functor}(D,T)) \cdot F)$-indexing of C.

(24) Let C, D be categories, and let F be a functor from C to D, and let I be an indexing of D. Then every target category of I is a target category of $I \cdot F$.

(25) Let C, D be categories, and let F be a functor from C to D, and let I be an indexing of D, and let T be a target category of I. Then $\operatorname{rng}(I \cdot F)$ is a subcategory of T.

(26) Let C, D, E be categories, and let F be a functor from C to D, and let G be a functor from D to E, and let I be an indexing of E. Then $(I \cdot G) \cdot F = I \cdot (G \cdot F)$.

Let C be a category, let I be an indexing of C, and let D be a categorial category. Let us assume that D is a target category of I. Let E be a categorial category and let F be a functor from D to E. The functor $F \cdot I$ yielding an indexing of C is defined as follows:

(Def.17) For every target category T of I and for every functor G from T to E such that $T = D$ and $G = F$ holds $F \cdot I = (G \cdot (I\text{-functor}(C,T)))$-indexing of C.

One can prove the following propositions:

(27) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D, E be categorial categories, and let F be a functor from T to D, and let G be a functor from T to E. If $F = G$, then $F \cdot I = G \cdot I$.

(28) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D be a categorial category, and let F be a functor from T to D. Then $\operatorname{Im} F$ is a target category of $F \cdot I$.

(29) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D be a categorial category, and let F be a functor from T to D. Then D is a target category of $F \cdot I$.

(30) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D be a categorial category, and let F be a functor from T to D. Then $\mathrm{rng}(F \cdot I)$ is a subcategory of $\mathrm{Im}\,F$.

(31) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D, E be categorial categories, and let F be a functor from T to D, and let G be a functor from D to E. Then $(G \cdot F) \cdot I = G \cdot (F \cdot I)$.

Let C, D be categories, let I_1 be an indexing of C, and let I_2 be an indexing of D. The functor $I_2 \cdot I_1$ yielding an indexing of C is defined as follows:

(Def.18) $I_2 \cdot I_1 = I_2 \cdot (I_1\text{-functor}(C, \mathrm{rng}\, I_1))$.

We now state several propositions:

(32) Let C be a category, and let D be a categorial category, and let I_1 be an indexing of C, and let I_2 be an indexing of D, and let T be a target category of I_1. If D is a target category of I_1, then $I_2 \cdot I_1 = I_2 \cdot (I_1\text{-functor}(C, T))$.

(33) Let C be a category, and let D be a categorial category, and let I_1 be an indexing of C, and let I_2 be an indexing of D, and let T be a target category of I_2. If D is a target category of I_1, then $I_2 \cdot I_1 = (I_2\text{-functor}(D, T)) \cdot I_1$.

(34) Let C, D be categories, and let F be a functor from C to D, and let I be an indexing of D, and let T be a target category of I, and let E be a categorial category, and let G be a functor from T to E. Then $(G \cdot I) \cdot F = G \cdot (I \cdot F)$.

(35) Let C be a category, and let I be an indexing of C, and let T be a target category of I, and let D be a categorial category, and let F be a functor from T to D, and let J be an indexing of D. Then $(J \cdot F) \cdot I = J \cdot (F \cdot I)$.

(36) Let C be a category, and let I be an indexing of C, and let T_1 be a target category of I, and let J be an indexing of T_1, and let T_2 be a target category of J, and let D be a categorial category, and let F be a functor from T_2 to D. Then $(F \cdot J) \cdot I = F \cdot (J \cdot I)$.

(37) Let C, D be categories, and let F be a functor from C to D, and let I be an indexing of D, and let T be a target category of I, and let J be an indexing of T. Then $(J \cdot I) \cdot F = J \cdot (I \cdot F)$.

(38) Let C be a category, and let I be an indexing of C, and let D be a target category of I, and let J be an indexing of D, and let E be a target category of J, and let K be an indexing of E. Then $(K \cdot J) \cdot I = K \cdot (J \cdot I)$.

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Associated Matrix of Linear Map

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The notation and terminology used in this paper are introduced in the following articles: [13], [2], [11], [17], [18], [33], [21], [32], [3], [34], [8], [9], [4], [14], [15], [35], [36], [23], [31], [16], [30], [26], [24], [12], [29], [19], [27], [1], [7], [25], [6], [10], [5], [22], [28], and [20].

1. Preliminaries

For simplicity we follow the rules: k, t, i, j, m, n are natural numbers, x is arbitrary, A is a set, and D is a non empty set.

We now state two propositions:

(1) For every finite sequence p of elements of D and for every i holds $p_{\restriction i}$ is a finite sequence of elements of D.

(2) For every i and for every finite sequence p holds $\operatorname{rng}(p_{\restriction i}) \subseteq \operatorname{rng} p$.

Let D be a non empty set. A matrix over D is a tabular finite sequence of elements of D^*.

Let K be a field. A matrix over K is a matrix over the carrier of K.

Let D be a non empty set, let us consider k, and let M be a matrix over D. Then $M_{\restriction k}$ is a matrix over D.

Next we state four propositions:

(3) For every finite sequence M of elements of D such that $\operatorname{len} M = n + 1$ holds $\operatorname{len}(M_{\restriction n+1}) = n$.

(4) Let M be a matrix over D of dimension $n + 1 \times m$ and let M_1 be a matrix over D. Then if $n > 0$, then $\operatorname{width} M = \operatorname{width}(M_{\restriction n+1})$ and if $M_1 = \langle M(n + 1)\rangle$, then $\operatorname{width} M = \operatorname{width} M_1$.

(5) For every matrix M over D of dimension $n + 1 \times m$ holds $M_{\restriction n+1}$ is a matrix over D of dimension $n \times m$.

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(6) For every finite sequence M of elements of D such that $\operatorname{len} M = n+1$ holds $M = (M{\restriction}_{\operatorname{len} M}) \frown \langle M(\operatorname{len} M)\rangle$.

Let us consider D and let P be a finite sequence of elements of D. Then $\langle P\rangle$ is a matrix over D of dimension $1 \times \operatorname{len} P$.

2. More on Finite Sequence

One can prove the following propositions:

(7) For every set A and for every finite sequence F holds $(\operatorname{Sgm}(F^{-1}\,A)) \frown \operatorname{Sgm}(F^{-1}\,(\operatorname{rng} F \setminus A))$ is a permutation of $\operatorname{dom} F$.

(8) Let F be a finite sequence and let A be a subset of $\operatorname{rng} F$. Suppose F is one-to-one. Then there exists a permutation p of $\operatorname{dom} F$ such that $(F - A^{\mathrm{c}}) \frown (F - A) = F \cdot p$.

A function is finite sequence yielding if:

(Def.1) For every x such that $x \in \operatorname{dom}$ it holds $\operatorname{it}(x)$ is a finite sequence.

Let us observe that there exists a function which is finite sequence yielding.

Let F, G be finite sequence yielding functions. The functor $F \frown G$ yields a finite sequence yielding function and is defined by the conditions (Def.2).

(Def.2) (i) $\operatorname{dom}(F \frown G) = \operatorname{dom} F \cap \operatorname{dom} G$, and

(ii) for arbitrary i such that $i \in \operatorname{dom}(F \frown G)$ and for all finite sequences f, g such that $f = F(i)$ and $g = G(i)$ holds $(F \frown G)(i) = f \frown g$.

3. Matrices and Finite Sequences in Vector Space

For simplicity we adopt the following convention: K denotes a field, V denotes a vector space over K, a denotes an element of the carrier of K, W denotes an element of the carrier of V, K_1, K_2, K_3 denote linear combinations of V, and X denotes a subset of the carrier of V.

Next we state four propositions:

(9) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $\sum K_1 = \sum K_2$, then $K_1 = K_2$.

(10) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $\operatorname{support} K_3 \subseteq X$ and $\sum K_1 = \sum K_2 + \sum K_3$, then $K_1 = K_2 + K_3$.

(11) If X is linearly independent and $\operatorname{support} K_1 \subseteq X$ and $\operatorname{support} K_2 \subseteq X$ and $a \neq 0_K$ and $\sum K_1 = a \cdot \sum K_2$, then $K_1 = a \cdot K_2$.

(12) For every basis b_2 of V there exists a linear combination K_4 of V such that $W = \sum K_4$ and $\operatorname{support} K_4 \subseteq b_2$.

Let K be a field and let V be a vector space over K. We say that V is finite dimensional if and only if:

(Def.3) There exists finite subset of the carrier of V which is a basis of V.

Let K be a field. Note that there exists a vector space over K which is strict and finite dimensional.

Let K be a field and let V be a finite dimensional vector space over K. A finite sequence of elements of the carrier of V is called an ordered basis of V if:

(Def.4) It is one-to-one and rng it is a basis of V.

For simplicity we adopt the following convention: p will denote a finite sequence, M_1 will denote a matrix over D of dimension $n \times m$, M_2 will denote a matrix over D of dimension $k \times m$, V_1, V_2, V_3 will denote finite dimensional vector spaces over K, f, f_1, f_2 will denote maps from V_1 into V_2, g will denote a map from V_2 into V_3, b_1 will denote an ordered basis of V_1, b_2 will denote an ordered basis of V_2, b_3 will denote an ordered basis of V_3, b will denote a basis of V_1, v_1, v_2 will denote vectors of V_2, v will denote an element of the carrier of V_1, p_2, F will denote finite sequences of elements of the carrier of V_1, p_1, d will denote finite sequences of elements of the carrier of K, and K_4 will denote a linear combination of V_1.

Let us consider K, let us consider V_1, V_2, and let us consider f_1, f_2. The functor $f_1 + f_2$ yielding a map from V_1 into V_2 is defined as follows:

(Def.5) For every element v of the carrier of V_1 holds $(f_1+f_2)(v) = f_1(v)+f_2(v)$.

Let us consider K, let us consider V_1, V_2, let us consider f, and let a be an element of the carrier of K. The functor $a \cdot f$ yielding a map from V_1 into V_2 is defined as follows:

(Def.6) For every element v of the carrier of V_1 holds $(a \cdot f)(v) = a \cdot f(v)$.

The following propositions are true:

(13) Let a be an element of the carrier of V_1, and let F be a finite sequence of elements of the carrier of V_1, and let G be a finite sequence of elements of the carrier of K. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every k and for every element v of the carrier of K such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = v \cdot a$. Then $\sum F = \sum G \cdot a$.

(14) Let a be an element of the carrier of V_1, and let F be a finite sequence of elements of the carrier of K, and let G be a finite sequence of elements of the carrier of V_1. If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{dom} F$ holds $G(k) = \pi_k F \cdot a$, then $\sum G = \sum F \cdot a$.

(15) If for every k such that $k \in \operatorname{dom} F$ holds $\pi_k F = 0_{(V_1)}$, then $\sum F = 0_{(V_1)}$.

Let us consider K, let us consider V_1, and let us consider p_1, p_2. The functor $\operatorname{lmlt}(p_1, p_2)$ yielding a finite sequence of elements of the carrier of V_1 is defined as follows:

(Def.7) $\operatorname{lmlt}(p_1, p_2) = (\text{the left multiplication of } V_1)^{\circ}(p_1, p_2)$.

Next we state the proposition

(16) If $\operatorname{dom} p_1 = \operatorname{dom} p_2$, then $\operatorname{dom} \operatorname{lmlt}(p_1, p_2) = \operatorname{dom} p_1$ and $\operatorname{dom} \operatorname{lmlt}(p_1, p_2) = \operatorname{dom} p_2$.

Let us consider K, let us consider V_1, and let M be a matrix over the carrier of V_1. The functor $\sum M$ yields a finite sequence of elements of the carrier of V_1 and is defined as follows:

(Def.8) $\operatorname{len} \sum M = \operatorname{len} M$ and for every k such that $k \in \operatorname{dom} \sum M$ holds $\pi_k \sum M = \sum \operatorname{Line}(M, k)$.

The following propositions are true:

(17) For every matrix M over the carrier of V_1 such that $\operatorname{len} M = 0$ holds $\sum \sum M = 0_{(V_1)}$.

(18) For every matrix M over the carrier of V_1 of dimension $m+1 \times 0$ holds $\sum \sum M = 0_{(V_1)}$.

(19) For every element x of the carrier of V_1 holds $\langle\langle x \rangle\rangle = \langle\langle x \rangle\rangle^{\mathrm{T}}$.

(20) For every finite sequence p of elements of the carrier of V_1 such that f is linear holds $f(\sum p) = \sum(f \cdot p)$.

(21) Let a be a finite sequence of elements of the carrier of K and let p be a finite sequence of elements of the carrier of V_1. If $\operatorname{len} p = \operatorname{len} a$, then if f is linear, then $f \cdot \operatorname{lmlt}(a, p) = \operatorname{lmlt}(a, f \cdot p)$.

(22) Let a be a finite sequence of elements of the carrier of K. If $\operatorname{len} a = \operatorname{len} b_2$, then if g is linear, then $g(\sum \operatorname{lmlt}(a, b_2)) = \sum \operatorname{lmlt}(a, g \cdot b_2)$.

(23) Let F, F_1 be finite sequences of elements of the carrier of V_1, and let K_4 be a linear combination of V_1, and let p be a permutation of $\operatorname{dom} F$. If $F_1 = F \cdot p$, then $K_4\, F_1 = (K_4\, F) \cdot p$.

(24) If F is one-to-one and $\operatorname{support} K_4 \subseteq \operatorname{rng} F$, then $\sum(K_4\, F) = \sum K_4$.

(25) Let A be a set and let p be a finite sequence of elements of the carrier of V_1. Suppose $\operatorname{rng} p \subseteq A$. Suppose f_1 is linear and f_2 is linear and for every v such that $v \in A$ holds $f_1(v) = f_2(v)$. Then $f_1(\sum p) = f_2(\sum p)$.

(26) If f_1 is linear and f_2 is linear, then for every ordered basis b_1 of V_1 such that $\operatorname{len} b_1 > 0$ holds if $f_1 \cdot b_1 = f_2 \cdot b_1$, then $f_1 = f_2$.

Let D be a non empty set. Observe that every matrix over D is finite sequence yielding.

Let D be a non empty set and let F, G be matrices over D. Then $F \frown G$ is a matrix over D.

Let D be a non empty set, let us consider n, m, k, let M_1 be a matrix over D of dimension $n \times k$, and let M_2 be a matrix over D of dimension $m \times k$. Then $M_1 \frown M_2$ is a matrix over D of dimension $n + m \times k$.

One can prove the following propositions:

(27) Given i, and let M_1 be a matrix over D of dimension $n \times k$, and let M_2 be a matrix over D of dimension $m \times k$. If $i \in \operatorname{dom} M_1$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_1, i)$.

(28) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\operatorname{width} M_1 = \operatorname{width} M_2$, then $\operatorname{width}(M_1 \frown M_2) = \operatorname{width} M_1$ and $\operatorname{width}(M_1 \frown M_2) = \operatorname{width} M_2$.

(29) Given i, n, and let M_1 be a matrix over D of dimension $t \times k$, and let M_2 be a matrix over D of dimension $m \times k$. If $n \in \operatorname{dom} M_2$ and $i = \operatorname{len} M_1 + n$, then $\operatorname{Line}(M_1 \frown M_2, i) = \operatorname{Line}(M_2, n)$.

(30) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$, then for every i such that $i \in \mathrm{Seg}\, \mathrm{width}\, M_1$ holds $(M_1 \frown M_2)_{\square,i} = ((M_1)_{\square,i}) \frown ((M_2)_{\square,i})$.

(31) Let M_1 be a matrix over the carrier of V_1 of dimension $n \times k$ and let M_2 be a matrix over the carrier of V_1 of dimension $m \times k$. Then $\sum(M_1 \frown M_2) = (\sum M_1) \frown \sum M_2$.

(32) Let M_1 be a matrix over D of dimension $n \times k$ and let M_2 be a matrix over D of dimension $m \times k$. If $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$, then $(M_1 \frown M_2)^{\mathrm{T}} = ({M_1}^{\mathrm{T}}) \frown {M_2}^{\mathrm{T}}$.

(33) For all matrices M_1, M_2 over the carrier of V_1 holds (the addition of $V_1)^\circ(\sum M_1, \sum M_2) = \sum(M_1 \frown M_2)$.

Let D be a non empty set, let F be a binary operation on D, and let P_1, P_2 be finite sequences of elements of D. Then $F^\circ(P_1, P_2)$ is a finite sequence of elements of D.

Next we state several propositions:

(34) Let P_1, P_2 be finite sequences of elements of the carrier of V_1. If $\mathrm{len}\, P_1 = \mathrm{len}\, P_2$, then $\sum((\text{the addition of } V_1)^\circ(P_1, P_2)) = \sum P_1 + \sum P_2$.

(35) For all matrices M_1, M_2 over the carrier of V_1 such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ holds $\sum\sum M_1 + \sum\sum M_2 = \sum\sum(M_1 \frown M_2)$.

(36) For every finite sequence P of elements of the carrier of V_1 holds $\sum\sum\langle P\rangle = \sum\sum(\langle P\rangle^{\mathrm{T}})$.

(37) For every n and for every matrix M over the carrier of V_1 such that $\mathrm{len}\, M = n$ holds $\sum\sum M = \sum\sum(M^{\mathrm{T}})$.

(38) Let M be a matrix over the carrier of K of dimension $n \times m$. Suppose $n > 0$ and $m > 0$. Let p, d be finite sequences of elements of the carrier of K. Suppose $\mathrm{len}\, p = n$ and $\mathrm{len}\, d = m$ and for every j such that $j \in \mathrm{dom}\, d$ holds $\pi_j d = \sum(p \bullet M_{\square,j})$. Let b, c be finite sequences of elements of the carrier of V_1. Suppose $\mathrm{len}\, b = m$ and $\mathrm{len}\, c = n$ and for every i such that $i \in \mathrm{dom}\, c$ holds $\pi_i c = \sum \mathrm{lmlt}(\mathrm{Line}(M, i), b)$. Then $\sum \mathrm{lmlt}(p, c) = \sum \mathrm{lmlt}(d, b)$.

4. Decomposition of a Vector in Basis

Let K be a field, let V be a finite dimensional vector space over K, let b_1 be an ordered basis of V, and let W be an element of the carrier of V. The functor $W \rightarrow b_1$ yielding a finite sequence of elements of the carrier of K is defined by the conditions (Def.9).

(Def.9) (i) $\mathrm{len}(W \rightarrow b_1) = \mathrm{len}\, b_1$, and

(ii) there exists a linear combination K_4 of V such that $W = \sum K_4$ and $\mathrm{support}\, K_4 \subseteq \mathrm{rng}\, b_1$ and for every k such that $1 \leq k$ and $k \leq \mathrm{len}(W \rightarrow b_1)$ holds $\pi_k(W \rightarrow b_1) = K_4(\pi_k b_1)$.

The following four propositions are true:

(39) If $v_1 \to b_2 = v_2 \to b_2$, then $v_1 = v_2$.

(40) $v = \sum \mathrm{lmlt}(v \to b_1, b_1)$.

(41) If $\mathrm{len}\, d = \mathrm{len}\, b_1$, then $d = \sum \mathrm{lmlt}(d, b_1) \to b_1$.

(42) Let a be a finite sequence of elements of the carrier of K. Suppose $\mathrm{len}\, a = \mathrm{len}\, b_2$. Let j be a natural number. Suppose $j \in \mathrm{dom}\, b_3$. Let d be a finite sequence of elements of the carrier of K. Suppose $\mathrm{len}\, d = \mathrm{len}\, b_2$ and for every k such that $k \in \mathrm{dom}\, b_2$ holds $d(k) = \pi_j(g(\pi_k b_2) \to b_3)$. If $\mathrm{len}\, b_2 > 0$ and $\mathrm{len}\, b_3 > 0$, then $\pi_j(\sum \mathrm{lmlt}(a, g \cdot b_2) \to b_3) = \sum(a \bullet d)$.

5. Associated Matrix of Linear Map

Let K be a field, let V_1, V_2 be finite dimensional vector spaces over K, let f be a function from the carrier of V_1 into the carrier of V_2, let b_1 be a finite sequence of elements of the carrier of V_1, and let b_2 be an ordered basis of V_2. The functor $\mathrm{AutMt}(f, b_1, b_2)$ yielding a matrix over K is defined as follows:

(Def.10) $\mathrm{len}\, \mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_1$ and for every k such that $k \in \mathrm{dom}\, b_1$ holds $\pi_k\, \mathrm{AutMt}(f, b_1, b_2) = f(\pi_k b_1) \to b_2$.

One can prove the following propositions:

(43) If $\mathrm{len}\, b_1 = 0$, then $\mathrm{AutMt}(f, b_1, b_2) = \varepsilon$.

(44) If $\mathrm{len}\, b_1 > 0$, then $\mathrm{width}\, \mathrm{AutMt}(f, b_1, b_2) = \mathrm{len}\, b_2$.

(45) If f_1 is linear and f_2 is linear, then if $\mathrm{AutMt}(f_1, b_1, b_2) = \mathrm{AutMt}(f_2, b_1, b_2)$ and $\mathrm{len}\, b_1 > 0$, then $f_1 = f_2$.

(46) If f is linear and g is linear and $\mathrm{len}\, b_1 > 0$ and $\mathrm{len}\, b_2 > 0$ and $\mathrm{len}\, b_3 > 0$, then $\mathrm{AutMt}(g \cdot f, b_1, b_3) = \mathrm{AutMt}(f, b_1, b_2) \cdot \mathrm{AutMt}(g, b_2, b_3)$.

(47) $\mathrm{AutMt}(f_1 + f_2, b_1, b_2) = \mathrm{AutMt}(f_1, b_1, b_2) + \mathrm{AutMt}(f_2, b_1, b_2)$.

(48) If $a \neq 0_K$, then $\mathrm{AutMt}(a \cdot f, b_1, b_2) = a \cdot \mathrm{AutMt}(f, b_1, b_2)$.

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On the Geometry of a Go-Board

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The articles [15], [17], [7], [1], [14], [16], [12], [4], [2], [8], [9], [13], [18], [3], [5], [6], [10], and [11] provide the notation and terminology for this paper.

For simplicity we follow the rules: i, j, n will be natural numbers, r, s, r_1, s_1, r_2, s_2 will be real numbers, p will be a point of $\mathcal{E}^2_{\mathrm{T}}$, G will be a Go-board, M will be a metric space, and u will be a point of $\mathcal{E}^2$.

One can prove the following propositions:

(4)[1] For every metric space M and for every point u of M such that $r > 0$ holds $u \in \mathrm{Ball}(u, r)$.

(6)[2] For every subset B of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and for every point u of $\mathcal{E}^n$ such that $B = \mathrm{Ball}(u, r)$ holds B is open.

(7) Let M be a metric space, and let u be a point of M, and let P be a subset of the carrier of M_{top}. Then $u \in \mathrm{Int}\, P$ if and only if there exists r such that $r > 0$ and $\mathrm{Ball}(u, r) \subseteq P$.

(8) Let u be a point of $\mathcal{E}^n$ and let P be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. Then $u \in \mathrm{Int}\, P$ if and only if there exists r such that $r > 0$ and $\mathrm{Ball}(u, r) \subseteq P$.

(9) For all points u, v of $\mathcal{E}^2$ such that $u = [r_1, s_1]$ and $v = [r_2, s_2]$ holds $\rho(u, v) = \sqrt{(r_1 - r_2)^{\mathbf{2}} + (s_1 - s_2)^{\mathbf{2}}}$.

(10) For every point u of $\mathcal{E}^2$ such that $u = [r, s]$ holds if $0 \leq r_2$ and $r_2 < r_1$, then $[r + r_2, s] \in \mathrm{Ball}(u, r_1)$.

(11) For every point u of $\mathcal{E}^2$ such that $u = [r, s]$ holds if $0 \leq s_2$ and $s_2 < s_1$, then $[r, s + s_2] \in \mathrm{Ball}(u, s_1)$.

(12) For every point u of $\mathcal{E}^2$ such that $u = [r, s]$ holds if $0 \leq r_2$ and $r_2 < r_1$, then $[r - r_2, s] \in \mathrm{Ball}(u, r_1)$.

[1]The propositions (1)–(3) have been removed.
[2]The proposition (5) has been removed.

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(13) For every point u of $\mathcal{E}^2$ such that $u = [r, s]$ holds if $0 \leq s_2$ and $s_2 < s_1$, then $[r, s - s_2] \in \mathrm{Ball}(u, s_1)$.

(14) If $1 \leq i$ and $i < \mathrm{len}\, G$ and $1 \leq j$ and $j < \mathrm{width}\, G$, then $G_{i,j} + G_{i+1,j+1} = G_{i,j+1} + G_{i+1,j}$.

(15) $\mathrm{Int}\, \mathrm{vstrip}(G, 0) = \{[r, s] : r < (G_{1,1})_{\mathbf{1}}\}$.

(16) $\mathrm{Int}\, \mathrm{vstrip}(G, \mathrm{len}\, G) = \{[r, s] : (G_{\mathrm{len}\, G,1})_{\mathbf{1}} < r\}$.

(17) If $1 \leq i$ and $i < \mathrm{len}\, G$, then $\mathrm{Int}\, \mathrm{vstrip}(G, i) = \{[r, s] : (G_{i,1})_{\mathbf{1}} < r \ \wedge\ r < (G_{i+1,1})_{\mathbf{1}}\}$.

(18) $\mathrm{Int}\, \mathrm{hstrip}(G, 0) = \{[r, s] : s < (G_{1,1})_{\mathbf{2}}\}$.

(19) $\mathrm{Int}\, \mathrm{hstrip}(G, \mathrm{width}\, G) = \{[r, s] : (G_{1,\mathrm{width}\, G})_{\mathbf{2}} < s\}$.

(20) If $1 \leq j$ and $j < \mathrm{width}\, G$, then $\mathrm{Int}\, \mathrm{hstrip}(G, j) = \{[r, s] : (G_{1,j})_{\mathbf{2}} < s \ \wedge\ s < (G_{1,j+1})_{\mathbf{2}}\}$.

(21) $\mathrm{Int}\, \mathrm{cell}(G, 0, 0) = \{[r, s] : r < (G_{1,1})_{\mathbf{1}} \ \wedge\ s < (G_{1,1})_{\mathbf{2}}\}$.

(22) $\mathrm{Int}\, \mathrm{cell}(G, 0, \mathrm{width}\, G) = \{[r, s] : r < (G_{1,1})_{\mathbf{1}} \ \wedge\ (G_{1,\mathrm{width}\, G})_{\mathbf{2}} < s\}$.

(23) If $1 \leq j$ and $j < \mathrm{width}\, G$, then $\mathrm{Int}\, \mathrm{cell}(G, 0, j) = \{[r, s] : r < (G_{1,1})_{\mathbf{1}} \ \wedge\ (G_{1,j})_{\mathbf{2}} < s \ \wedge\ s < (G_{1,j+1})_{\mathbf{2}}\}$.

(24) $\mathrm{Int}\, \mathrm{cell}(G, \mathrm{len}\, G, 0) = \{[r, s] : (G_{\mathrm{len}\, G,1})_{\mathbf{1}} < r \ \wedge\ s < (G_{1,1})_{\mathbf{2}}\}$.

(25) $\mathrm{Int}\, \mathrm{cell}(G, \mathrm{len}\, G, \mathrm{width}\, G) = \{[r, s] : (G_{\mathrm{len}\, G,1})_{\mathbf{1}} < r \ \wedge\ (G_{1,\mathrm{width}\, G})_{\mathbf{2}} < s\}$.

(26) If $1 \leq j$ and $j < \mathrm{width}\, G$, then $\mathrm{Int}\, \mathrm{cell}(G, \mathrm{len}\, G, j) = \{[r, s] : (G_{\mathrm{len}\, G,1})_{\mathbf{1}} < r \ \wedge\ (G_{1,j})_{\mathbf{2}} < s \ \wedge\ s < (G_{1,j+1})_{\mathbf{2}}\}$.

(27) If $1 \leq i$ and $i < \mathrm{len}\, G$, then $\mathrm{Int}\, \mathrm{cell}(G, i, 0) = \{[r, s] : (G_{i,1})_{\mathbf{1}} < r \ \wedge\ r < (G_{i+1,1})_{\mathbf{1}} \ \wedge\ s < (G_{1,1})_{\mathbf{2}}\}$.

(28) If $1 \leq i$ and $i < \mathrm{len}\, G$, then $\mathrm{Int}\, \mathrm{cell}(G, i, \mathrm{width}\, G) = \{[r, s] : (G_{i,1})_{\mathbf{1}} < r \ \wedge\ r < (G_{i+1,1})_{\mathbf{1}} \ \wedge\ (G_{1,\mathrm{width}\, G})_{\mathbf{2}} < s\}$.

(29) If $1 \leq i$ and $i < \mathrm{len}\, G$ and $1 \leq j$ and $j < \mathrm{width}\, G$, then $\mathrm{Int}\, \mathrm{cell}(G, i, j) = \{[r, s] : (G_{i,1})_{\mathbf{1}} < r \ \wedge\ r < (G_{i+1,1})_{\mathbf{1}} \ \wedge\ (G_{1,j})_{\mathbf{2}} < s \ \wedge\ s < (G_{1,j+1})_{\mathbf{2}}\}$.

(30) If $1 \leq j$ and $j \leq \mathrm{width}\, G$ and $p \in \mathrm{Int}\, \mathrm{hstrip}(G, j)$, then $p_{\mathbf{2}} > (G_{1,j})_{\mathbf{2}}$.

(31) If $j < \mathrm{width}\, G$ and $p \in \mathrm{Int}\, \mathrm{hstrip}(G, j)$, then $p_{\mathbf{2}} < (G_{1,j+1})_{\mathbf{2}}$.

(32) If $1 \leq i$ and $i \leq \mathrm{len}\, G$ and $p \in \mathrm{Int}\, \mathrm{vstrip}(G, i)$, then $p_{\mathbf{1}} > (G_{i,1})_{\mathbf{1}}$.

(33) If $i < \mathrm{len}\, G$ and $p \in \mathrm{Int}\, \mathrm{vstrip}(G, i)$, then $p_{\mathbf{1}} < (G_{i+1,1})_{\mathbf{1}}$.

(34) If $1 \leq i$ and $i + 1 \leq \mathrm{len}\, G$ and $1 \leq j$ and $j + 1 \leq \mathrm{width}\, G$, then $\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}) \in \mathrm{Int}\, \mathrm{cell}(G, i, j)$.

(35) If $1 \leq i$ and $i + 1 \leq \mathrm{len}\, G$, then $\frac{1}{2} \cdot (G_{i,\mathrm{width}\, G} + G_{i+1,\mathrm{width}\, G}) + [0, 1] \in \mathrm{Int}\, \mathrm{cell}(G, i, \mathrm{width}\, G)$.

(36) If $1 \leq i$ and $i+1 \leq \mathrm{len}\, G$, then $\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1}) - [0, 1] \in \mathrm{Int}\, \mathrm{cell}(G, i, 0)$.

(37) If $1 \leq j$ and $j + 1 \leq \mathrm{width}\, G$, then $\frac{1}{2} \cdot (G_{\mathrm{len}\, G,j} + G_{\mathrm{len}\, G,j+1}) + [1, 0] \in \mathrm{Int}\, \mathrm{cell}(G, \mathrm{len}\, G, j)$.

(38) If $1 \leq j$ and $j + 1 \leq \mathrm{width}\, G$, then $\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1}) - [1, 0] \in \mathrm{Int}\, \mathrm{cell}(G, 0, j)$.

(39) $G_{1,1} - [1, 1] \in \mathrm{Int}\, \mathrm{cell}(G, 0, 0)$.

(40) $G_{\operatorname{len} G, \operatorname{width} G} + [1, 1] \in \operatorname{Int} \operatorname{cell}(G, \operatorname{len} G, \operatorname{width} G)$.

(41) $G_{1, \operatorname{width} G} + [-1, 1] \in \operatorname{Int} \operatorname{cell}(G, 0, \operatorname{width} G)$.

(42) $G_{\operatorname{len} G, 1} + [1, -1] \in \operatorname{Int} \operatorname{cell}(G, \operatorname{len} G, 0)$.

(43) If $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}), \frac{1}{2} \cdot (G_{i,j} + G_{i,j+1})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, j) \cup \{\frac{1}{2} \cdot (G_{i,j} + G_{i,j+1})\}$.

(44) Suppose $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$. Then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}), \frac{1}{2} \cdot (G_{i,j+1} + G_{i+1,j+1})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, j) \cup \{\frac{1}{2} \cdot (G_{i,j+1} + G_{i+1,j+1})\}$.

(45) Suppose $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$. Then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}), \frac{1}{2} \cdot (G_{i+1,j} + G_{i+1,j+1})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, j) \cup \{\frac{1}{2} \cdot (G_{i+1,j} + G_{i+1,j+1})\}$.

(46) If $1 \leq i$ and $i < \operatorname{len} G$ and $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}), \frac{1}{2} \cdot (G_{i,j} + G_{i+1,j})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, j) \cup \{\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j})\}$.

(47) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1}) - [1, 0], \frac{1}{2} \cdot (G_{1,j} + G_{1,j+1})) \subseteq \operatorname{Int} \operatorname{cell}(G, 0, j) \cup \{\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1})\}$.

(48) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1}) + [1, 0], \frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1})) \subseteq \operatorname{Int} \operatorname{cell}(G, \operatorname{len} G, j) \cup \{\frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1})\}$.

(49) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1}) - [0, 1], \frac{1}{2} \cdot (G_{i,1} + G_{i+1,1})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, 0) \cup \{\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1})\}$.

(50) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G}) + [0, 1], \frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G})) \subseteq \operatorname{Int} \operatorname{cell}(G, i, \operatorname{width} G) \cup \{\frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G})\}$.

(51) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1}) - [1, 0], G_{1,j} - [1, 0]) \subseteq \operatorname{Int} \operatorname{cell}(G, 0, j) \cup \{G_{1,j} - [1, 0]\}$.

(52) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1}) - [1, 0], G_{1,j+1} - [1, 0]) \subseteq \operatorname{Int} \operatorname{cell}(G, 0, j) \cup \{G_{1,j+1} - [1, 0]\}$.

(53) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1}) + [1, 0], G_{\operatorname{len} G,j} + [1, 0]) \subseteq \operatorname{Int} \operatorname{cell}(G, \operatorname{len} G, j) \cup \{G_{\operatorname{len} G,j} + [1, 0]\}$.

(54) If $1 \leq j$ and $j < \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1}) + [1, 0], G_{\operatorname{len} G,j+1} + [1, 0]) \subseteq \operatorname{Int} \operatorname{cell}(G, \operatorname{len} G, j) \cup \{G_{\operatorname{len} G,j+1} + [1, 0]\}$.

(55) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1}) - [0, 1], G_{i,1} - [0, 1]) \subseteq \operatorname{Int} \operatorname{cell}(G, i, 0) \cup \{G_{i,1} - [0, 1]\}$.

(56) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1}) - [0, 1], G_{i+1,1} - [0, 1]) \subseteq \operatorname{Int} \operatorname{cell}(G, i, 0) \cup \{G_{i+1,1} - [0, 1]\}$.

(57) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G}) + [0, 1], G_{i,\operatorname{width} G} + [0, 1]) \subseteq \operatorname{Int} \operatorname{cell}(G, i, \operatorname{width} G) \cup \{G_{i,\operatorname{width} G} + [0, 1]\}$.

(58) If $1 \leq i$ and $i < \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G}) + [0, 1], G_{i+1,\operatorname{width} G} + [0, 1]) \subseteq \operatorname{Int} \operatorname{cell}(G, i, \operatorname{width} G) \cup \{G_{i+1,\operatorname{width} G} + [0, 1]\}$.

(59) $\mathcal{L}(G_{1,1} - [1, 1], G_{1,1} - [1, 0]) \subseteq \operatorname{Int} \operatorname{cell}(G, 0, 0) \cup \{G_{1,1} - [1, 0]\}$.

(60) $\mathcal{L}(G_{\mathrm{len}\,G,1}+[1,-1],G_{\mathrm{len}\,G,1}+[1,0])\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G,0)\cup\{G_{\mathrm{len}\,G,1}+[1,0]\}$.

(61) $\mathcal{L}(G_{1,\mathrm{width}\,G}+[-1,1],G_{1,\mathrm{width}\,G}-[1,0])\subseteq \mathrm{Int\,cell}(G,0,\mathrm{width}\,G)\cup\{G_{1,\mathrm{width}\,G}-[1,0]\}$.

(62) $\mathcal{L}(G_{\mathrm{len}\,G,\mathrm{width}\,G}+[1,1],G_{\mathrm{len}\,G,\mathrm{width}\,G}+[1,0])\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G,\mathrm{width}\,G)\cup\{G_{\mathrm{len}\,G,\mathrm{width}\,G}+[1,0]\}$.

(63) $\mathcal{L}(G_{1,1}-[1,1],G_{1,1}-[0,1])\subseteq \mathrm{Int\,cell}(G,0,0)\cup\{G_{1,1}-[0,1]\}$.

(64) $\mathcal{L}(G_{\mathrm{len}\,G,1}+[1,-1],G_{\mathrm{len}\,G,1}-[0,1])\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G,0)\cup\{G_{\mathrm{len}\,G,1}-[0,1]\}$.

(65) $\mathcal{L}(G_{1,\mathrm{width}\,G}+[-1,1],G_{1,\mathrm{width}\,G}+[0,1])\subseteq \mathrm{Int\,cell}(G,0,\mathrm{width}\,G)\cup\{G_{1,\mathrm{width}\,G}+[0,1]\}$.

(66) $\mathcal{L}(G_{\mathrm{len}\,G,\mathrm{width}\,G}+[1,1],G_{\mathrm{len}\,G,\mathrm{width}\,G}+[0,1])\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G,\mathrm{width}\,G)\cup\{G_{\mathrm{len}\,G,\mathrm{width}\,G}+[0,1]\}$.

(67) Suppose $1\leq i$ and $i<\mathrm{len}\,G$ and $1\leq j$ and $j+1<\mathrm{width}\,G$. Then $\mathcal{L}(\frac{1}{2}\cdot(G_{i,j}+G_{i+1,j+1}),\frac{1}{2}\cdot(G_{i,j+1}+G_{i+1,j+2}))\subseteq \mathrm{Int\,cell}(G,i,j)\cup\mathrm{Int\,cell}(G,i,j+1)\cup\{\frac{1}{2}\cdot(G_{i,j+1}+G_{i+1,j+1})\}$.

(68) Suppose $1\leq j$ and $j<\mathrm{width}\,G$ and $1\leq i$ and $i+1<\mathrm{len}\,G$. Then $\mathcal{L}(\frac{1}{2}\cdot(G_{i,j}+G_{i+1,j+1}),\frac{1}{2}\cdot(G_{i+1,j}+G_{i+2,j+1}))\subseteq \mathrm{Int\,cell}(G,i,j)\cup\mathrm{Int\,cell}(G,i+1,j)\cup\{\frac{1}{2}\cdot(G_{i+1,j}+G_{i+1,j+1})\}$.

(69) If $1\leq i$ and $i<\mathrm{len}\,G$ and $1<\mathrm{width}\,G$, then $\mathcal{L}(\frac{1}{2}\cdot(G_{i,1}+G_{i+1,1})-[0,1],\frac{1}{2}\cdot(G_{i,1}+G_{i+1,2}))\subseteq \mathrm{Int\,cell}(G,i,0)\cup\mathrm{Int\,cell}(G,i,1)\cup\{\frac{1}{2}\cdot(G_{i,1}+G_{i+1,1})\}$.

(70) Suppose $1\leq i$ and $i<\mathrm{len}\,G$ and $1<\mathrm{width}\,G$. Then $\mathcal{L}(\frac{1}{2}\cdot(G_{i,\mathrm{width}\,G}+G_{i+1,\mathrm{width}\,G})+[0,1],\frac{1}{2}\cdot(G_{i,\mathrm{width}\,G}+G_{i+1,\mathrm{width}\,G-'1}))\subseteq \mathrm{Int\,cell}(G,i,\mathrm{width}\,G-'1)\cup\mathrm{Int\,cell}(G,i,\mathrm{width}\,G)\cup\{\frac{1}{2}\cdot(G_{i,\mathrm{width}\,G}+G_{i+1,\mathrm{width}\,G})\}$.

(71) If $1\leq j$ and $j<\mathrm{width}\,G$ and $1<\mathrm{len}\,G$, then $\mathcal{L}(\frac{1}{2}\cdot(G_{1,j}+G_{1,j+1})-[1,0],\frac{1}{2}\cdot(G_{1,j}+G_{2,j+1}))\subseteq \mathrm{Int\,cell}(G,0,j)\cup\mathrm{Int\,cell}(G,1,j)\cup\{\frac{1}{2}\cdot(G_{1,j}+G_{1,j+1})\}$.

(72) Suppose $1\leq j$ and $j<\mathrm{width}\,G$ and $1<\mathrm{len}\,G$. Then $\mathcal{L}(\frac{1}{2}\cdot(G_{\mathrm{len}\,G,j}+G_{\mathrm{len}\,G,j+1})+[1,0],\frac{1}{2}\cdot(G_{\mathrm{len}\,G,j}+G_{\mathrm{len}\,G-'1,j+1}))\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G-'1,j)\cup\mathrm{Int\,cell}(G,\mathrm{len}\,G,j)\cup\{\frac{1}{2}\cdot(G_{\mathrm{len}\,G,j}+G_{\mathrm{len}\,G,j+1})\}$.

(73) If $1<\mathrm{len}\,G$ and $1\leq j$ and $j+1<\mathrm{width}\,G$, then $\mathcal{L}(\frac{1}{2}\cdot(G_{1,j}+G_{1,j+1})-[1,0],\frac{1}{2}\cdot(G_{1,j+1}+G_{1,j+2})-[1,0])\subseteq \mathrm{Int\,cell}(G,0,j)\cup\mathrm{Int\,cell}(G,0,j+1)\cup\{G_{1,j+1}-[1,0]\}$.

(74) Suppose $1<\mathrm{len}\,G$ and $1\leq j$ and $j+1<\mathrm{width}\,G$. Then $\mathcal{L}(\frac{1}{2}\cdot(G_{\mathrm{len}\,G,j}+G_{\mathrm{len}\,G,j+1})+[1,0],\frac{1}{2}\cdot(G_{\mathrm{len}\,G,j+1}+G_{\mathrm{len}\,G,j+2})+[1,0])\subseteq \mathrm{Int\,cell}(G,\mathrm{len}\,G,j)\cup\mathrm{Int\,cell}(G,\mathrm{len}\,G,j+1)\cup\{G_{\mathrm{len}\,G,j+1}+[1,0]\}$.

(75) If $1<\mathrm{width}\,G$ and $1\leq i$ and $i+1<\mathrm{len}\,G$, then $\mathcal{L}(\frac{1}{2}\cdot(G_{i,1}+G_{i+1,1})-[0,1],\frac{1}{2}\cdot(G_{i+1,1}+G_{i+2,1})-[0,1])\subseteq \mathrm{Int\,cell}(G,i,0)\cup\mathrm{Int\,cell}(G,i+1,0)\cup\{G_{i+1,1}-[0,1]\}$.

(76) Suppose $1 < \operatorname{width} G$ and $1 \leq i$ and $i+1 < \operatorname{len} G$. Then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G}) + [0,1], \frac{1}{2} \cdot (G_{i+1,\operatorname{width} G} + G_{i+2,\operatorname{width} G}) + [0,1]) \subseteq \operatorname{Int}\operatorname{cell}(G, i, \operatorname{width} G) \cup \operatorname{Int}\operatorname{cell}(G, i+1, \operatorname{width} G) \cup \{G_{i+1,\operatorname{width} G} + [0,1]\}$.

(77) If $1 < \operatorname{len} G$ and $1 < \operatorname{width} G$, then $\mathcal{L}(G_{1,1} - [1,1], \frac{1}{2} \cdot (G_{1,1} + G_{1,2}) - [1,0]) \subseteq \operatorname{Int}\operatorname{cell}(G, 0, 0) \cup \operatorname{Int}\operatorname{cell}(G, 0, 1) \cup \{G_{1,1} - [1,0]\}$.

(78) If $1 < \operatorname{len} G$ and $1 < \operatorname{width} G$, then $\mathcal{L}(G_{\operatorname{len} G,1} + [1,-1], \frac{1}{2} \cdot (G_{\operatorname{len} G,1} + G_{\operatorname{len} G,2}) + [1,0]) \subseteq \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, 0) \cup \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, 1) \cup \{G_{\operatorname{len} G,1} + [1,0]\}$.

(79) If $1 < \operatorname{len} G$ and $1 < \operatorname{width} G$, then $\mathcal{L}(G_{1,\operatorname{width} G} + [-1,1], \frac{1}{2} \cdot (G_{1,\operatorname{width} G} + G_{1,\operatorname{width} G -' 1}) - [1,0]) \subseteq \operatorname{Int}\operatorname{cell}(G, 0, \operatorname{width} G) \cup \operatorname{Int}\operatorname{cell}(G, 0, \operatorname{width} G -' 1) \cup \{G_{1,\operatorname{width} G} - [1,0]\}$.

(80) If $1 < \operatorname{len} G$ and $1 < \operatorname{width} G$, then $\mathcal{L}(G_{\operatorname{len} G,\operatorname{width} G} + [1,1], \frac{1}{2} \cdot (G_{\operatorname{len} G,\operatorname{width} G} + G_{\operatorname{len} G,\operatorname{width} G -' 1}) + [1,0]) \subseteq \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, \operatorname{width} G) \cup \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, \operatorname{width} G -' 1) \cup \{G_{\operatorname{len} G,\operatorname{width} G} + [1,0]\}$.

(81) If $1 < \operatorname{width} G$ and $1 < \operatorname{len} G$, then $\mathcal{L}(G_{1,1} - [1,1], \frac{1}{2} \cdot (G_{1,1} + G_{2,1}) - [0,1]) \subseteq \operatorname{Int}\operatorname{cell}(G, 0, 0) \cup \operatorname{Int}\operatorname{cell}(G, 1, 0) \cup \{G_{1,1} - [0,1]\}$.

(82) If $1 < \operatorname{width} G$ and $1 < \operatorname{len} G$, then $\mathcal{L}(G_{1,\operatorname{width} G} + [-1,1], \frac{1}{2} \cdot (G_{1,\operatorname{width} G} + G_{2,\operatorname{width} G}) + [0,1]) \subseteq \operatorname{Int}\operatorname{cell}(G, 0, \operatorname{width} G) \cup \operatorname{Int}\operatorname{cell}(G, 1, \operatorname{width} G) \cup \{G_{1,\operatorname{width} G} + [0,1]\}$.

(83) If $1 < \operatorname{width} G$ and $1 < \operatorname{len} G$, then $\mathcal{L}(G_{\operatorname{len} G,1} + [1,-1], \frac{1}{2} \cdot (G_{\operatorname{len} G,1} + G_{\operatorname{len} G -' 1,1}) - [0,1]) \subseteq \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, 0) \cup \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G -' 1, 0) \cup \{G_{\operatorname{len} G,1} - [0,1]\}$.

(84) If $1 < \operatorname{width} G$ and $1 < \operatorname{len} G$, then $\mathcal{L}(G_{\operatorname{len} G,\operatorname{width} G} + [1,1], \frac{1}{2} \cdot (G_{\operatorname{len} G,\operatorname{width} G} + G_{\operatorname{len} G -' 1,\operatorname{width} G}) + [0,1]) \subseteq \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, \operatorname{width} G) \cup \operatorname{Int}\operatorname{cell}(G, \operatorname{len} G -' 1, \operatorname{width} G) \cup \{G_{\operatorname{len} G,\operatorname{width} G} + [0,1]\}$.

(85) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}), p)$ meets $\operatorname{Int}\operatorname{cell}(G, i, j)$.

(86) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$, then $\mathcal{L}(p, \frac{1}{2} \cdot (G_{i,\operatorname{width} G} + G_{i+1,\operatorname{width} G}) + [0,1])$ meets $\operatorname{Int}\operatorname{cell}(G, i, \operatorname{width} G)$.

(87) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{i,1} + G_{i+1,1}) - [0,1], p)$ meets $\operatorname{Int}\operatorname{cell}(G, i, 0)$.

(88) If $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(\frac{1}{2} \cdot (G_{1,j} + G_{1,j+1}) - [1,0], p)$ meets $\operatorname{Int}\operatorname{cell}(G, 0, j)$.

(89) If $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(p, \frac{1}{2} \cdot (G_{\operatorname{len} G,j} + G_{\operatorname{len} G,j+1}) + [1,0])$ meets $\operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, j)$.

(90) $\mathcal{L}(p, G_{1,1} - [1,1])$ meets $\operatorname{Int}\operatorname{cell}(G, 0, 0)$.

(91) $\mathcal{L}(p, G_{\operatorname{len} G,\operatorname{width} G} + [1,1])$ meets $\operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, \operatorname{width} G)$.

(92) $\mathcal{L}(p, G_{1,\operatorname{width} G} + [-1,1])$ meets $\operatorname{Int}\operatorname{cell}(G, 0, \operatorname{width} G)$.

(93) $\mathcal{L}(p, G_{\operatorname{len} G,1} + [1,-1])$ meets $\operatorname{Int}\operatorname{cell}(G, \operatorname{len} G, 0)$.

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The Theorem of Weierstrass

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Summary. The basic purpose of this article is to prove the important Weierstrass' theorem which states that a real valued continuous function f on a topological space T assumes a maximum and a minimum value on the compact subset S of T, i.e., there exist points x_1, x_2 of T being elements of S, such that $f(x_1)$ and $f(x_2)$ are the supremum and the infimum, respectively, of $f(S)$, which is the image of S under the function f. The paper is divided into three parts. In the first part, we prove some auxiliary theorems concerning properties of balls in metric spaces and define special families of subsets of topological spaces. These concepts are used in the next part of the paper which contains the essential part of the article, namely the formalization of the proof of Weierstrass' theorem. Here, we also prove a theorem concerning the compactness of images of compact sets of T under a continuous function. The final part of this work is developed for the purpose of defining some measures of the distance between compact subsets of topological metric spaces. Some simple theorems about these measures are also proved.

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The papers [31], [36], [9], [32], [30], [35], [29], [37], [7], [8], [5], [6], [27], [2], [15], [1], [14], [17], [10], [21], [19], [20], [18], [25], [33], [34], [3], [13], [22], [24], [38], [12], [26], [11], [4], [23], [28], and [16] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) Let M be a metric space, and let x_1, x_2 be points of M, and let r_1, r_2 be real numbers. Then there exists a point x of M and there exists a real number r such that $\mathrm{Ball}(x_1, r_1) \cup \mathrm{Ball}(x_2, r_2) \subseteq \mathrm{Ball}(x, r)$.

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(2) Let M be a metric space, and let n be a natural number, and let F be a family of subsets of M, and let p be a finite sequence. Suppose F is finite and a family of balls and $\operatorname{rng} p = F$ and $\operatorname{dom} p = \operatorname{Seg}(n+1)$. Then there exists a family G of subsets of M such that

(i) G is finite and a family of balls, and

(ii) there exists a finite sequence q such that $\operatorname{rng} q = G$ and $\operatorname{dom} q = \operatorname{Seg} n$ and there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \bigcup G \cup \operatorname{Ball}(x, r)$.

(3) Let M be a metric space and let F be a family of subsets of M. Suppose F is finite and a family of balls. Then there exists a point x of M and there exists a real number r such that $\bigcup F \subseteq \operatorname{Ball}(x, r)$.

Let T, S be topological spaces, let f be a map from T into S, and let G be a family of subsets of S. The functor $f^{-1} G$ yields a family of subsets of T and is defined by the condition (Def.1).

(Def.1) Let A be a subset of the carrier of T. Then $A \in f^{-1} G$ if and only if there exists a subset B of the carrier of S such that $B \in G$ and $A = f^{-1} B$.

Next we state two propositions:

(4) Let T, S be topological spaces, and let f be a map from T into S, and let A, B be families of subsets of S. If $A \subseteq B$, then $f^{-1} A \subseteq f^{-1} B$.

(5) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of S. If f is continuous and B is open, then $f^{-1} B$ is open.

Let T, S be topological spaces, let f be a map from T into S, and let G be a family of subsets of T. The functor $f^{\circ} G$ yields a family of subsets of S and is defined by the condition (Def.2).

(Def.2) Let A be a subset of the carrier of S. Then $A \in f^{\circ} G$ if and only if there exists a subset B of the carrier of T such that $B \in G$ and $A = f^{\circ} B$.

One can prove the following propositions:

(6) Let T, S be topological spaces, and let f be a map from T into S, and let A, B be families of subsets of T. If $A \subseteq B$, then $f^{\circ} A \subseteq f^{\circ} B$.

(7) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of S, and let P be a subset of the carrier of S. If $f^{\circ} f^{-1} B$ is a cover of P, then B is a cover of P.

(8) Let T, S be topological spaces, and let f be a map from T into S, and let B be a family of subsets of T, and let P be a subset of the carrier of T. If B is a cover of P, then $f^{-1} f^{\circ} B$ is a cover of P.

(9) Let T, S be topological spaces, and let f be a map from T into S, and let Q be a family of subsets of S. Then $\bigcup (f^{\circ} f^{-1} Q) \subseteq \bigcup Q$.

(10) Let T, S be topological spaces, and let f be a map from T into S, and let P be a family of subsets of T. Then $\bigcup P \subseteq \bigcup (f^{-1} f^{\circ} P)$.

(11) Let T, S be topological spaces, and let f be a map from T into S, and let Q be a family of subsets of S. If Q is finite, then $f^{-1} Q$ is finite.

(12) Let T, S be topological spaces, and let f be a map from T into S, and let P be a family of subsets of T. If P is finite, then $f^{\circ}P$ is finite.

(13) Let T, S be topological spaces, and let f be a map from T into S, and let P be a subset of the carrier of T, and let F be a family of subsets of S. Given a family B of subsets of T such that $B \subseteq f^{-1}F$ and B is a cover of P and finite. Then there exists a family G of subsets of S such that $G \subseteq F$ and G is a cover of $f^{\circ}P$ and finite.

2. The Weierstrass' Theorem

One can prove the following three propositions:

(14) Let T, S be topological spaces, and let f be a map from T into S, and let P be a subset of the carrier of T. If P is compact and f is continuous, then $f^{\circ}P$ is compact.

(15) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. If P is compact and f is continuous, then $f^{\circ}P$ is compact.

(16) Let f be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ and let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$. If P is compact and f is continuous, then $f^{\circ}P$ is compact.

Let P be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$. The functor Ω_P yields a subset of $\mathbb{R}$ and is defined as follows:

(Def.3) $\Omega_P = P$.

Next we state three propositions:

(17) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is bounded.

(18) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is closed.

(19) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that P is compact holds Ω_P is compact.

Let P be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$. The functor $\sup P$ yields a real number and is defined as follows:

(Def.4) $\sup P = \sup(\Omega_P)$.

The functor $\inf P$ yielding a real number is defined by:

(Def.5) $\inf P = \inf(\Omega_P)$.

We now state two propositions:

(20) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_1 of T such that $x_1 \in P$ and $f(x_1) = \sup(f^{\circ}P)$.

(21) Let T be a topological space, and let f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let P be a subset of the carrier of T. Suppose $P \neq \emptyset$ and P is compact and f is continuous. Then there exists a point x_2 of T such that $x_2 \in P$ and $f(x_2) = \inf(f^\circ P)$.

3. The Measure of the Distance Between Compact Sets

Let M be a metric space and let x be a point of M. The functor $\mathrm{dist}(x)$ yielding a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def.6) For every point y of M holds $(\mathrm{dist}(x))(y) = \rho(y, x)$.

The following three propositions are true:

(22) For every metric space M and for every point x of M holds $\mathrm{dist}(x)$ is continuous.

(23) Let M be a metric space, and let x be a point of M, and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in P$ and $(\mathrm{dist}(x))(x_1) = \sup((\mathrm{dist}(x))^\circ P)$.

(24) Let M be a metric space, and let x be a point of M, and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in P$ and $(\mathrm{dist}(x))(x_2) = \inf((\mathrm{dist}(x))^\circ P)$.

Let M be a metric space and let P be a subset of the carrier of M_{top}. Let us assume that $P \neq \emptyset$ and P is compact. The functor $\mathrm{dist}_{\max}(P)$ yielding a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def.7) For every point x of M holds $(\mathrm{dist}_{\max}(P))(x) = \sup((\mathrm{dist}(x))^\circ P)$.

The functor $\mathrm{dist}_{\min}(P)$ yields a map from M_{top} into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def.8) For every point x of M holds $(\mathrm{dist}_{\min}(P))(x) = \inf((\mathrm{dist}(x))^\circ P)$.

One can prove the following propositions:

(25) Let M be a metric space and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. If $p_1 \in P$, then $\rho(p_1, p_2) \leq \sup((\mathrm{dist}(p_2))^\circ P)$ and $\inf((\mathrm{dist}(p_2))^\circ P) \leq \rho(p_1, p_2)$.

(26) Let M be a metric space and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\sup((\mathrm{dist}(p_1))^\circ P) - \sup((\mathrm{dist}(p_2))^\circ P)| \leq \rho(p_1, p_2)$.

(27) Let M be a metric space and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and let x_1, x_2 be real numbers. If $x_1 = (\mathrm{dist}_{\max}(P))(p_1)$ and $x_2 = (\mathrm{dist}_{\max}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(28) Let M be a metric space and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M. Then $|\inf((\mathrm{dist}(p_1))^\circ P) - \inf((\mathrm{dist}(p_2))^\circ P)| \leq \rho(p_1, p_2)$.

(29) Let M be a metric space and let P be a subset of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact. Let p_1, p_2 be points of M and let x_1,

x_2 be real numbers. If $x_1 = (\mathrm{dist}_{\mathrm{min}}(P))(p_1)$ and $x_2 = (\mathrm{dist}_{\mathrm{min}}(P))(p_2)$, then $|x_1 - x_2| \leq \rho(p_1, p_2)$.

(30) Let M be a metric space and let X be a subset of the carrier of M_{top}. If $X \neq \emptyset$ and X is compact, then $\mathrm{dist}_{\mathrm{max}}(X)$ is continuous.

(31) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in Q$ and $(\mathrm{dist}_{\mathrm{max}}(P))(x_1) = \sup((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

(32) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in Q$ and $(\mathrm{dist}_{\mathrm{max}}(P))(x_2) = \inf((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

(33) Let M be a metric space and let X be a subset of the carrier of M_{top}. If $X \neq \emptyset$ and X is compact, then $\mathrm{dist}_{\mathrm{min}}(X)$ is continuous.

(34) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_1 of M_{top} such that $x_1 \in Q$ and $(\mathrm{dist}_{\mathrm{min}}(P))(x_1) = \sup((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

(35) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exists a point x_2 of M_{top} such that $x_2 \in Q$ and $(\mathrm{dist}_{\mathrm{min}}(P))(x_2) = \inf((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Let us assume that $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. The functor $\mathrm{dist}_{\mathrm{min}}^{\mathrm{min}}(P, Q)$ yields a real number and is defined as follows:

(Def.9) $\mathrm{dist}_{\mathrm{min}}^{\mathrm{min}}(P, Q) = \inf((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}_{\mathrm{min}}^{\mathrm{max}}(P, Q)$ yielding a real number is defined as follows:

(Def.10) $\mathrm{dist}_{\mathrm{min}}^{\mathrm{max}}(P, Q) = \sup((\mathrm{dist}_{\mathrm{min}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}_{\mathrm{max}}^{\mathrm{min}}(P, Q)$ yielding a real number is defined as follows:

(Def.11) $\mathrm{dist}_{\mathrm{max}}^{\mathrm{min}}(P, Q) = \inf((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

The functor $\mathrm{dist}_{\mathrm{max}}^{\mathrm{max}}(P, Q)$ yielding a real number is defined as follows:

(Def.12) $\mathrm{dist}_{\mathrm{max}}^{\mathrm{max}}(P, Q) = \sup((\mathrm{dist}_{\mathrm{max}}(P))^{\circ}Q)$.

One can prove the following propositions:

(36) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}_{\mathrm{min}}^{\mathrm{min}}(P, Q)$.

(37) Let M be a metric space and let P, Q be subsets of the carrier of M_{top}. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = \mathrm{dist}_{\mathrm{max}}^{\mathrm{min}}(P, Q)$.

(38) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = {\rm dist}_{\rm min}^{\rm max}(P, Q)$.

(39) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then there exist points x_1, x_2 of M such that $x_1 \in P$ and $x_2 \in Q$ and $\rho(x_1, x_2) = {\rm dist}_{\rm max}^{\rm max}(P, Q)$.

(40) Let M be a metric space and let P, Q be subsets of the carrier of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Let x_1, x_2 be points of M. If $x_1 \in P$ and $x_2 \in Q$, then ${\rm dist}_{\rm min}^{\rm min}(P, Q) \leq \rho(x_1, x_2)$ and $\rho(x_1, x_2) \leq {\rm dist}_{\rm max}^{\rm max}(P, Q)$.

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Dyadic Numbers and T_4 Topological Spaces

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Summary. This article is the first part of a paper proving the fundamental Urysohn's Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into four parts. In the first part, we prove some auxiliary theorems concerning properties of natural numbers and prove two useful schemes about recurrently defined functions; in the second part, we define a special set of rational numbers, which we call dyadic, and prove some of its properties. The next part of the paper contains the definitions of T_1 space and normal space, and we prove related theorems used in later parts of the paper. The final part of this work is developed for proving the theorem about the existence of some special family of subsets of a topological space. This theorem is essential in proving Urysohn's Lemma.

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The notation and terminology used in this paper have been introduced in the following articles: [24], [30], [9], [25], [23], [22], [31], [6], [7], [4], [2], [16], [3], [5], [28], [29], [1], [10], [13], [19], [32], [12], [18], [14], [15], [11], [20], [21], [8], [17], [27], and [26].

1. Preliminaries

The following propositions are true:

(1) $0 \neq \frac{1}{2}$ and $1 \neq \frac{1}{2}$.

(2) $0 < \frac{1}{2}$ and $\frac{1}{2} < 1$.

(3) For every natural number n holds $1 \leq 2^n$.

(4) For every natural number n holds $0 < 2^n$.

In this article we present several logical schemes. The scheme *FuncEx2DChoice* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a non empty set $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

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There exists a function F from $[\!:\mathcal{A}, \mathcal{B}:\!]$ into $\mathcal{C}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\mathcal{P}[x, y, F(\langle x, y\rangle)]$

provided the parameters meet the following requirement:

- For every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ there exists an element z of $\mathcal{C}$ such that $\mathcal{P}[x, y, z]$.

The scheme *RecExDN RD* concerns a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a function F from $\mathbb{N}$ into $\mathcal{A}$ such that $F(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, F(n), F(n+1)]$

provided the parameters satisfy the following condition:

- For every natural number n and for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$.

2. Dyadic Numbers

The subset $\mathbb{R}_{<0}$ of $\mathbb{R}$ is defined by:

(Def.1) For every real number x holds $x \in \mathbb{R}_{<0}$ iff $x < 0$.

The subset $\mathbb{R}_{>1}$ of $\mathbb{R}$ is defined by:

(Def.2) For every real number x holds $x \in \mathbb{R}_{>1}$ iff $1 < x$.

Let n be a natural number. The functor $\mathrm{dyadic}(n)$ yields a subset of $\mathbb{R}$ and is defined by:

(Def.3) For every real number x holds $x \in \mathrm{dyadic}(n)$ iff there exists a natural number i such that $0 \leq i$ and $i \leq 2^n$ and $x = \frac{i}{2^n}$.

The subset DYADIC of $\mathbb{R}$ is defined by:

(Def.4) For every real number a holds $a \in \mathrm{DYADIC}$ iff there exists a natural number n such that $a \in \mathrm{dyadic}(n)$.

The subset DOM of $\mathbb{R}$ is defined by:

(Def.5) $\mathrm{DOM} = \mathbb{R}_{<0} \cup \mathrm{DYADIC} \cup \mathbb{R}_{>1}$.

Let T be a topological space, let A be a non empty subset of $\mathbb{R}$, let F be a function from A into $2^{\text{the carrier of } T}$, and let r be an element of A. Then $F(r)$ is a subset of the carrier of T.

One can prove the following three propositions:

(5) For every natural number n and for every real number x such that $x \in \mathrm{dyadic}(n)$ holds $0 \leq x$ and $x \leq 1$.

(6) $\mathrm{dyadic}(0) = \{0, 1\}$.

(7) $\mathrm{dyadic}(1) = \{0, \frac{1}{2}, 1\}$.

Let n be a natural number. Note that $\mathrm{dyadic}(n)$ is non empty.

Next we state the proposition

(8) For every natural number x and for every natural number n holds x^n is a natural number.

Let x, n be natural numbers. Then x^n is a natural number.

The following proposition is true

(9) Let n be a natural number. Then there exists a finite sequence F_1 such that $\operatorname{dom} F_1 = \operatorname{Seg}(2^n + 1)$ and for every natural number i such that $i \in \operatorname{dom} F_1$ holds $F_1(i) = \frac{i-1}{2^n}$.

Let n be a natural number. The functor $\operatorname{dyad}(n)$ yielding a finite sequence is defined by:

(Def.6) $\operatorname{dom}\operatorname{dyad}(n) = \operatorname{Seg}(2^n + 1)$ and for every natural number i such that $i \in \operatorname{dom}\operatorname{dyad}(n)$ holds $(\operatorname{dyad}(n))(i) = \frac{i-1}{2^n}$.

We now state the proposition

(10) For every natural number n holds $\operatorname{dom}\operatorname{dyad}(n) = \operatorname{Seg}(2^n + 1)$ and $\operatorname{rng}\operatorname{dyad}(n) = \operatorname{dyadic}(n)$.

Let us note that DYADIC is non empty.

Let us observe that DOM is non empty.

One can prove the following propositions:

(11) For every natural number n holds $\operatorname{dyadic}(n) \subseteq \operatorname{dyadic}(n+1)$.

(12) For every natural number n holds $0 \in \operatorname{dyadic}(n)$ and $1 \in \operatorname{dyadic}(n)$.

(13) For every natural number n and for every natural number i such that $0 < i$ and $i \leq 2^n$ holds $\frac{i \cdot 2 - 1}{2^{n+1}} \in \operatorname{dyadic}(n+1) \setminus \operatorname{dyadic}(n)$.

(14) For every natural number n and for every natural number i such that $0 \leq i$ and $i < 2^n$ holds $\frac{i \cdot 2 + 1}{2^{n+1}} \in \operatorname{dyadic}(n+1) \setminus \operatorname{dyadic}(n)$.

(15) For every natural number n holds $\frac{1}{2^{n+1}} \in \operatorname{dyadic}(n+1) \setminus \operatorname{dyadic}(n)$.

Let n be a natural number and let x be an element of $\operatorname{dyadic}(n)$. The functor $\operatorname{axis}(x, n)$ yields a natural number and is defined by:

(Def.7) $x = \frac{\operatorname{axis}(x,n)}{2^n}$.

One can prove the following propositions:

(16) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $x = \frac{\operatorname{axis}(x,n)}{2^n}$ and $0 \leq \operatorname{axis}(x, n)$ and $\operatorname{axis}(x, n) \leq 2^n$.

(17) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $\frac{\operatorname{axis}(x,n)-1}{2^n} < x$ and $x < \frac{\operatorname{axis}(x,n)+1}{2^n}$.

(18) For every natural number n and for every element x of $\operatorname{dyadic}(n)$ holds $\frac{\operatorname{axis}(x,n)-1}{2^n} < \frac{\operatorname{axis}(x,n)+1}{2^n}$.

(19) For every natural number n there exists a natural number k such that $n = k \cdot 2$ or $n = k \cdot 2 + 1$.

(20) Let n be a natural number and let x be an element of $\operatorname{dyadic}(n + 1)$. If $x \notin \operatorname{dyadic}(n)$, then $\frac{\operatorname{axis}(x,n+1)-1}{2^{n+1}} \in \operatorname{dyadic}(n)$ and $\frac{\operatorname{axis}(x,n+1)+1}{2^{n+1}} \in \operatorname{dyadic}(n)$.

(21) For every natural number n and for all elements x_1, x_2 of $\operatorname{dyadic}(n)$ such that $x_1 < x_2$ holds $\operatorname{axis}(x_1, n) < \operatorname{axis}(x_2, n)$.

(22) For every natural number n and for all elements x_1, x_2 of dyadic(n) such that $x_1 < x_2$ holds $x_1 \leq \frac{\mathrm{axis}(x_2,n)-1}{2^n}$ and $\frac{\mathrm{axis}(x_1,n)+1}{2^n} \leq x_2$.

(23) Let n be a natural number and let x_1, x_2 be elements of dyadic$(n+1)$. If $x_1 < x_2$ and $x_1 \notin$ dyadic(n) and $x_2 \notin$ dyadic(n), then $\frac{\mathrm{axis}(x_1,n+1)+1}{2^{n+1}} \leq \frac{\mathrm{axis}(x_2,n+1)-1}{2^{n+1}}$.

3. Normal Spaces

Let T be a topological space and let x be a point of T. A subset of the carrier of T is said to be a neighbourhood of x in T if:

(Def.8) There exists a subset A of the carrier of T such that A is open and $x \in A$ and $A \subseteq$ it.

One can prove the following propositions:

(24) Let T be a topological space and let A be a subset of the carrier of T. Then A is open if and only if for every point x of T such that $x \in A$ there exists a neighbourhood B of x in T such that $B \subseteq A$.

(25) Let T be a topological space, and let A be a subset of the carrier of T, and let x be a point of T. If A is open and $x \in A$, then A is a neighbourhood of x in T.

(26) Let T be a topological space and let A be a subset of the carrier of T. Suppose that for every point x of T such that $x \in A$ holds A is a neighbourhood of x in T. Then A is open.

Let T be a topological space. We say that T is a T_1 space if and only if the condition (Def.9) is satisfied.

(Def.9) Let p, q be points of T. Suppose $p \neq q$. Then there exist subsets W, V of the carrier of T such that W is open and V is open and $p \in W$ and $q \notin W$ and $q \in V$ and $p \notin V$.

Next we state the proposition

(27) For every topological space T holds T is a T_1 space iff for every point p of T holds $\{p\}$ is closed.

Let T be a topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let x be a point of T. Then $F(x)$ is a real number.

The following four propositions are true:

(28) Let T be a topological space. Suppose T is a T_4 space. Let A, B be subsets of the carrier of T. Suppose $A \neq \emptyset$ and A is open and B is open and $\overline{A} \subseteq B$. Then there exists a subset C of the carrier of T such that $C \neq \emptyset$ and C is open and $\overline{A} \subseteq C$ and $\overline{C} \subseteq B$.

(29) Let T be a topological space. Then T is a T_3 space if and only if for every subset A of the carrier of T and for every point p of T such that A is open and $p \in A$ there exists a subset B of the carrier of T such that $p \in B$ and B is open and $\overline{B} \subseteq A$.

(30) Let T be a topological space. Suppose T is a T_4 space and a T_1 space. Let A be a subset of the carrier of T. Suppose A is open and $A \neq \emptyset$. Then there exists a subset B of the carrier of T such that $B \neq \emptyset$ and $\overline{B} \subseteq A$.

(31) Let T be a topological space. Suppose T is a T_4 space. Let A be a subset of the carrier of T. Suppose A is open and $A \neq \emptyset$. Let B be a subset of the carrier of T. Suppose B is closed and $B \neq \emptyset$ and $B \subseteq A$. Then there exists a subset C of the carrier of T such that C is open and $B \subseteq C$ and $\overline{C} \subseteq A$.

4. Some Increasing Family of Sets in Normal Space

Let T be a topological space and let A, B, C be subsets of the carrier of T. We say that C is between A and B if and only if:

(Def.10) $C \neq \emptyset$ and C is open and $\overline{A} \subseteq C$ and $\overline{C} \subseteq B$.

One can prove the following proposition

(32) Let T be a topological space. Suppose T is a T_4 space. Let A, B be subsets of the carrier of T. Suppose $A \neq \emptyset$ and A is closed and B is closed and $A \cap B = \emptyset$. Let n be a natural number and let G be a function from dyadic(n) into $2^{\text{the carrier of } T}$. Suppose that for all elements r_1, r_2 of dyadic(n) such that $r_1 < r_2$ holds $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$ and $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$. Then there exists a function F from dyadic($n+1$) into $2^{\text{the carrier of } T}$ such that for all elements r_1, r_2, r of dyadic($n+1$) if $r_1 < r_2$, then $F(r_1)$ is open and $F(r_2)$ is open and $\overline{F(r_1)} \subseteq F(r_2)$ and $A \subseteq F(0)$ and $B = \Omega_T \setminus F(1)$ and if $r \in$ dyadic(n), then $F(r) = G(r)$.

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Full Adder Circuit. Part I [1]

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Summary. We continue the formalisation of circuits started by Piotr Rudnicki, Andrzej Trybulec, Pauline Kawamoto, and the second author in [16,17,14,15]. The first step in proving properties of full n-bit adder circuit, i.e. 1-bit adder, is presented. We employ the notation of combining circuits introduced in [13].

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The terminology and notation used in this paper are introduced in the following papers: [23], [25], [20], [1], [24], [27], [7], [8], [5], [11], [6], [19], [9], [26], [18], [3], [2], [4], [10], [12], [22], [21], [16], [17], [14], [15], and [13].

1. Combining of Many Sorted Signatures

A set is pair if:

(Def.1) There exist sets x, y such that it $= \langle x, y \rangle$.

Let us mention that every set which is pair is also non empty.

Let x, y be sets. Observe that $\langle x, y \rangle$ is pair.

Let us mention that there exists a set which is pair and there exists a set which is non pair.

Let us observe that every natural number is non pair.

A set has a pair if:

(Def.2) There exists a pair set x such that $x \in$ it.

Note that every set which is empty has no pairs. Let x be a non pair set. Note that $\{x\}$ has no pairs. Let y be a non pair set. Observe that $\{x, y\}$ has no pairs. Let z be a non pair set. One can check that $\{x, y, z\}$ has no pairs.

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Let us note that there exists a non empty set which has no pairs.

Let X, Y be sets with no pairs. One can verify that $X \cup Y$ has no pairs.

Let X be a set with no pairs and let Y be a set. One can verify the following observations:

* $X \setminus Y$ has no pairs,
* $X \cap Y$ has no pairs, and
* $Y \cap X$ has no pairs.

One can verify that every set which is empty is also relation-like. Let x be a pair set. One can check that $\{x\}$ is relation-like. Let y be a pair set. Observe that $\{x, y\}$ is relation-like. Let z be a pair set. One can check that $\{x, y, z\}$ is relation-like.

Let us note that every set which is relation-like and has no pairs is also empty.

A function is nonpair yielding if:

(Def.3) For every set x such that $x \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(x)$ is non pair.

Let x be a non pair set. Observe that $\langle x \rangle$ is nonpair yielding. Let y be a non pair set. One can check that $\langle x, y \rangle$ is nonpair yielding. Let z be a non pair set. Observe that $\langle x, y, z \rangle$ is nonpair yielding.

One can prove the following proposition

(1) For every function f such that f is nonpair yielding holds $\operatorname{rng} f$ has no pairs.

Let n be a natural number. Observe that there exists a finite sequence with length n which is one-to-one and nonpair yielding.

One can check that there exists a finite sequence which is one-to-one and nonpair yielding.

Let f be a nonpair yielding function. Note that $\operatorname{rng} f$ has no pairs.

The following propositions are true:

(2) Let S_1, S_2 be non empty many sorted signatures. Suppose $S_1 \approx S_2$ and $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InnerVertices}(S_1 {+\cdot} S_2)$ is a binary relation.

(3) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ is a binary relation and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InnerVertices}(S_1 {+\cdot} S_2)$ is a binary relation.

(4) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ and $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$ holds $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_1 {+\cdot} S_2)$ and $\mathrm{InputVertices}(S_1 {+\cdot} S_2) = \mathrm{InputVertices}(S_1) \cup (\mathrm{InputVertices}(S_2) \setminus \mathrm{InnerVertices}(S_1))$.

(5) For all sets X, R such that X has no pairs and R is a binary relation holds X misses R.

(6) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose $\mathrm{InputVertices}(S_1)$ has no pairs and $\mathrm{InnerVertices}(S_2)$ is a binary relation. Then $\mathrm{InputVertices}(S_1) \subseteq \mathrm{InputVertices}(S_1 {+\cdot} S_2)$

and InputVertices(S_1+·S_2) = InputVertices(S_1) ∪ (InputVertices(S_2) \ InnerVertices(S_1)).

(7) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. Suppose InputVertices(S_1) has no pairs and InnerVertices(S_1) is a binary relation and InputVertices(S_2) has no pairs and InnerVertices(S_2) is a binary relation. Then InputVertices(S_1+·S_2) = InputVertices(S_1) ∪ InputVertices(S_2).

(8) For all non empty many sorted signatures S_1, S_2 such that $S_1 \approx S_2$ and InputVertices(S_1) has no pairs and InputVertices(S_2) has no pairs holds InputVertices(S_1+·S_2) has no pairs.

(9) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates. If InputVertices(S_1) has no pairs and InputVertices(S_2) has no pairs, then InputVertices(S_1+·S_2) has no pairs.

2. Combinig of Circuits

In this article we present several logical schemes. The scheme *2AryBooleDef* concerns a binary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from $Boolean^2$ into *Boolean* such that for all elements x, y of *Boolean* holds $f(\langle x, y\rangle) = \mathcal{F}(x, y)$, and

(ii) for all functions f_1, f_2 from $Boolean^2$ into *Boolean* such that for all elements x, y of *Boolean* holds $f_1(\langle x, y\rangle) = \mathcal{F}(x, y)$ and for all elements x, y of *Boolean* holds $f_2(\langle x, y\rangle) = \mathcal{F}(x, y)$ holds $f_1 = f_2$

for all values of the parameter.

The scheme *3AryBooleDef* deals with a ternary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from $Boolean^3$ into *Boolean* such that for all elements x, y, z of *Boolean* holds $f(\langle x, y, z\rangle) = \mathcal{F}(x, y, z)$, and

(ii) for all functions f_1, f_2 from $Boolean^3$ into *Boolean* such that for all elements x, y, z of *Boolean* holds $f_1(\langle x, y, z\rangle) = \mathcal{F}(x, y, z)$ and for all elements x, y, z of *Boolean* holds $f_2(\langle x, y, z\rangle) = \mathcal{F}(x, y, z)$ holds $f_1 = f_2$

for all values of the parameter.

The function xor from $Boolean^2$ into *Boolean* is defined by:

(Def.4) For all elements x, y of *Boolean* holds $\mathrm{xor}(\langle x, y\rangle) = x \oplus y$.

The function or from $Boolean^2$ into *Boolean* is defined by:

(Def.5) For all elements x, y of *Boolean* holds $\mathrm{or}(\langle x, y\rangle) = x \vee y$.

The function & from $Boolean^2$ into *Boolean* is defined as follows:

(Def.6) For all elements x, y of *Boolean* holds $\&(\langle x, y\rangle) = x \wedge y$.

The function or_3 from $Boolean^3$ into *Boolean* is defined by:

(Def.7) For all elements x, y, z of *Boolean* holds $\mathrm{or}_3(\langle x, y, z\rangle) = x \vee y \vee z$.

Let x be a set. Then $\langle x \rangle$ is a finite sequence with length 1. Let y be a set. Then $\langle x, y \rangle$ is a finite sequence with length 2. Let z be a set. Then $\langle x, y, z \rangle$ is a finite sequence with length 3.

Let n, m be natural numbers, let p be a finite sequence with length n, and let q be a finite sequence with length m. Then $p \frown q$ is a finite sequence with length $n + m$.

3. Signatures with One Operation

The following proposition is true

(10) Let S be a circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let g be a gate of S. Then $(\mathrm{Following}(s))(\text{the result sort of } g) = (\mathrm{Den}(g, A))(s \cdot \mathrm{Arity}(g))$.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, let s be a state of A, and let n be a natural number. The functor $\mathrm{Following}(s, n)$ yielding a state of A is defined by the condition (Def.8).

(Def.8) There exists a function f from $\mathbb{N}$ into $\prod(\text{the sorts of } A)$ such that $\mathrm{Following}(s, n) = f(n)$ and $f(0) = s$ and for every natural number n and for every state x of A such that $x = f(n)$ holds $f(n + 1) = \mathrm{Following}(x)$.

The following propositions are true:

(11) Let S be a circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A. Then $\mathrm{Following}(s, 0) = s$.

(12) Let S be a circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let n be a natural number. Then $\mathrm{Following}(s, n + 1) = \mathrm{Following}(\mathrm{Following}(s, n))$.

(13) Let S be a circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let n, m be natural numbers. Then $\mathrm{Following}(s, n + m) = \mathrm{Following}(\mathrm{Following}(s, n), m)$.

(14) Let S be a non void circuit-like non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A. Then $\mathrm{Following}(s, 1) = \mathrm{Following}(s)$.

(15) Let S be a non void circuit-like non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A. Then $\mathrm{Following}(s, 2) = \mathrm{Following}(\mathrm{Following}(s))$.

(16) Let S be a circuit-like non void non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let n be a natural number. Then $\mathrm{Following}(s, n + 1) = \mathrm{Following}(\mathrm{Following}(s), n)$.

Let S be a non void circuit-like non empty many sorted signature, let A be a non-empty circuit of S, let s be a state of A, and let x be a set. We say that s is stable at x if and only if:

(Def.9) For every natural number n holds $(\mathrm{Following}(s,n))(x) = s(x)$.

The following propositions are true:

(17) Let S be a non void circuit-like non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let x be a set. If s is stable at x, then for every natural number n holds $\mathrm{Following}(s,n)$ is stable at x.

(18) Let S be a non void circuit-like non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let x be a set. If $x \in \mathrm{InputVertices}(S)$, then s is stable at x.

(19) Let S be a non void circuit-like non empty many sorted signature, and let A be a non-empty circuit of S, and let s be a state of A, and let g be a gate of S. Suppose that for every set x such that $x \in \mathrm{rng}\,\mathrm{Arity}(g)$ holds s is stable at x. Then $\mathrm{Following}(s)$ is stable at the result sort of g.

4. Unsplit Condition

The following propositions are true:

(20) Let S_1, S_2 be non empty many sorted signatures and let v be a vertex of S_1. Then $v \in$ the carrier of $S_1{+\cdot}S_2$ and $v \in$ the carrier of $S_2{+\cdot}S_1$.

(21) Let S_1, S_2 be unsplit non empty many sorted signatures with arity held in gates and let x be a set. If $x \in \mathrm{InnerVertices}(S_1)$, then $x \in \mathrm{InnerVertices}(S_1{+\cdot}S_2)$ and $x \in \mathrm{InnerVertices}(S_2{+\cdot}S_1)$.

(22) For all non empty many sorted signatures S_1, S_2 and for every set x such that $x \in \mathrm{InnerVertices}(S_2)$ holds $x \in \mathrm{InnerVertices}(S_1{+\cdot}S_2)$.

(23) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $S_1{+\cdot}S_2 = S_2{+\cdot}S_1$.

(24) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, and let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates. Then $A_1{+\cdot}A_2 = A_2{+\cdot}A_1$.

(25) Let S_1, S_2, S_3 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, and let A_1 be a Boolean circuit of S_1, and let A_2 be a Boolean circuit of S_2, and let A_3 be a Boolean circuit of S_3. Then $(A_1{+\cdot}A_2){+\cdot}A_3 = A_1{+\cdot}(A_2{+\cdot}A_3)$.

(26) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates, and let A_1 be a Boolean non-empty circuit of S_1 with denotation held in gates, and let

A_2 be a Boolean non-empty circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$. Then $s \restriction$ (the carrier of S_1) is a state of A_1 and $s \restriction$ (the carrier of S_2) is a state of A_2.

(27) For all unsplit non empty many sorted signatures S_1, S_2 with arity held in gates holds $\mathrm{InnerVertices}(S_1 {+\cdot} S_2) = \mathrm{InnerVertices}(S_1) \cup \mathrm{InnerVertices}(S_2)$.

(28) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$, and let s_1 be a state of A_1. If $s_1 = s \restriction$ (the carrier of S_1), then $\mathrm{Following}(s) \restriction$ (the carrier of S_1) $= \mathrm{Following}(s_1)$.

(29) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$, and let s_2 be a state of A_2. If $s_2 = s \restriction$ (the carrier of S_2), then $\mathrm{Following}(s) \restriction$ (the carrier of S_2) $= \mathrm{Following}(s_2)$.

(30) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$, and let s_1 be a state of A_1. Suppose $s_1 = s \restriction$ (the carrier of S_1). Let n be a natural number. Then $\mathrm{Following}(s, n) \restriction$ (the carrier of S_1) $= \mathrm{Following}(s_1, n)$.

(31) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_1)$ misses $\mathrm{InputVertices}(S_2)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$, and let s_2 be a state of A_2. Suppose $s_2 = s \restriction$ (the carrier of S_2). Let n be a natural number. Then $\mathrm{Following}(s, n) \restriction$ (the carrier of S_2) $= \mathrm{Following}(s_2, n)$.

(32) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose $\mathrm{InnerVertices}(S_2)$ misses $\mathrm{InputVertices}(S_1)$. Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 {+\cdot} A_2$, and let s_1 be a state of A_1. Suppose $s_1 = s \restriction$ (the carrier of S_1). Let v be a set. Suppose $v \in$ the carrier of S_1. Let n be a natural number. Then $(\mathrm{Following}(s, n))(v) = (\mathrm{Following}(s_1, n))(v)$.

(33) Let S_1, S_2 be unsplit non void non empty many sorted signatures with arity held in gates and Boolean denotation held in gates. Suppose

InnerVertices(S_1) misses InputVertices(S_2). Let A_1 be a Boolean circuit of S_1 with denotation held in gates, and let A_2 be a Boolean circuit of S_2 with denotation held in gates, and let s be a state of $A_1 +\cdot A_2$, and let s_2 be a state of A_2. Suppose $s_2 = s \restriction$ (the carrier of S_2). Let v be a set. Suppose $v \in$ the carrier of S_2. Let n be a natural number. Then (Following(s, n))(v) = (Following(s_2, n))(v).

Let S be a non void non empty many sorted signature with denotation held in gates and let g be a gate of S. One can verify that $g_{\mathbf{2}}$ is function-like and relation-like.

Next we state four propositions:

(34) Let S be a circuit-like non void non empty many sorted signature with denotation held in gates and let A be a non-empty circuit of S. Suppose A has denotation held in gates. Let s be a state of A and let g be a gate of S. Then (Following(s))(the result sort of g) = $g_{\mathbf{2}}(s \cdot \mathrm{Arity}(g))$.

(35) Let S be an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, and let A be a Boolean non-empty circuit of S with denotation held in gates, and let s be a state of A, and let p be a finite sequence, and let f be a function. If $\langle p, f \rangle \in$ the operation symbols of S, then (Following(s))($\langle p, f \rangle$) = $f(s \cdot p)$.

(36) Let S be an unsplit non void non empty many sorted signature with arity held in gates and Boolean denotation held in gates, and let A be a Boolean non-empty circuit of S with denotation held in gates, and let s be a state of A, and let p be a finite sequence, and let f be a function. Suppose $\langle p, f \rangle \in$ the operation symbols of S and for every set x such that $x \in \mathrm{rng}\, p$ holds s is stable at x. Then Following(s) is stable at $\langle p, f \rangle$.

(37) For every unsplit non empty many sorted signature S holds InnerVertices(S) = the operation symbols of S.

5. One Gate Circuits

We now state a number of propositions:

(38) For every set f and for every finite sequence p holds InnerVertices(1GateCircStr(p, f)) is a binary relation.

(39) For every set f and for every nonpair yielding finite sequence p holds InputVertices(1GateCircStr(p, f)) has no pairs.

(40) For every set f and for all sets x, y holds InputVertices(1GateCircStr($\langle x, y \rangle, f$)) = $\{x, y\}$.

(41) For every set f and for all non pair sets x, y holds InputVertices(1GateCircStr($\langle x, y \rangle, f$)) has no pairs.

(42) For every set f and for all sets x, y, z holds InputVertices(1GateCircStr($\langle x, y, z \rangle, f$)) = $\{x, y, z\}$.

(43) Let x, y, f be sets. Then $x \in$ the carrier of 1GateCircStr($\langle x, y\rangle, f$) and $y \in$ the carrier of 1GateCircStr($\langle x, y\rangle, f$) and $\langle\langle x, y\rangle, f\rangle \in$ the carrier of 1GateCircStr($\langle x, y\rangle, f$).

(44) Let x, y, z, f be sets. Then $x \in$ the carrier of 1GateCircStr($\langle x, y, z\rangle, f$) and $y \in$ the carrier of 1GateCircStr($\langle x, y, z\rangle, f$) and $z \in$ the carrier of 1GateCircStr($\langle x, y, z\rangle, f$).

(45) Let f, x be sets and let p be a finite sequence. Then $x \in$ the carrier of 1GateCircStr(p, f, x) and for every set y such that $y \in \operatorname{rng} p$ holds $y \in$ the carrier of 1GateCircStr(p, f, x).

(46) For all sets f, x and for every finite sequence p holds 1GateCircStr(p, f, x) is circuit-like and has arity held in gates.

(47) For every finite sequence p and for every set f holds $\langle p, f\rangle \in$ InnerVertices(1GateCircStr(p, f)).

Let x, y be sets and let f be a function from *Boolean*^{2} into *Boolean*. The functor 1GateCircuit(x, y, f) yielding a Boolean strict circuit of 1GateCircStr($\langle x, y\rangle, f$) with denotation held in gates is defined by:

(Def.10) 1GateCircuit(x, y, f) = 1GateCircuit($\langle x, y\rangle, f$).

We adopt the following convention: x, y, z, c denote sets and f denotes a function from *Boolean*^{2} into *Boolean*.

We now state four propositions:

(48) Let X be a finite non empty set, and let f be a function from X^2 into X, and let s be a state of 1GateCircuit($\langle x, y\rangle, f$). Then (Following(s))($\langle\langle x, y\rangle, f\rangle$) $= f(\langle s(x), s(y)\rangle)$ and (Following(s))(x) $= s(x)$ and (Following(s))(y) $= s(y)$.

(49) Let X be a finite non empty set, and let f be a function from X^2 into X, and let s be a state of 1GateCircuit($\langle x, y\rangle, f$). Then Following(s) is stable.

(50) For every state s of 1GateCircuit(x, y, f) holds (Following(s))($\langle\langle x, y\rangle, f\rangle$) $= f(\langle s(x), s(y)\rangle)$ and (Following(s))(x) $= s(x)$ and (Following(s))(y) $= s(y)$.

(51) For every state s of 1GateCircuit(x, y, f) holds Following(s) is stable.

Let x, y, z be sets and let f be a function from *Boolean*^{3} into *Boolean*. The functor 1GateCircuit(x, y, z, f) yields a Boolean strict circuit of 1GateCircStr($\langle x, y, z\rangle, f$) with denotation held in gates and is defined by:

(Def.11) 1GateCircuit(x, y, z, f) = 1GateCircuit($\langle x, y, z\rangle, f$).

We now state four propositions:

(52) Let X be a finite non empty set, and let f be a function from X^3 into X, and let s be a state of 1GateCircuit($\langle x, y, z\rangle, f$). Then (Following($s$))($\langle\langle x, y, z\rangle, f\rangle$) $= f(\langle s(x), s(y), s(z)\rangle)$ and (Following(s))(x) $= s(x)$ and (Following(s))(y) $= s(y)$ and (Following(s))(z) $= s(z)$.

(53) Let X be a finite non empty set, and let f be a function from X^3 into X, and let s be a state of 1GateCircuit($\langle x, y, z\rangle, f$). Then Following($s$) is

stable.

(54) Let f be a function from $Boolean^3$ into $Boolean$ and let s be a state of 1GateCircuit(x, y, z, f). Then $(\mathrm{Following}(s))(\langle\langle x, y, z\rangle, f\rangle) = f(\langle s(x), s(y), s(z)\rangle)$ and $(\mathrm{Following}(s))(x) = s(x)$ and $(\mathrm{Following}(s))(y) = s(y)$ and $(\mathrm{Following}(s))(z) = s(z)$.

(55) For every function f from $Boolean^3$ into $Boolean$ and for every state s of 1GateCircuit(x, y, z, f) holds Following(s) is stable.

6. Boolean Circuits

Let x, y, c be sets and let f be a function from $Boolean^2$ into $Boolean$. The functor 2GatesCircStr(x, y, c, f) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def.12) 2GatesCircStr(x, y, c, f) = 1GateCircStr$(\langle x, y\rangle, f)$+· 1GateCircStr$(\langle\langle\langle x, y\rangle, f\rangle, c\rangle, f)$.

Let x, y, c be sets and let f be a function from $Boolean^2$ into $Boolean$. The functor 2GatesCircOutput(x, y, c, f) yields an element of InnerVertices(2GatesCircStr(x, y, c, f)) and is defined as follows:

(Def.13) 2GatesCircOutput$(x, y, c, f) = \langle\langle\langle\langle x, y\rangle, f\rangle, c\rangle, f\rangle$.

Let x, y, c be sets and let f be a function from $Boolean^2$ into $Boolean$. One can verify that 2GatesCircOutput(x, y, c, f) is pair.

One can prove the following two propositions:

(56) InnerVertices(2GatesCircStr(x, y, c, f)) = $\{\langle\langle x, y\rangle, f\rangle$, 2GatesCircOutput$(x, y, c, f)\}$.

(57) If $c \neq \langle\langle x, y\rangle, f\rangle$, then InputVertices(2GatesCircStr(x, y, c, f)) = $\{x, y, c\}$.

Let x, y, c be sets and let f be a function from $Boolean^2$ into $Boolean$. The functor 2GatesCircuit(x, y, c, f) yields a strict Boolean circuit of 2GatesCircStr(x, y, c, f) with denotation held in gates and is defined by:

(Def.14) 2GatesCircuit(x, y, c, f) = 1GateCircuit(x, y, f)+· 1GateCircuit$(\langle\langle x, y\rangle, f\rangle, c, f)$.

We now state four propositions:

(58) InnerVertices(2GatesCircStr(x, y, c, f)) is a binary relation.

(59) For all non pair sets x, y, c holds InputVertices(2GatesCircStr(x, y, c, f)) has no pairs.

(60) $x \in$ the carrier of 2GatesCircStr(x, y, c, f) and $y \in$ the carrier of 2GatesCircStr(x, y, c, f) and $c \in$ the carrier of 2GatesCircStr(x, y, c, f).

(61) $\langle\langle x, y\rangle, f\rangle \in$ the carrier of 2GatesCircStr(x, y, c, f) and $\langle\langle\langle\langle x, y\rangle, f\rangle, c\rangle, f\rangle \in$ the carrier of 2GatesCircStr(x, y, c, f).

Let S be an unsplit non void non empty many sorted signature, let A be a Boolean circuit of S, let s be a state of A, and let v be a vertex of S. Then $s(v)$ is an element of *Boolean*.

In the sequel s will be a state of 2GatesCircuit(x, y, c, f).

One can prove the following propositions:

(62) Suppose $c \neq \langle\langle x, y\rangle, f\rangle$. Then (Following$(s, 2)$)(2GatesCircOutput$(x, y, c, f)) = f(\langle f(\langle s(x), s(y)\rangle), s(c)\rangle)$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, f\rangle) = f(\langle s(x), s(y)\rangle)$ and (Following$(s, 2)$)$(x) = s(x)$ and (Following$(s, 2)$)$(y) = s(y)$ and (Following$(s, 2)$)$(c) = s(c)$.

(63) If $c \neq \langle\langle x, y\rangle, f\rangle$, then Following$(s, 2)$ is stable.

(64) Suppose $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$. Let s be a state of 2GatesCircuit(x, y, c, xor) and let a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then (Following$(s, 2)$)(2GatesCircOutput$(x, y, c, \mathrm{xor})) = a_1 \oplus a_2 \oplus a_3$.

(65) Suppose $c \neq \langle\langle x, y\rangle, \mathrm{or}\rangle$. Let s be a state of 2GatesCircuit(x, y, c, or) and let a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then (Following$(s, 2)$)(2GatesCircOutput$(x, y, c, \mathrm{or})) = a_1 \vee a_2 \vee a_3$.

(66) Suppose $c \neq \langle\langle x, y\rangle, \&\rangle$. Let s be a state of 2GatesCircuit$(x, y, c, \&)$ and let a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then (Following$(s, 2)$)(2GatesCircOutput$(x, y, c, \&)) = a_1 \wedge a_2 \wedge a_3$.

7. One Bit Adder

Let x, y, c be sets. The functor BitAdderOutput(x, y, c) yields an element of InnerVertices(2GatesCircStr(x, y, c, xor)) and is defined as follows:

(Def.15) BitAdderOutput(x, y, c) = 2GatesCircOutput(x, y, c, xor).

Let x, y, c be sets. The functor BitAdderCirc(x, y, c) yields a strict Boolean circuit of 2GatesCircStr(x, y, c, xor) with denotation held in gates and is defined as follows:

(Def.16) BitAdderCirc(x, y, c) = 2GatesCircuit(x, y, c, xor).

Let x, y, c be sets. The functor MajorityIStr(x, y, c) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def.17) MajorityIStr(x, y, c) = 1GateCircStr$(\langle x, y\rangle, \&)$+· 1GateCircStr$(\langle y, c\rangle, \&)$+· 1GateCircStr$(\langle c, x\rangle, \&)$.

Let x, y, c be sets. The functor MajorityStr(x, y, c) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def.18) MajorityStr(x, y, c) = MajorityIStr(x, y, c)+· 1GateCircStr$(\langle\langle\langle x, y\rangle, \&\rangle, \langle\langle y, c\rangle, \&\rangle, \langle\langle c, x\rangle, \&\rangle\rangle, \mathrm{or}_3)$.

Let x, y, c be sets. The functor $\mathrm{MajorityICirc}(x,y,c)$ yields a strict Boolean circuit of $\mathrm{MajorityIStr}(x,y,c)$ with denotation held in gates and is defined as follows:

(Def.19) $\mathrm{MajorityICirc}(x,y,c) = \mathrm{1GateCircuit}(x,y,\&) +\cdot \mathrm{1GateCircuit}(y,c,\&) +\cdot \mathrm{1GateCircuit}(c,x,\&)$.

Next we state several propositions:

(67) $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ is a binary relation.

(68) For all non pair sets x, y, c holds $\mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ has no pairs.

(69) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(x)$ and $b = s(y)$ holds $(\mathrm{Following}(s))(\langle\langle x,y\rangle, \&\rangle) = a \wedge b$.

(70) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(y)$ and $b = s(c)$ holds $(\mathrm{Following}(s))(\langle\langle y,c\rangle, \&\rangle) = a \wedge b$.

(71) For every state s of $\mathrm{MajorityICirc}(x,y,c)$ and for all elements a, b of *Boolean* such that $a = s(c)$ and $b = s(x)$ holds $(\mathrm{Following}(s))(\langle\langle c,x\rangle, \&\rangle) = a \wedge b$.

Let x, y, c be sets. The functor $\mathrm{MajorityOutput}(x,y,c)$ yields an element of $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and is defined by:

(Def.20) $\mathrm{MajorityOutput}(x,y,c) = \langle\langle\langle\langle x,y\rangle, \&\rangle, \langle\langle y,c\rangle, \&\rangle, \langle\langle c,x\rangle, \&\rangle\rangle, \mathrm{or}_3\rangle$.

Let x, y, c be sets. The functor $\mathrm{MajorityCirc}(x,y,c)$ yielding a strict Boolean circuit of $\mathrm{MajorityStr}(x,y,c)$ with denotation held in gates is defined by:

(Def.21) $\mathrm{MajorityCirc}(x,y,c) = \mathrm{MajorityICirc}(x,y,c) +\cdot \mathrm{1GateCircuit}(\langle\langle x,y\rangle, \&\rangle, \langle\langle y,c\rangle, \&\rangle, \langle\langle c,x\rangle, \&\rangle, \mathrm{or}_3)$.

Next we state a number of propositions:

(72) $x \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$ and $y \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$ and $c \in$ the carrier of $\mathrm{MajorityStr}(x,y,c)$.

(73) $\langle\langle x,y\rangle, \&\rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and $\langle\langle y,c\rangle, \&\rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$ and $\langle\langle c,x\rangle, \&\rangle \in \mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c))$.

(74) For all non pair sets x, y, c holds $x \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ and $y \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$ and $c \in \mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c))$.

(75) For all non pair sets x, y, c holds $\mathrm{InputVertices}(\mathrm{MajorityStr}(x,y,c)) = \{x,y,c\}$ and $\mathrm{InnerVertices}(\mathrm{MajorityStr}(x,y,c)) = \{\langle\langle x,y\rangle, \&\rangle, \langle\langle y,c\rangle, \&\rangle, \langle\langle c,x\rangle, \&\rangle\} \cup \{\mathrm{MajorityOutput}(x,y,c)\}$.

(76) Let x, y, c be non pair sets, and let s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and let a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$, then $(\mathrm{Following}(s))(\langle\langle x,y\rangle, \&\rangle) = a_1 \wedge a_2$.

(77) Let x, y, c be non pair sets, and let s be a state of $\mathrm{MajorityCirc}(x,y,c)$, and let a_2, a_3 be elements of *Boolean*. If $a_2 = s(y)$ and $a_3 = s(c)$, then

(Following(s))($\langle\langle y, c\rangle$, &$\rangle$) $= a_2 \wedge a_3$.

(78) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_1, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_3 = s(c)$, then (Following(s))($\langle\langle c, x\rangle$, &$\rangle$) $= a_3 \wedge a_1$.

(79) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle$, &$\rangle)$ and $a_2 = s(\langle\langle y, c\rangle$, &$\rangle)$ and $a_3 = s(\langle\langle c, x\rangle$, &$\rangle)$, then (Following($s$))(MajorityOutput($x, y, c$)) $= a_1 \vee a_2 \vee a_3$.

(80) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_1, a_2 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$, then (Following($s, 2$))($\langle\langle x, y\rangle$, &$\rangle$) $= a_1 \wedge a_2$.

(81) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_2, a_3 be elements of *Boolean*. If $a_2 = s(y)$ and $a_3 = s(c)$, then (Following($s, 2$))($\langle\langle y, c\rangle$, &$\rangle$) $= a_2 \wedge a_3$.

(82) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_1, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_3 = s(c)$, then (Following($s, 2$))($\langle\langle c, x\rangle$, &$\rangle$) $= a_3 \wedge a_1$.

(83) Let x, y, c be non pair sets, and let s be a state of MajorityCirc(x, y, c), and let a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$, then (Following($s, 2$))(MajorityOutput(x, y, c)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(84) For all non pair sets x, y, c and for every state s of MajorityCirc(x, y, c) holds Following($s, 2$) is stable.

Let x, y, c be sets. The functor BitAdderWithOverflowStr(x, y, c) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def.22) BitAdderWithOverflowStr(x, y, c) = 2GatesCircStr(x, y, c, xor) +· MajorityStr(x, y, c).

The following three propositions are true:

(85) For all non pair sets x, y, c holds InputVertices(BitAdderWithOverflowStr (x, y, c)) $= \{x, y, c\}$.

(86) For all non pair sets x, y, c holds InnerVertices(BitAdderWithOverflowStr (x, y, c)) $= \{\langle\langle x, y\rangle$, xor$\rangle$, 2GatesCircOutput($x, y, c$, xor)$\} \cup \{\langle\langle x, y\rangle$, &$\rangle$, $\langle\langle y, c\rangle$, &$\rangle$, $\langle\langle c, x\rangle$, &$\rangle\} \cup \{$MajorityOutput($x, y, c$)$\}$.

(87) Let S be a non empty many sorted signature. Suppose S = BitAdderWithOverflowStr(x, y, c). Then $x \in$ the carrier of S and $y \in$ the carrier of S and $c \in$ the carrier of S.

Let x, y, c be sets. The functor BitAdderWithOverflowCirc(x, y, c) yielding a strict Boolean circuit of BitAdderWithOverflowStr(x, y, c) with denotation held in gates is defined as follows:

(Def.23) BitAdderWithOverflowCirc(x, y, c) = BitAdderCirc(x, y, c) +· MajorityCirc(x, y, c).

We now state several propositions:

(88) InnerVertices(BitAdderWithOverflowStr(x, y, c)) is a binary relation.

(89) For all non pair sets x, y, c holds InputVertices(BitAdderWithOverflowStr (x, y, c)) has no pairs.

(90) BitAdderOutput$(x, y, c) \in$ InnerVertices(BitAdderWithOverflowStr(x, y, c)) and MajorityOutput$(x, y, c) \in$ InnerVertices(BitAdderWithOverflow Str(x, y, c)).

(91) Let x, y, c be non pair sets, and let s be a state of BitAdderWithOverflowCirc(x, y, c), and let a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then (Following$(s, 2)$)(BitAdderOutput(x, y, c)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)(MajorityOutput(x, y, c)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(92) For all non pair sets x, y, c and for every state s of BitAdderWithOverflowCirc(x, y, c) holds Following$(s, 2)$ is stable.

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Continuous, Stable, and Linear Maps of Coherence Spaces

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The papers [18], [21], [9], [14], [16], [11], [3], [19], [22], [7], [6], [10], [20], [12], [13], [17], [1], [2], [5], [8], [15], and [4] provide the terminology and notation for this paper.

1. Directed Sets

One can check that there exists a coherent space which is finite. Let us observe that a set is binary complete if:

(Def.1) For every set A such that for all sets a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in$ it holds $\bigcup A \in$ it.

Let X be a set. The functor FlatCoh(X) yielding a set is defined as follows:

(Def.2) $\mathrm{FlatCoh}(X) = \mathrm{CohSp}(\triangle_X)$.

The functor SubFin(X) yielding a subset of X is defined by:

(Def.3) For every set x holds $x \in \mathrm{SubFin}(X)$ iff $x \in X$ and x is finite.

One can prove the following three propositions:

(1) For all sets X, x holds $x \in \mathrm{FlatCoh}(X)$ iff $x = \emptyset$ or there exists a set y such that $x = \{y\}$ and $y \in X$.

(2) For every set X holds $\bigcup \mathrm{FlatCoh}(X) = X$.

(3) For every finite down-closed set X holds $\mathrm{SubFin}(X) = X$.

One can check that $\{\emptyset\}$ is down-closed and binary complete. Let X be a set. One can check that 2^X is down-closed and binary complete and FlatCoh(X) is non empty down-closed and binary complete.

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Let C be a non empty down-closed set. Observe that $\mathrm{SubFin}(C)$ is non empty and down-closed.

We now state the proposition

(4) $\mathrm{Web}(\{\emptyset\}) = \emptyset$.

The scheme *MinimalElement wrt Incl* concerns sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a set a such that $a \in \mathcal{B}$ and $\mathcal{P}[a]$ and for every set b such that $b \in \mathcal{B}$ and $\mathcal{P}[b]$ and $b \subseteq a$ holds $b = a$

provided the following requirements are met:

- $\mathcal{A} \in \mathcal{B}$,
- $\mathcal{P}[\mathcal{A}]$,
- $\mathcal{A}$ is finite.

Let X be a set. One can check that there exists a subset of X which is finite.

Let C be a coherent space. Observe that there exists an element of C which is finite.

Let X be a set. We say that X is $\cup$-directed if and only if:

(Def.4) For every finite subset Y of X there exists a set a such that $\bigcup Y \subseteq a$ and $a \in X$.

We say that X is $\cap$-directed if and only if:

(Def.5) For every finite subset Y of X there exists a set a such that for every set y such that $y \in Y$ holds $a \subseteq y$ and $a \in X$.

Let us note that every set which is $\cup$-directed is also non empty and every set which is $\cap$-directed is also non empty.

We now state several propositions:

(5) Let X be a set. Suppose X is $\cup$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $a \cup b \subseteq c$ and $c \in X$.

(6) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $a \cup b \subseteq c$ and $c \in X$. Then X is $\cup$-directed.

(7) Let X be a set. Suppose X is $\cap$-directed. Let a, b be sets. If $a \in X$ and $b \in X$, then there exists a set c such that $c \subseteq a \cap b$ and $c \in X$.

(8) Let X be a non empty set. Suppose that for all sets a, b such that $a \in X$ and $b \in X$ there exists a set c such that $c \subseteq a \cap b$ and $c \in X$. Then X is $\cap$-directed.

(9) For every set x holds $\{x\}$ is $\cup$-directed and $\cap$-directed.

(10) For all sets x, y holds $\{x, y, x \cup y\}$ is $\cup$-directed.

(11) For all sets x, y holds $\{x, y, x \cap y\}$ is $\cap$-directed.

Let us observe that there exists a set which is $\cup$-directed $\cap$-directed and finite.

Let C be a non empty set. Observe that there exists a subset of C which is $\cup$-directed $\cap$-directed and finite.

We now state the proposition

(12) For every set X holds $\operatorname{Fin} X$ is $\cup$-directed and $\cap$-directed.

Let X be a set. Observe that $\operatorname{Fin} X$ is $\cup$-directed and $\cap$-directed.

Let C be a down-closed non empty set. Note that there exists a subset of C which is preboolean and non empty.

Let C be a down-closed non empty set and let a be an element of C. Then $\operatorname{Fin} a$ is a preboolean non empty subset of C.

One can prove the following proposition

(13) Let X be a non empty set and let Y be a set. Suppose X is $\cup$-directed and $Y \subseteq \bigcup X$ and Y is finite. Then there exists a set Z such that $Z \in X$ and $Y \subseteq Z$.

Let X be a set. We say that X is $\cap$-closed if and only if:

(Def.6) For all sets x, y such that $x \in X$ and $y \in X$ holds $x \cap y \in X$.

We say that X is closed under directed unions if and only if:

(Def.7) For every subset A of X such that A is $\cup$-directed holds $\bigcup A \in X$.

One can check that every set which is down-closed is also $\cap$-closed.

Next we state two propositions:

(14) For every coherent space C and for all elements x, y of C holds $x \cap y \in C$.

(15) For every coherent space C and for every $\cup$-directed subset A of C holds $\bigcup A \in C$.

Let us note that every coherent space is closed under directed unions.

Let us note that there exists a coherent space which is $\cap$-closed and closed under directed unions.

Let C be a closed under directed unions non empty set and let A be a $\cup$-directed subset of C. Then $\bigcup A$ is an element of C.

Let X, Y be sets. We say that X includes lattice of Y if and only if:

(Def.8) For all sets a, b such that $a \in Y$ and $b \in Y$ holds $a \cap b \in X$ and $a \cup b \in X$.

The following proposition is true

(16) For every non empty set X such that X includes lattice of X holds X is $\cup$-directed and $\cap$-directed.

Let X, x, y be sets. We say that X includes lattice of x, y if and only if:

(Def.9) X includes lattice of $\{x, y\}$.

One can prove the following proposition

(17) For all sets X, x, y holds X includes lattice of x, y iff $x \in X$ and $y \in X$ and $x \cap y \in X$ and $x \cup y \in X$.

2. Continuous, Stable, and Linear Functions

Let f be a function. We say that f is preserving arbitrary unions if and only if:

(Def.10) For every subset A of $\operatorname{dom} f$ such that $\bigcup A \in \operatorname{dom} f$ holds $f(\bigcup A) = \bigcup (f^\circ A)$.

We say that f is preserving directed unions if and only if:

(Def.11) For every subset A of $\operatorname{dom} f$ such that A is $\cup$-directed and $\bigcup A \in \operatorname{dom} f$ holds $f(\bigcup A) = \bigcup(f^{\circ}A)$.

Let f be a function. We say that f is $\subseteq$-monotone if and only if:

(Def.12) For all sets a, b such that $a \in \operatorname{dom} f$ and $b \in \operatorname{dom} f$ and $a \subseteq b$ holds $f(a) \subseteq f(b)$.

We say that f is preserving binary intersections if and only if:

(Def.13) For all sets a, b such that $\operatorname{dom} f$ includes lattice of a, b holds $f(a \cap b) = f(a) \cap f(b)$.

Let us note that every function which is preserving directed unions is also $\subseteq$-monotone and every function which is preserving arbitrary unions is also preserving directed unions.

Next we state two propositions:

(18) Let f be a function. Suppose f is preserving arbitrary unions. Let x, y be sets. If $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $x \cup y \in \operatorname{dom} f$, then $f(x \cup y) = f(x) \cup f(y)$.

(19) For every function f such that f is preserving arbitrary unions holds $f(\emptyset) = \emptyset$.

Let C_1, C_2 be coherent spaces. Note that there exists a function from C_1 into C_2 which is preserving arbitrary unions and preserving binary intersections.

Let C be a coherent space. One can verify that there exists a many sorted set indexed by C which is preserving arbitrary unions and preserving binary intersections.

Let f be a function. We say that f is continuous if and only if:

(Def.14) $\operatorname{dom} f$ is closed under directed unions and f is preserving directed unions.

Let f be a function. We say that f is stable if and only if:

(Def.15) $\operatorname{dom} f$ is $\cap$-closed and f is continuous and preserving binary intersections.

Let f be a function. We say that f is linear if and only if:

(Def.16) f is stable and preserving arbitrary unions.

One can check the following observations:

* every function which is continuous is also preserving directed unions,
* every function which is stable is also preserving binary intersections and continuous, and
* every function which is linear is also preserving arbitrary unions and stable.

Let X be a closed under directed unions set. Note that every many sorted set indexed by X which is preserving directed unions is also continuous.

Let X be a $\cap$-closed set. Observe that every many sorted set indexed by X which is continuous and preserving binary intersections is also stable.

Let us note that every function which is stable and preserving arbitrary unions is also linear.

Note that there exists a function which is linear. Let C be a coherent space. One can check that there exists a many sorted set indexed by C which is linear. Let B be a coherent space. One can check that there exists a function from B into C which is linear.

Let f be a continuous function. One can verify that $\operatorname{dom} f$ is closed under directed unions.

Let f be a stable function. One can verify that $\operatorname{dom} f$ is $\cap$-closed.

We now state several propositions:

(20) For every set X holds $\bigcup \operatorname{Fin} X = X$.

(21) For every continuous function f such that $\operatorname{dom} f$ is down-closed and for every set a such that $a \in \operatorname{dom} f$ holds $f(a) = \bigcup (f^\circ \operatorname{Fin} a)$.

(22) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed. Then f is continuous if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f$ is closed under directed unions,

(ii) f is $\subseteq$-monotone, and

(iii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$.

(23) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed and closed under directed unions. Then f is stable if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set b such that b is finite and $b \subseteq a$ and $y \in f(b)$ and for every set c such that $c \subseteq a$ and $y \in f(c)$ holds $b \subseteq c$.

(24) Let f be a function. Suppose $\operatorname{dom} f$ is down-closed and closed under directed unions. Then f is linear if and only if the following conditions are satisfied:

(i) f is $\subseteq$-monotone, and

(ii) for all sets a, y such that $a \in \operatorname{dom} f$ and $y \in f(a)$ there exists a set x such that $x \in a$ and $y \in f(\{x\})$ and for every set b such that $b \subseteq a$ and $y \in f(b)$ holds $x \in b$.

3. Graph of Continuous Function

Let f be a function. The functor $\operatorname{graph}(f)$ yielding a set is defined as follows:

(Def.17) For every set x holds $x \in \operatorname{graph}(f)$ iff there exists a finite set y and there exists a set z such that $x = \langle y, z \rangle$ and $y \in \operatorname{dom} f$ and $z \in f(y)$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\operatorname{graph}(f)$ is a subset of $[\!: C_1, \bigcup C_2 :\!]$.

Let f be a function. Note that $\operatorname{graph}(f)$ is relation-like.

Next we state several propositions:

(25) For every function f and for all sets x, y holds $\langle x, y\rangle \in \operatorname{graph}(f)$ iff x is finite and $x \in \operatorname{dom} f$ and $y \in f(x)$.

(26) Let f be a $\subseteq$-monotone function and let a, b be sets. Suppose $b \in \operatorname{dom} f$ and $a \subseteq b$ and b is finite. Let y be a set. If $\langle a, y\rangle \in \operatorname{graph}(f)$, then $\langle b, y\rangle \in \operatorname{graph}(f)$.

(27) Let C_1, C_2 be coherent spaces, and let f be a function from C_1 into C_2, and let a be an element of C_1, and let y_1, y_2 be sets. If $\langle a, y_1\rangle \in \operatorname{graph}(f)$ and $\langle a, y_2\rangle \in \operatorname{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(28) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let a, b be elements of C_1. Suppose $a \cup b \in C_1$. Let y_1, y_2 be sets. If $\langle a, y_1\rangle \in \operatorname{graph}(f)$ and $\langle b, y_2\rangle \in \operatorname{graph}(f)$, then $\{y_1, y_2\} \in C_2$.

(29) For all coherent spaces C_1, C_2 and for all continuous functions f, g from C_1 into C_2 such that $\operatorname{graph}(f) = \operatorname{graph}(g)$ holds $f = g$.

(30) Let C_1, C_2 be coherent spaces and let X be a subset of $[: C_1, \bigcup C_2 :]$. Suppose that

- (i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,
- (ii) for all finite elements a, b of C_1 such that $a \subseteq b$ and for every set y such that $\langle a, y\rangle \in X$ holds $\langle b, y\rangle \in X$, and
- (iii) for every finite element a of C_1 and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle a, y_2\rangle \in X$ holds $\{y_1, y_2\} \in C_2$.

Then there exists a continuous function f from C_1 into C_2 such that $X = \operatorname{graph}(f)$.

(31) Let C_1, C_2 be coherent spaces, and let f be a continuous function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\operatorname{graph}(f))^{\circ} \operatorname{Fin} a$.

4. Trace of Stable Function

Let f be a function. The functor $\operatorname{Trace}(f)$ yields a set and is defined by the condition (Def.18).

(Def.18) Let x be a set. Then $x \in \operatorname{Trace}(f)$ if and only if there exist sets a, y such that $x = \langle a, y\rangle$ and $a \in \operatorname{dom} f$ and $y \in f(a)$ and for every set b such that $b \in \operatorname{dom} f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Next we state the proposition

(32) Let f be a function and let a, y be sets. Then $\langle a, y\rangle \in \operatorname{Trace}(f)$ if and only if the following conditions are satisfied:

- (i) $a \in \operatorname{dom} f$,
- (ii) $y \in f(a)$, and
- (iii) for every set b such that $b \in \operatorname{dom} f$ and $b \subseteq a$ and $y \in f(b)$ holds $a = b$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\operatorname{Trace}(f)$ is a subset of $[: C_1, \bigcup C_2 :]$.

Let f be a function. One can check that $\mathrm{Trace}(f)$ is relation-like.

Next we state a number of propositions:

(33) For every continuous function f such that $\mathrm{dom}\, f$ is down-closed holds $\mathrm{Trace}(f) \subseteq \mathrm{graph}(f)$.

(34) Let f be a continuous function. Suppose $\mathrm{dom}\, f$ is down-closed. Let a, y be sets. If $\langle a, y \rangle \in \mathrm{Trace}(f)$, then a is finite.

(35) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let a_1, a_2 be sets. Suppose $a_1 \cup a_2 \in C_1$. Let y_1, y_2 be sets. If $\langle a_1, y_1 \rangle \in \mathrm{Trace}(f)$ and $\langle a_2, y_2 \rangle \in \mathrm{Trace}(f)$, then $\{y_1, y_2\} \in C_2$.

(36) Let C_1, C_2 be coherent spaces, and let f be a preserving binary intersections function from C_1 into C_2, and let a_1, a_2 be sets. If $a_1 \cup a_2 \in C_1$, then for every set y such that $\langle a_1, y \rangle \in \mathrm{Trace}(f)$ and $\langle a_2, y \rangle \in \mathrm{Trace}(f)$ holds $a_1 = a_2$.

(37) Let C_1, C_2 be coherent spaces and let f, g be stable functions from C_1 into C_2. If $\mathrm{Trace}(f) \subseteq \mathrm{Trace}(g)$, then for every element a of C_1 holds $f(a) \subseteq g(a)$.

(38) For all coherent spaces C_1, C_2 and for all stable functions f, g from C_1 into C_2 such that $\mathrm{Trace}(f) = \mathrm{Trace}(g)$ holds $f = g$.

(39) Let C_1, C_2 be coherent spaces and let X be a subset of $[\!: C_1, \bigcup C_2 :\!]$. Suppose that

(i) for every set x such that $x \in X$ holds $x_{\mathbf{1}}$ is finite,

(ii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1 \rangle \in X$ and $\langle b, y_2 \rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and

(iii) for all elements a, b of C_1 such that $a \cup b \in C_1$ and for every set y such that $\langle a, y \rangle \in X$ and $\langle b, y \rangle \in X$ holds $a = b$.

Then there exists a stable function f from C_1 into C_2 such that $X = \mathrm{Trace}(f)$.

(40) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\mathrm{Trace}(f))^{\circ}\, \mathrm{Fin}\, a$.

(41) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let a be an element of C_1, and let y be a set. Then $y \in f(a)$ if and only if there exists an element b of C_1 such that $\langle b, y \rangle \in \mathrm{Trace}(f)$ and $b \subseteq a$.

(42) For all coherent spaces C_1, C_2 there exists a stable function f from C_1 into C_2 such that $\mathrm{Trace}(f) = \emptyset$.

(43) Let C_1, C_2 be coherent spaces, and let a be a finite element of C_1, and let y be a set. If $y \in \bigcup C_2$, then there exists a stable function f from C_1 into C_2 such that $\mathrm{Trace}(f) = \{\langle a, y \rangle\}$.

(44) Let C_1, C_2 be coherent spaces, and let a be an element of C_1, and let y be a set. Suppose $y \in \bigcup C_2$. Let f be a stable function from C_1 into C_2. Suppose $\mathrm{Trace}(f) = \{\langle a, y \rangle\}$. Let b be an element of C_1. Then if $a \subseteq b$, then $f(b) = \{y\}$ and if $a \not\subseteq b$, then $f(b) = \emptyset$.

(45) Let C_1, C_2 be coherent spaces, and let f be a stable function from C_1 into C_2, and let X be a subset of $\mathrm{Trace}(f)$. Then there exists a stable function g from C_1 into C_2 such that $\mathrm{Trace}(g) = X$.

(46) Let C_1, C_2 be coherent spaces and let A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a stable function f from C_1 into C_2 such that $x \cup y = \mathrm{Trace}(f)$. Then there exists a stable function f from C_1 into C_2 such that $\bigcup A = \mathrm{Trace}(f)$.

Let C_1, C_2 be coherent spaces. The functor $\mathrm{StabCoh}(C_1, C_2)$ yielding a set is defined as follows:

(Def.19) For every set x holds $x \in \mathrm{StabCoh}(C_1, C_2)$ iff there exists a stable function f from C_1 into C_2 such that $x = \mathrm{Trace}(f)$.

Let C_1, C_2 be coherent spaces. Note that $\mathrm{StabCoh}(C_1, C_2)$ is non empty down-closed and binary complete.

We now state three propositions:

(47) For all coherent spaces C_1, C_2 and for every stable function f from C_1 into C_2 holds $\mathrm{Trace}(f) \subseteq [\!\!: \mathrm{SubFin}(C_1), \bigcup C_2 :\!\!]$.

(48) For all coherent spaces C_1, C_2 holds $\bigcup \mathrm{StabCoh}(C_1, C_2) = [\!\!: \mathrm{SubFin}(C_1), \bigcup C_2 :\!\!]$.

(49) Let C_1, C_2 be coherent spaces, and let a, b be finite elements of C_1, and let y_1, y_2 be sets. Then $\langle\langle a, y_1\rangle, \langle b, y_2\rangle\rangle \in \mathrm{Web}(\mathrm{StabCoh}(C_1, C_2))$ if and only if one of the following conditions is satisfied:

(i) $a \cup b \notin C_1$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$, or

(ii) $\langle y_1, y_2\rangle \in \mathrm{Web}(C_2)$ and if $y_1 = y_2$, then $a = b$.

5. Trace of Linear Function

The following proposition is true

(50) Let C_1, C_2 be coherent spaces and let f be a stable function from C_1 into C_2. Then f is linear if and only if for all sets a, y such that $\langle a, y\rangle \in \mathrm{Trace}(f)$ there exists a set x such that $a = \{x\}$.

Let f be a function. The functor $\mathrm{LinTrace}(f)$ yielding a set is defined as follows:

(Def.20) For every set x holds $x \in \mathrm{LinTrace}(f)$ iff there exist sets y, z such that $x = \langle y, z\rangle$ and $\langle\{y\}, z\rangle \in \mathrm{Trace}(f)$.

Next we state three propositions:

(51) For every function f and for all sets x, y holds $\langle x, y\rangle \in \mathrm{LinTrace}(f)$ iff $\langle\{x\}, y\rangle \in \mathrm{Trace}(f)$.

(52) For every function f such that $f(\emptyset) = \emptyset$ and for all sets x, y such that $\{x\} \in \mathrm{dom}\, f$ and $y \in f(\{x\})$ holds $\langle x, y\rangle \in \mathrm{LinTrace}(f)$.

(53) For every function f and for all sets x, y such that $\langle x, y\rangle \in \mathrm{LinTrace}(f)$ holds $\{x\} \in \mathrm{dom}\, f$ and $y \in f(\{x\})$.

Let C_1, C_2 be non empty sets and let f be a function from C_1 into C_2. Then $\mathrm{LinTrace}(f)$ is a subset of $[\!:\bigcup C_1, \bigcup C_2\,:\!]$.

Let f be a function. One can verify that $\mathrm{LinTrace}(f)$ is relation-like.

Let C_1, C_2 be coherent spaces. The functor $\mathrm{LinCoh}(C_1, C_2)$ yielding a set is defined as follows:

(Def.21) For every set x holds $x \in \mathrm{LinCoh}(C_1, C_2)$ iff there exists a linear function f from C_1 into C_2 such that $x = \mathrm{LinTrace}(f)$.

Next we state a number of propositions:

(54) Let C_1, C_2 be coherent spaces, and let f be a $\subseteq$-monotone function from C_1 into C_2, and let x_1, x_2 be sets. Suppose $\{x_1, x_2\} \in C_1$. Let y_1, y_2 be sets. If $\langle x_1, y_1\rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y_2\rangle \in \mathrm{LinTrace}(f)$, then $\{y_1, y_2\} \in C_2$.

(55) Let C_1, C_2 be coherent spaces, and let f be a preserving binary intersections function from C_1 into C_2, and let x_1, x_2 be sets. If $\{x_1, x_2\} \in C_1$, then for every set y such that $\langle x_1, y\rangle \in \mathrm{LinTrace}(f)$ and $\langle x_2, y\rangle \in \mathrm{LinTrace}(f)$ holds $x_1 = x_2$.

(56) For all coherent spaces C_1, C_2 and for all linear functions f, g from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \mathrm{LinTrace}(g)$ holds $f = g$.

(57) Let C_1, C_2 be coherent spaces and let X be a subset of $[\!:\bigcup C_1, \bigcup C_2\,:\!]$. Suppose that

 (i) for all sets a, b such that $\{a, b\} \in C_1$ and for all sets y_1, y_2 such that $\langle a, y_1\rangle \in X$ and $\langle b, y_2\rangle \in X$ holds $\{y_1, y_2\} \in C_2$, and

 (ii) for all sets a, b such that $\{a, b\} \in C_1$ and for every set y such that $\langle a, y\rangle \in X$ and $\langle b, y\rangle \in X$ holds $a = b$.

 Then there exists a linear function f from C_1 into C_2 such that $X = \mathrm{LinTrace}(f)$.

(58) Let C_1, C_2 be coherent spaces, and let f be a linear function from C_1 into C_2, and let a be an element of C_1. Then $f(a) = (\mathrm{LinTrace}(f))^\circ a$.

(59) For all coherent spaces C_1, C_2 there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \emptyset$.

(60) Let C_1, C_2 be coherent spaces, and let x be a set, and let y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Then there exists a linear function f from C_1 into C_2 such that $\mathrm{LinTrace}(f) = \{\langle x, y\rangle\}$.

(61) Let C_1, C_2 be coherent spaces, and let x be a set, and let y be a set. Suppose $x \in \bigcup C_1$ and $y \in \bigcup C_2$. Let f be a linear function from C_1 into C_2. Suppose $\mathrm{LinTrace}(f) = \{\langle x, y\rangle\}$. Let a be an element of C_1. Then if $x \in a$, then $f(a) = \{y\}$ and if $x \notin a$, then $f(a) = \emptyset$.

(62) Let C_1, C_2 be coherent spaces, and let f be a linear function from C_1 into C_2, and let X be a subset of $\mathrm{LinTrace}(f)$. Then there exists a linear function g from C_1 into C_2 such that $\mathrm{LinTrace}(g) = X$.

(63) Let C_1, C_2 be coherent spaces and let A be a set. Suppose that for all sets x, y such that $x \in A$ and $y \in A$ there exists a linear function f

from C_1 into C_2 such that $x \cup y = \mathrm{LinTrace}(f)$. Then there exists a linear function f from C_1 into C_2 such that $\bigcup A = \mathrm{LinTrace}(f)$.

Let C_1, C_2 be coherent spaces. One can check that $\mathrm{LinCoh}(C_1, C_2)$ is non empty down-closed and binary complete.

One can prove the following propositions:

(64) For all coherent spaces C_1, C_2 holds $\bigcup \mathrm{LinCoh}(C_1, C_2) = [\!: \bigcup C_1, \bigcup C_2 :\!]$.

(65) Let C_1, C_2 be coherent spaces, and let x_1, x_2 be sets, and let y_1, y_2 be sets. Then $\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in \mathrm{Web}(\mathrm{LinCoh}(C_1, C_2))$ if and only if the following conditions are satisfied:

(i) $x_1 \in \bigcup C_1$,

(ii) $x_2 \in \bigcup C_1$, and

(iii) $\langle x_1, x_2\rangle \notin \mathrm{Web}(C_1)$ and $y_1 \in \bigcup C_2$ and $y_2 \in \bigcup C_2$ or $\langle y_1, y_2\rangle \in \mathrm{Web}(C_2)$ and if $y_1 = y_2$, then $x_1 = x_2$.

6. Negation of Coherence Spaces

Let C be a coherent space. The functor $\neg C$ yielding a set is defined by:

(Def.22) $\neg C = \{a : a$ ranges over subsets of $\bigcup C$, $\bigwedge_{b\,:\,\text{element of } C} \bigvee_{x\,:\,\text{set}}\ a \cap b \subseteq \{x\}\}$.

One can prove the following proposition

(66) Let C be a coherent space and let x be a set. Then $x \in \neg C$ if and only if the following conditions are satisfied:

(i) $x \subseteq \bigcup C$, and

(ii) for every element a of C there exists a set z such that $x \cap a \subseteq \{z\}$.

Let C be a coherent space. Observe that $\neg C$ is non empty down-closed and binary complete.

Next we state several propositions:

(67) For every coherent space C holds $\bigcup \neg C = \bigcup C$.

(68) For every coherent space C and for all sets x, y such that $x \neq y$ and $\{x, y\} \in C$ holds $\{x, y\} \notin \neg C$.

(69) For every coherent space C and for all sets x, y such that $\{x, y\} \subseteq \bigcup C$ and $\{x, y\} \notin C$ holds $\{x, y\} \in \neg C$.

(70) For every coherent space C and for all sets x, y holds $\langle x, y\rangle \in \mathrm{Web}(\neg C)$ iff $x \in \bigcup C$ but $y \in \bigcup C$ but $x = y$ or $\langle x, y\rangle \notin \mathrm{Web}(C)$.

(71) For every coherent space C holds $\neg\neg C = C$.

(72) $\neg\{\emptyset\} = \{\emptyset\}$.

(73) For every set X holds $\neg\,\mathrm{FlatCoh}(X) = 2^X$ and $\neg(2^X) = \mathrm{FlatCoh}(X)$.

7. Product and Coproduct on Coherence Spaces

Let x, y be sets. The functor $x \uplus y$ yielding a set is defined by:

(Def.23) $x \uplus y = \bigcup \mathrm{disjoint}\langle x, y\rangle$.

We now state a number of propositions:

(74) For all sets x, y holds $x \uplus y = [\!: x, \{1\} :\!] \cup [\!: y, \{2\} :\!]$.

(75) For every set x holds $x \uplus \emptyset = [\!: x, \{1\} :\!]$ and $\emptyset \uplus x = [\!: x, \{2\} :\!]$.

(76) For all sets x, y, z such that $z \in x \uplus y$ holds $z = \langle z_{\mathbf{1}}, z_{\mathbf{2}}\rangle$ but $z_{\mathbf{2}} = 1$ and $z_{\mathbf{1}} \in x$ or $z_{\mathbf{2}} = 2$ and $z_{\mathbf{1}} \in y$.

(77) For all sets x, y, z holds $\langle z, 1\rangle \in x \uplus y$ iff $z \in x$.

(78) For all sets x, y, z holds $\langle z, 2\rangle \in x \uplus y$ iff $z \in y$.

(79) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 \subseteq x_2 \uplus y_2$ iff $x_1 \subseteq x_2$ and $y_1 \subseteq y_2$.

(80) For all sets x, y, z such that $z \subseteq x \uplus y$ there exist sets x_1, y_1 such that $z = x_1 \uplus y_1$ and $x_1 \subseteq x$ and $y_1 \subseteq y$.

(81) For all sets x_1, y_1, x_2, y_2 holds $x_1 \uplus y_1 = x_2 \uplus y_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

(82) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cup (x_2 \uplus y_2) = x_1 \cup x_2 \uplus y_1 \cup y_2$.

(83) For all sets x_1, y_1, x_2, y_2 holds $(x_1 \uplus y_1) \cap (x_2 \uplus y_2) = x_1 \cap x_2 \uplus y_1 \cap y_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \sqcap C_2$ yields a set and is defined by:

(Def.24) $C_1 \sqcap C_2 = \{a \uplus b : a$ ranges over elements of C_1, b ranges over elements of $C_2\}$.

The functor $C_1 \sqcup C_2$ yielding a set is defined as follows:

(Def.25) $C_1 \sqcup C_2 = \{a \uplus \emptyset : a$ ranges over elements of $C_1\} \cup \{\emptyset \uplus b : b$ ranges over elements of $C_2\}$.

The following propositions are true:

(84) Let C_1, C_2 be coherent spaces and let x be a set. Then $x \in C_1 \sqcap C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$.

(85) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcap C_2$ iff $x \in C_1$ and $y \in C_2$.

(86) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcap C_2) = \bigcup C_1 \uplus \bigcup C_2$.

(87) For all coherent spaces C_1, C_2 and for all sets x, y holds $x \uplus y \in C_1 \sqcup C_2$ iff $x \in C_1$ and $y = \emptyset$ or $x = \emptyset$ and $y \in C_2$.

(88) Let C_1, C_2 be coherent spaces and let x be a set. Suppose $x \in C_1 \sqcup C_2$. Then there exists an element a of C_1 and there exists an element b of C_2 such that $x = a \uplus b$ but $a = \emptyset$ or $b = \emptyset$.

(89) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \sqcup C_2) = \bigcup C_1 \uplus \bigcup C_2$.

Let C_1, C_2 be coherent spaces. Observe that $C_1 \sqcap C_2$ is non empty down-closed and binary complete and $C_1 \sqcup C_2$ is non empty down-closed and binary complete.

In the sequel C_1, C_2 will be coherent spaces.

We now state several propositions:

(90) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(91) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(92) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcap C_2)$.

(93) For all sets x, y holds $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_1)$.

(94) For all sets x, y holds $\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle \in \mathrm{Web}(C_1 \sqcup C_2)$ iff $\langle x, y\rangle \in \mathrm{Web}(C_2)$.

(95) For all sets x, y such that $x \in \bigcup C_1$ and $y \in \bigcup C_2$ holds $\langle\langle x, 1\rangle, \langle y, 2\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$ and $\langle\langle y, 2\rangle, \langle x, 1\rangle\rangle \notin \mathrm{Web}(C_1 \sqcup C_2)$.

(96) $\neg(C_1 \sqcap C_2) = \neg C_1 \sqcup \neg C_2$.

Let C_1, C_2 be coherent spaces. The functor $C_1 \otimes C_2$ yielding a set is defined as follows:

(Def.26) $C_1 \otimes C_2 = \bigcup\{2^{[: a, b :]} : a$ ranges over elements of C_1, b ranges over elements of $C_2\}$.

We now state the proposition

(97) Let C_1, C_2 be coherent spaces and let x be a set. Then $x \in C_1 \otimes C_2$ if and only if there exists an element a of C_1 and there exists an element b of C_2 such that $x \subseteq [: a, b :]$.

Let C_1, C_2 be coherent spaces. One can check that $C_1 \otimes C_2$ is non empty.

Next we state the proposition

(98) For all coherent spaces C_1, C_2 and for every element a of $C_1 \otimes C_2$ holds $\pi_1(a) \in C_1$ and $\pi_2(a) \in C_2$ and $a \subseteq [: \pi_1(a), \pi_2(a) :]$.

Let C_1, C_2 be coherent spaces. One can check that $C_1 \otimes C_2$ is down-closed and binary complete.

Next we state two propositions:

(99) For all coherent spaces C_1, C_2 holds $\bigcup(C_1 \otimes C_2) = [: \bigcup C_1, \bigcup C_2 :]$.

(100) For all sets x_1, y_1, x_2, y_2 holds $\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in \mathrm{Web}(C_1 \otimes C_2)$ iff $\langle x_1, y_1\rangle \in \mathrm{Web}(C_1)$ and $\langle x_2, y_2\rangle \in \mathrm{Web}(C_2)$.

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Some Basic Properties of Many Sorted Sets

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The notation and terminology used here are introduced in the following papers: [11], [12], [5], [13], [2], [3], [4], [6], [1], [10], [9], [8], and [7].

1. Preliminaries

For simplicity we follow a convention: i will be arbitrary, I will be a set, f will be a function, x, x_1, x_2, y, A, B, X, Y, Z will be many sorted sets indexed by I, J will be a non empty set, and N_1 will be a many sorted set indexed by J.

We now state three propositions:

(1) For every set X and for every many sorted set M indexed by I such that $i \in I$ holds $\mathrm{dom}(M {+\cdot} (i \dot{\longmapsto} X)) = I$.

(2) If $f = \emptyset$, then f is a many sorted set indexed by $\emptyset$.

(3) If I is non empty, then there exists no X which is empty yielding and non-empty.

2. Singelton and unordered pairs

Let us consider I, A. The functor $\{A\}$ yielding a many sorted set indexed by I is defined as follows:

(Def.1) For every i such that $i \in I$ holds $\{A\}(i) = \{A(i)\}$.

Let us consider I, A. Observe that $\{A\}$ is non-empty and locally-finite.

Let us consider I, A, B. The functor $\{A, B\}$ yields a many sorted set indexed by I and is defined as follows:

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(Def.2) For every i such that $i \in I$ holds $\{A, B\}(i) = \{A(i), B(i)\}$.

Let us consider I, A, B. One can verify that $\{A, B\}$ is non-empty and locally-finite.

We now state a number of propositions:

(4) $X = \{y\}$ iff for every x holds $x \in X$ iff $x = y$.

(5) If for every x holds $x \in X$ iff $x = x_1$ or $x = x_2$, then $X = \{x_1, x_2\}$.

(6) If $X = \{x_1, x_2\}$, then for every x such that $x = x_1$ or $x = x_2$ holds $x \in X$.

(7) $\{N_1\} \neq \emptyset_I$.

(8) If $x \in \{A\}$, then $x = A$.

(9) $x \in \{x\}$.

(10) If $x = A$ or $x = B$, then $x \in \{A, B\}$.

(11) $\{A\} \cup \{B\} = \{A, B\}$.

(12) $\{x, x\} = \{x\}$.

(13) $\{A, B\} = \{B, A\}$.

(14) If $\{A\} \subseteq \{B\}$, then $A = B$.

(15) If $\{x\} = \{y\}$, then $x = y$.

(16) If $\{x\} = \{A, B\}$, then $x = A$ and $x = B$.

(17) If $\{x\} = \{A, B\}$, then $A = B$.

(18) $\{x\} \subseteq \{x, y\}$ and $\{y\} \subseteq \{x, y\}$.

(19) If $\{x\} \cup \{y\} = \{x\}$ or $\{x\} \cup \{y\} = \{y\}$, then $x = y$.

(20) $\{x\} \cup \{x, y\} = \{x, y\}$.

(21) $\{y\} \cup \{x, y\} = \{x, y\}$.

(22) If I is non empty and $\{x\} \cap \{y\} = \emptyset_I$, then $x \neq y$.

(23) If $\{x\} \cap \{y\} = \{x\}$ or $\{x\} \cap \{y\} = \{y\}$, then $x = y$.

(24) $\{x\} \cap \{x, y\} = \{x\}$ and $\{y\} \cap \{x, y\} = \{y\}$.

(25) If I is non empty and $\{x\} \setminus \{y\} = \{x\}$, then $x \neq y$.

(26) If $\{x\} \setminus \{y\} = \emptyset_I$, then $x = y$.

(27) $\{x\} \setminus \{x, y\} = \emptyset_I$ and $\{y\} \setminus \{x, y\} = \emptyset_I$.

(28) If $\{x\} \subseteq \{y\}$, then $\{x\} = \{y\}$.

(29) If $\{x, y\} \subseteq \{A\}$, then $x = A$ and $y = A$.

(30) If $\{x, y\} \subseteq \{A\}$, then $\{x, y\} = \{A\}$.

(31) $2^{\{x\}} = \{\emptyset_I, \{x\}\}$.

(32) $\{A\} \subseteq 2^A$.

(33) $\bigcup\{x\} = x$.

(34) $\bigcup\{\{x\}, \{y\}\} = \{x, y\}$.

(35) $\bigcup\{A, B\} = A \cup B$.

(36) $\{x\} \subseteq X$ iff $x \in X$.

(37) $\{x_1, x_2\} \subseteq X$ iff $x_1 \in X$ and $x_2 \in X$.

(38) If $A = \emptyset_I$ or $A = \{x_1\}$ or $A = \{x_2\}$ or $A = \{x_1, x_2\}$, then $A \subseteq \{x_1, x_2\}$.

3. Sum of unordered pairs (or a singelton) and a set

One can prove the following propositions:

(39) If $x \in A$ or $x = B$, then $x \in A \cup \{B\}$.

(40) $A \cup \{x\} \subseteq B$ iff $x \in B$ and $A \subseteq B$.

(41) If $\{x\} \cup X = X$, then $x \in X$.

(42) If $x \in X$, then $\{x\} \cup X = X$.

(43) $\{x, y\} \cup A = A$ iff $x \in A$ and $y \in A$.

(44) If I is non empty, then $\{x\} \cup X \neq \emptyset_I$.

(45) If I is non empty, then $\{x, y\} \cup X \neq \emptyset_I$.

4. Intersection of unordered pairs (or a singelton) and a set

We now state several propositions:

(46) If $X \cap \{x\} = \{x\}$, then $x \in X$.

(47) If $x \in X$, then $X \cap \{x\} = \{x\}$.

(48) $x \in X$ and $y \in X$ iff $\{x, y\} \cap X = \{x, y\}$.

(49) If I is non empty and $\{x\} \cap X = \emptyset_I$, then $x \notin X$.

(50) If I is non empty and $\{x, y\} \cap X = \emptyset_I$, then $x \notin X$ and $y \notin X$.

5. Difference of unordered pairs (or a singelton) and a set

The following propositions are true:

(51) If $y \in X \setminus \{x\}$, then $y \in X$.

(52) If I is non empty and $y \in X \setminus \{x\}$, then $y \neq x$.

(53) If I is non empty and $X \setminus \{x\} = X$, then $x \notin X$.

(54) If I is non empty and $\{x\} \setminus X = \{x\}$, then $x \notin X$.

(55) $\{x\} \setminus X = \emptyset_I$ iff $x \in X$.

(56) If I is non empty and $\{x, y\} \setminus X = \{x\}$, then $x \notin X$.

(57) If I is non empty and $\{x, y\} \setminus X = \{y\}$, then $y \notin X$.

(58) If I is non empty and $\{x, y\} \setminus X = \{x, y\}$, then $x \notin X$ and $y \notin X$.

(59) $\{x, y\} \setminus X = \emptyset_I$ iff $x \in X$ and $y \in X$.

(60) If $X = \emptyset_I$ or $X = \{x\}$ or $X = \{y\}$ or $X = \{x, y\}$, then $X \setminus \{x, y\} = \emptyset_I$.

6. Cartesian product

One can prove the following propositions:

(61) If $X = \emptyset_I$ or $Y = \emptyset_I$, then $[\![X, Y]\!] = \emptyset_I$.

(62) If X is non-empty and Y is non-empty and $[\![X, Y]\!] = [\![Y, X]\!]$, then $X = Y$.

(63) If $[\![X, X]\!] = [\![Y, Y]\!]$, then $X = Y$.

(64) If Z is non-empty and if $[\![X, Z]\!] \subseteq [\![Y, Z]\!]$ or $[\![Z, X]\!] \subseteq [\![Z, Y]\!]$, then $X \subseteq Y$.

(65) If $X \subseteq Y$, then $[\![X, Z]\!] \subseteq [\![Y, Z]\!]$ and $[\![Z, X]\!] \subseteq [\![Z, Y]\!]$.

(66) If $x_1 \subseteq A$ and $x_2 \subseteq B$, then $[\![x_1, x_2]\!] \subseteq [\![A, B]\!]$.

(67) $[\![X \cup Y, Z]\!] = [\![X, Z]\!] \cup [\![Y, Z]\!]$ and $[\![Z, X \cup Y]\!] = [\![Z, X]\!] \cup [\![Z, Y]\!]$.

(68) $[\![x_1 \cup x_2, A \cup B]\!] = [\![x_1, A]\!] \cup [\![x_1, B]\!] \cup [\![x_2, A]\!] \cup [\![x_2, B]\!]$.

(69) $[\![X \cap Y, Z]\!] = [\![X, Z]\!] \cap [\![Y, Z]\!]$ and $[\![Z, X \cap Y]\!] = [\![Z, X]\!] \cap [\![Z, Y]\!]$.

(70) $[\![x_1 \cap x_2, A \cap B]\!] = [\![x_1, A]\!] \cap [\![x_2, B]\!]$.

(71) If $A \subseteq X$ and $B \subseteq Y$, then $[\![A, Y]\!] \cap [\![X, B]\!] = [\![A, B]\!]$.

(72) $[\![X \setminus Y, Z]\!] = [\![X, Z]\!] \setminus [\![Y, Z]\!]$ and $[\![Z, X \setminus Y]\!] = [\![Z, X]\!] \setminus [\![Z, Y]\!]$.

(73) $[\![x_1, x_2]\!] \setminus [\![A, B]\!] = [\![x_1 \setminus A, x_2]\!] \cup [\![x_1, x_2 \setminus B]\!]$.

(74) If $x_1 \cap x_2 = \emptyset_I$ or $A \cap B = \emptyset_I$, then $[\![x_1, A]\!] \cap [\![x_2, B]\!] = \emptyset_I$.

(75) If X is non-empty, then $[\![\{x\}, X]\!]$ is non-empty and $[\![X, \{x\}]\!]$ is non-empty.

(76) $[\![\{x, y\}, X]\!] = [\![\{x\}, X]\!] \cup [\![\{y\}, X]\!]$ and $[\![X, \{x, y\}]\!] = [\![X, \{x\}]\!] \cup [\![X, \{y\}]\!]$.

(77) If x_1 is non-empty and A is non-empty and $[\![x_1, A]\!] = [\![x_2, B]\!]$, then $x_1 = x_2$ and $A = B$.

(78) If $X \subseteq [\![X, Y]\!]$ or $X \subseteq [\![Y, X]\!]$, then $X = \emptyset_I$.

(79) If $A \in [\![x, y]\!]$ and $A \in [\![X, Y]\!]$, then $A \in [\![x \cap X, y \cap Y]\!]$.

(80) If $[\![x, X]\!] \subseteq [\![y, Y]\!]$ and $[\![x, X]\!]$ is non-empty, then $x \subseteq y$ and $X \subseteq Y$.

(81) If $A \subseteq X$, then $[\![A, A]\!] \subseteq [\![X, X]\!]$.

(82) If $X \cap Y = \emptyset_I$, then $[\![X, Y]\!] \cap [\![Y, X]\!] = \emptyset_I$.

(83) If A is non-empty and if $[\![A, B]\!] \subseteq [\![X, Y]\!]$ or $[\![B, A]\!] \subseteq [\![Y, X]\!]$, then $B \subseteq Y$.

(84) If $x \subseteq [\![A, B]\!]$ and $y \subseteq [\![X, Y]\!]$, then $x \cup y \subseteq [\![A \cup X, B \cup Y]\!]$.

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Replacement of Subtrees in a Tree

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Summary. This paper is based on previous works [1], [3] in which the operation replacement of subtree in a tree has been defined. We extend this notion for arbitrary non empty antichain.

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The notation and terminology used in this paper are introduced in the following papers: [8], [9], [6], [10], [5], [7], [4], [1], [3], and [2].

We follow the rules: T, T_1 will denote trees, P will denote an antichain of prefixes of T, and p, q, r will denote finite sequences of elements of $\mathbb{N}$.

We now state the proposition

(1) For all finite sequences p, q, r, s such that $p \frown q = s \frown r$ holds p and s are comparable.

Let us consider T, T_1 and let us consider P. Let us assume that $P \neq \emptyset$. The functor $T(P/T_1)$ yields a tree and is defined as follows:

(Def.1) $q \in T(P/T_1)$ iff $q \in T$ and for every p such that $p \in P$ holds $p \not\preceq q$ or there exist p, r such that $p \in P$ and $r \in T_1$ and $q = p \frown r$.

One can prove the following propositions:

(2) Suppose $P \neq \emptyset$. Then $T(P/T_1) = \{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\preceq t_1\} \cup \{p \frown s : p$ ranges over elements of T, s ranges over elements of T_1, $p \in P\}$.

(3) $\{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\preceq t_1\} \subseteq \{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\prec t_1\}$.

(4) $P \subseteq \{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\prec t_1\}$.

(5) $\{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\prec t_1\} \setminus \{t_1 : t_1$ ranges over elements of T, $\bigwedge_p \ p \in P \ \Rightarrow \ p \not\preceq t_1\} = P$.

(6) For all T, T_1, P holds $P \subseteq \{p \frown s : p$ ranges over elements of T, s ranges over elements of T_1, $p \in P\}$.

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(7) Suppose $P \neq \emptyset$. Then $T(P/T_1) = \{t_1 : t_1$ ranges over elements of T, $\bigwedge_p\ p \in P \Rightarrow p \not\preceq t_1\} \cup \{p \frown s : p$ ranges over elements of T, s ranges over elements of T_1, $p \in P\}$.

(8) If $p \in P$ and $q \in T_1$, then $p \frown q \in T(P/T_1)$.

(9) If $p \in P$, then $T_1 = T(P/T_1) \restriction p$.

Let us consider T. Observe that there exists an antichain of prefixes of T which is non empty.

Let us consider T and let t be an element of T. Then $\{t\}$ is a non empty antichain of prefixes of T.

In the sequel t will be an element of T.

We now state the proposition

(10) $T(\{t\}/T_1) = T(t/T_1)$.

In the sequel T, T_1 denote decorated trees, P denotes an antichain of prefixes of $\operatorname{dom} T$, and t denotes an element of $\operatorname{dom} T$.

Let us consider T, P, T_1. Let us assume that $P \neq \emptyset$. The functor $T(P/T_1)$ yields a decorated tree and is defined by the conditions (Def.2).

(Def.2) (i) $\operatorname{dom}(T(P/T_1)) = (\operatorname{dom} T)(P/\operatorname{dom} T_1)$, and

(ii) for every q such that $q \in (\operatorname{dom} T)(P/\operatorname{dom} T_1)$ holds for every p such that $p \in P$ holds $p \not\preceq q$ and $T(P/T_1)(q) = T(q)$ or there exist p, r such that $p \in P$ and $r \in \operatorname{dom} T_1$ and $q = p \frown r$ and $T(P/T_1)(q) = T_1(r)$.

We now state several propositions:

(11) If $P \neq \emptyset$, then $\operatorname{dom}(T(P/T_1)) = (\operatorname{dom} T)(P/\operatorname{dom} T_1)$.

(12) If $p \in \operatorname{dom} T$, then $\operatorname{dom}(T(p/T_1)) = (\operatorname{dom} T)(p/\operatorname{dom} T_1)$.

(13) Suppose $P \neq \emptyset$. Given q. Suppose $q \in \operatorname{dom}(T(P/T_1))$. Then for every p such that $p \in P$ holds $p \not\preceq q$ and $T(P/T_1)(q) = T(q)$ or there exist p, r such that $p \in P$ and $r \in \operatorname{dom} T_1$ and $q = p \frown r$ and $T(P/T_1)(q) = T_1(r)$.

(14) Suppose $p \in \operatorname{dom} T$. Given q. Suppose $q \in \operatorname{dom}(T(p/T_1))$. Then $p \not\preceq q$ and $T(p/T_1)(q) = T(q)$ or there exists r such that $r \in \operatorname{dom} T_1$ and $q = p \frown r$ and $T(p/T_1)(q) = T_1(r)$.

(15) Suppose $P \neq \emptyset$. Given q. Suppose $q \in \operatorname{dom}(T(P/T_1))$ and $q \in \{t_1 : t_1$ ranges over elements of $\operatorname{dom} T$, $\bigwedge_p\ p \in P \Rightarrow p \not\preceq t_1\}$. Then $T(P/T_1)(q) = T(q)$.

(16) If $p \in \operatorname{dom} T$, then for every q such that $q \in \operatorname{dom}(T(p/T_1))$ and $q \in \{t_1 : t_1$ ranges over elements of $\operatorname{dom} T, p \not\preceq t_1\}$ holds $T(p/T_1)(q) = T(q)$.

(17) Suppose $P \neq \emptyset$. Given q. Suppose $q \in \operatorname{dom}(T(P/T_1))$ and $q \in \{p \frown s : p$ ranges over elements of $\operatorname{dom} T, s$ ranges over elements of $\operatorname{dom} T_1, p \in P\}$. Then there exists an element p' of $\operatorname{dom} T$ and there exists an element r of $\operatorname{dom} T_1$ such that $q = p' \frown r$ and $p' \in P$ and $T(P/T_1)(q) = T_1(r)$.

(18) Suppose $p \in \operatorname{dom} T$. Given q. Suppose $q \in \operatorname{dom}(T(p/T_1))$ and $q \in \{p \frown s : s$ ranges over elements of $\operatorname{dom} T_1, s = s\}$. Then there exists an element r of $\operatorname{dom} T_1$ such that $q = p \frown r$ and $T(p/T_1)(q) = T_1(r)$.

(19) $T(\{t\}/T_1) = T(t/T_1)$.

In the sequel D will denote a non empty set, T, T_1 will denote trees decorated with elements of D, and P will denote an antichain of prefixes of $\operatorname{dom} T$.

Let us consider D, T, P, T_1. Let us assume that $P \neq \emptyset$. The functor $T(P/T_1)$ yields a tree decorated with elements of D and is defined by:

(Def.3) $T(P/T_1) = T(P/T_1)$.

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Minimal Signature for Partial Algebra

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Summary. The concept of characterizing of partial algebras by many sorted signature is introduced, i.e. we say that a signature S characterizes a partial algebra A if there is an S-algebra whose sorts form a partition of the carrier of algebra A and operations are formed from operations of A by the partition. The main result is that for any partial algebra there is the minimal many sorted signature which characterizes the algebra. The minimality means that there are signature endomorphisms from any signature which characterizes the algebra A onto the minimal one.

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The papers [16], [18], [9], [1], [12], [19], [20], [6], [17], [3], [5], [7], [21], [13], [8], [11], [2], [4], [15], [14], and [10] provide the notation and terminology for this paper.

1. Preliminary

Let f be a non empty binary relation. Observe that $\operatorname{dom} f$ is non empty and $\operatorname{rng} f$ is non empty.

Let f be a non-empty function. One can verify that $\operatorname{rng} f$ has non empty elements.

Let X, Y be non empty sets. One can verify that there exists a partial function from X to Y which is non empty.

Let X be a set with non empty elements. Note that every finite sequence of elements of X is non-empty.

Let A be a non empty set. One can verify that there exists a finite sequence of operational functions of A which is homogeneous quasi total non-empty and non empty.

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Let us observe that every universal algebra structure which is non-empty is also non empty.

Let X be a non empty set with non empty elements. One can verify that every element of X is non empty.

Next we state two propositions:

(1) For all non-empty functions f, g such that $\prod f \subseteq \prod g$ holds $\operatorname{dom} f = \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \subseteq g(x)$.

(2) For all non-empty functions f, g such that $\prod f = \prod g$ holds $f = g$.

Let A be a non empty set and let f be a finite sequence of operational functions of A. Then $\operatorname{rng} f$ is a subset of $A^* \dot{\rightarrow} A$.

Let A, B be non empty sets and let S be a non empty subset of $A \dot{\rightarrow} B$. We see that the element of S is a partial function from A to B.

Let A be a non-empty universal algebra structure. An operation symbol of A is an element of dom (the characteristic of A). An operation of A is an element of rng (the characteristic of A).

Let A be a non-empty universal algebra structure and let o be an operation symbol of A. The functor $\operatorname{Den}(o, A)$ yielding an operation of A is defined by:

(Def.1) $\operatorname{Den}(o, A) =$ (the characteristic of A)(o).

2. Partitions

Let X be a set. Note that every partition of X has non empty elements.

Let X be a non empty set. One can verify that every partition of X is non empty.

Let X be a set and let R be an equivalence relation of X. Then Classes R is a partition of X.

Next we state a number of propositions:

(3) Let X be a set, and let P be a partition of X, and let x, a, b be sets. If $x \in a$ and $a \in P$ and $x \in b$ and $b \in P$, then $a = b$.

(4) Let X, Y be sets. Suppose X is finer than Y. Let p be a finite sequence of elements of X. Then there exists a finite sequence q of elements of Y such that $\prod p \subseteq \prod q$.

(5) Let X be a set, and let P, Q be partitions of X, and let f be a function from P into Q. Suppose that for every set a such that $a \in P$ holds $a \subseteq f(a)$. Let p be a finite sequence of elements of P and let q be a finite sequence of elements of Q. Then $\prod p \subseteq \prod q$ if and only if $f \cdot p = q$.

(6) For every set P and for every function f such that $\operatorname{rng} f \subseteq \bigcup P$ there exists a function p such that $\operatorname{dom} p = \operatorname{dom} f$ and $\operatorname{rng} p \subseteq P$ and $f \in \prod p$.

(7) Let X be a set, and let P be a partition of X, and let f be a finite sequence of elements of X. Then there exists a finite sequence p of elements of P such that $f \in \prod p$.

(8) Let X, Y be non empty sets, and let P be a partition of X, and let Q be a partition of Y. Then $\{[\!:p, q:\!] : p$ ranges over elements of P, q ranges over elements of $Q\}$ is a partition of $[\!:X, Y:\!]$.

(9) For every non empty set X and for every partition P of X holds $\{\prod p : p$ ranges over elements of $P^*\}$ is a partition of X^*.

(10) Let X be a non empty set, and let n be a natural number, and let P be a partition of X. Then $\{\prod p : p$ ranges over elements of $P^n\}$ is a partition of X^n.

(11) Let X be a non empty set and let Y be a set. Suppose $Y \subseteq X$. Let P be a partition of X. Then $\{a \cap Y : a$ ranges over elements of P, a meets $Y\}$ is a partition of Y.

(12) Let f be a non empty function and let P be a partition of $\operatorname{dom} f$. Then $\{f \restriction a : a$ ranges over elements of $P\}$ is a partition of f.

Let X be a set. The functor SmallestPartition(X) yielding a partition of X is defined as follows:

(Def.2) SmallestPartition$(X) = \text{Classes}(\triangle_X)$.

One can prove the following propositions:

(13) For every non empty set X holds SmallestPartition$(X) = \{\{x\} : x$ ranges over elements of $X\}$.

(14) Let X be a set and let p be a finite sequence of elements of SmallestPartition(X). Then there exists a finite sequence q of elements of X such that $\prod p = \{q\}$.

Let X be a set. A function is said to be an indexed partition of X if:

(Def.3) rng it is a partition of X and it is one-to-one.

Let X be a set. Note that every indexed partition of X is one-to-one and non-empty. Let P be an indexed partition of X. Then $\operatorname{rng} P$ is a partition of X.

Let X be a non empty set. Observe that every indexed partition of X is non empty.

Let X be a set and let P be a partition of X. Then $\triangle_P$ is an indexed partition of X.

Let X be a set, let P be an indexed partition of X, and let x be a set. Let us assume that $x \in X$. The P-index of x is a set and is defined by:

(Def.4) The P-index of $x \in \operatorname{dom} P$ and $x \in P($the P-index of $x)$.

Next we state two propositions:

(15) Let X be a set and let P be a non-empty function. Suppose $\bigcup P = X$ and for all sets x, y such that $x \in \operatorname{dom} P$ and $y \in \operatorname{dom} P$ and $x \neq y$ holds $P(x)$ misses $P(y)$. Then P is an indexed partition of X.

(16) Let X, Y be non empty sets, and let P be a partition of Y, and let f be a function from X into P. If $P \subseteq \operatorname{rng} f$ and f is one-to-one, then f is an indexed partition of Y.

3. Relations Generated by Operations of Partial Algebra

In this article we present several logical schemes. The scheme *RelationEx* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a relation R between $\mathcal{A}$ and $\mathcal{B}$ such that for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R$ if and only if $\mathcal{P}[x, y]$

for all values of the parameters.

The scheme *IndRelationEx* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$, a natural number $\mathcal{C}$, a relation $\mathcal{D}$ between $\mathcal{A}$ and $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a relation between $\mathcal{A}$ and $\mathcal{B}$, and states that:

> There exists a relation R between $\mathcal{A}$ and $\mathcal{B}$ and there exists a many sorted set F indexed by $\mathbb{N}$ such that
> (i) $R = F(\mathcal{C})$,
> (ii) $F(0) = \mathcal{D}$, and
> (iii) for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R, i)$

for all values of the parameters.

The scheme *RelationUniq* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

> Let R_1, R_2 be relations between $\mathcal{A}$ and $\mathcal{B}$. Suppose that
> (i) for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R_1$ iff $\mathcal{P}[x, y]$, and
> (ii) for every element x of $\mathcal{A}$ and for every element y of $\mathcal{B}$ holds $\langle x, y\rangle \in R_2$ iff $\mathcal{P}[x, y]$.
> Then $R_1 = R_2$

for all values of the parameters.

The scheme *IndRelationUniq* concerns non empty sets $\mathcal{A}$, $\mathcal{B}$, a natural number $\mathcal{C}$, a relation $\mathcal{D}$ between $\mathcal{A}$ and $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a relation between $\mathcal{A}$ and $\mathcal{B}$, and states that:

> Let R_1, R_2 be relations between $\mathcal{A}$ and $\mathcal{B}$. Suppose that
> (i) there exists a many sorted set F indexed by $\mathbb{N}$ such that $R_1 = F(\mathcal{C})$ and $F(0) = \mathcal{D}$ and for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R, i)$, and
> (ii) there exists a many sorted set F indexed by $\mathbb{N}$ such that $R_2 = F(\mathcal{C})$ and $F(0) = \mathcal{D}$ and for every natural number i and for every relation R between $\mathcal{A}$ and $\mathcal{B}$ such that $R = F(i)$ holds $F(i+1) = \mathcal{F}(R, i)$.
> Then $R_1 = R_2$

for all values of the parameters.

Let A be a partial non-empty universal algebra structure. The functor $\mathrm{DomRel}(A)$ yields a binary relation on the carrier of A and is defined by the condition (Def.5).

(Def.5) Let x, y be elements of the carrier of A. Then $\langle x, y\rangle \in \mathrm{DomRel}(A)$ if and only if for every operation f of A and for all finite sequences p, q holds $p \frown \langle x\rangle \frown q \in \mathrm{dom}\, f$ iff $p \frown \langle y\rangle \frown q \in \mathrm{dom}\, f$.

Let A be a partial non-empty universal algebra structure. Note that $\mathrm{DomRel}(A)$ is equivalence relation-like.

Let A be a non-empty partial universal algebra structure and let R be a binary relation on the carrier of A. The functor R^A yielding a binary relation on the carrier of A is defined by the condition (Def.6).

(Def.6) Let x, y be elements of the carrier of A. Then $\langle x, y\rangle \in R^A$ if and only if the following conditions are satisfied:

(i) $\langle x, y\rangle \in R$, and

(ii) for every operation f of A and for all finite sequences p, q such that $p\frown\langle x\rangle\frown q \in \mathrm{dom}\, f$ and $p\frown\langle y\rangle\frown q \in \mathrm{dom}\, f$ holds $\langle f(p\frown\langle x\rangle\frown q),\, f(p\frown\langle y\rangle\frown q)\rangle \in R$.

Let A be a non-empty partial universal algebra structure, let R be a binary relation on the carrier of A, and let i be a natural number. The functor $R^{A,i}$ yielding a binary relation on the carrier of A is defined by the condition (Def.7).

(Def.7) There exists a many sorted set F indexed by $\mathbb{N}$ such that

(i) $R^{A,i} = F(i)$,

(ii) $F(0) = R$, and

(iii) for every natural number i and for every binary relation R on the carrier of A such that $R = F(i)$ holds $F(i + 1) = R^A$.

Next we state several propositions:

(17) Let A be a non-empty partial universal algebra structure and let R be a binary relation on the carrier of A. Then $R^{A,0} = R$ and $R^{A,1} = R^A$.

(18) Let A be a non-empty partial universal algebra structure, and let i be a natural number, and let R be a binary relation on the carrier of A. Then $R^{A,i+1} = (R^{A,i})^A$.

(19) Let A be a non-empty partial universal algebra structure, and let i, j be natural numbers, and let R be a binary relation on the carrier of A. Then $R^{A,i+j} = (R^{A,i})^{A,j}$.

(20) Let A be a non-empty partial universal algebra structure and let R be an equivalence relation of the carrier of A. If $R \subseteq \mathrm{DomRel}(A)$, then R^A is equivalence relation-like.

(21) Let A be a non-empty partial universal algebra structure and let R be a binary relation on the carrier of A. Then $R^A \subseteq R$.

(22) Let A be a non-empty partial universal algebra structure and let R be an equivalence relation of the carrier of A. Suppose $R \subseteq \mathrm{DomRel}(A)$. Let i be a natural number. Then $R^{A,i}$ is equivalence relation-like.

Let A be a non-empty partial universal algebra structure. The functor $\mathrm{LimDomRel}(A)$ yields a binary relation on the carrier of A and is defined by the condition (Def.8).

(Def.8) Let x, y be elements of the carrier of A. Then $\langle x, y \rangle \in \mathrm{LimDomRel}(A)$ if and only if for every natural number i holds $\langle x, y \rangle \in (\mathrm{DomRel}(A))^{A,i}$.

The following proposition is true

(23) For every non-empty partial universal algebra structure A holds $\mathrm{LimDomRel}(A) \subseteq \mathrm{DomRel}(A)$.

Let A be a non-empty partial universal algebra structure. Note that $\mathrm{LimDomRel}(A)$ is equivalence relation-like.

4. Partitability

Let X be a non empty set, let f be a partial function from X^* to X, and let P be a partition of X. We say that f is partitable w.r.t. P if and only if:

(Def.9) For every finite sequence p of elements of P there exists an element a of P such that $f^\circ \prod p \subseteq a$.

Let X be a non empty set, let f be a partial function from X^* to X, and let P be a partition of X. We say that f is exactly partitable w.r.t. P if and only if:

(Def.10) f is partitable w.r.t. P and for every finite sequence p of elements of P such that $\prod p$ meets $\mathrm{dom}\, f$ holds $\prod p \subseteq \mathrm{dom}\, f$.

We now state the proposition

(24) Let A be a non-empty partial universal algebra structure. Then every operation of A is exactly partitable w.r.t. SmallestPartition(the carrier of A).

The scheme *FiniteTransitivity* concerns finite sequences $\mathcal{A}$, $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{A}]$,
- $\mathrm{len}\, \mathcal{A} = \mathrm{len}\, \mathcal{B}$,
- For all finite sequences p, q and for all sets z_1, z_2 such that $\mathcal{P}[p \frown \langle z_1 \rangle \frown q]$ and $\mathcal{Q}[z_1, z_2]$ holds $\mathcal{P}[p \frown \langle z_2 \rangle \frown q]$,
- For every natural number i such that $i \in \mathrm{dom}\, \mathcal{A}$ holds $\mathcal{Q}[\mathcal{A}(i), \mathcal{B}(i)]$.

One can prove the following proposition

(25) For every non-empty partial universal algebra structure A holds every operation of A is exactly partitable w.r.t. $\mathrm{Classes}\, \mathrm{LimDomRel}(A)$.

Let A be a partial non-empty universal algebra structure. A partition of the carrier of A is said to be a partition of A if:

(Def.11) Every operation of A is exactly partitable w.r.t. it.

Let A be a partial non-empty universal algebra structure. An indexed partition of the carrier of A is called an indexed partition of A if:

(Def.12) $\mathrm{rng}\, \mathrm{it}$ is a partition of A.

Let A be a partial non-empty universal algebra structure and let P be an indexed partition of A. Then $\operatorname{rng} P$ is a partition of A.

One can prove the following propositions:

(26) For every non-empty partial universal algebra structure A holds $\mathrm{Classes}\,\mathrm{LimDomRel}(A)$ is a partition of A.

(27) Let X be a non empty set, and let P be a partition of X, and let p be a finite sequence of elements of P, and let q_1, q_2 be finite sequences, and let x, y be sets. Suppose $q_1 \frown \langle x \rangle \frown q_2 \in \prod p$ and there exists an element a of P such that $x \in a$ and $y \in a$. Then $q_1 \frown \langle y \rangle \frown q_2 \in \prod p$.

(28) For every partial non-empty universal algebra structure A holds every partition of A is finer than $\mathrm{Classes}\,\mathrm{LimDomRel}(A)$.

5. Signature Morphisms

Let S_1, S_2 be many sorted signatures and let f, g be functions. We say that f and g form morphism between S_1 and S_2 if and only if the conditions (Def.13) are satisfied.

(Def.13) (i) $\operatorname{dom} f =$ the carrier of S_1,

(ii) $\operatorname{dom} g =$ the operation symbols of S_1,

(iii) $\operatorname{rng} f \subseteq$ the carrier of S_2,

(iv) $\operatorname{rng} g \subseteq$ the operation symbols of S_2,

(v) $f \cdot (\text{the result sort of } S_1) = (\text{the result sort of } S_2) \cdot (g)$, and

(vi) for every set o and for every function p such that $o \in$ the operation symbols of S_1 and $p = (\text{the arity of } S_1)(o)$ holds $f \cdot p = (\text{the arity of } S_2)(g(o))$.

Next we state two propositions:

(29) Let S be a non void non empty many sorted signature. Then $\mathrm{id}_{(\text{the carrier of } S)}$ and $\mathrm{id}_{(\text{the operation symbols of } S)}$ form morphism between S and S.

(30) Let S_1, S_2, S_3 be many sorted signatures and let f_1, f_2, g_1, g_2 be functions. Suppose f_1 and g_1 form morphism between S_1 and S_2 and f_2 and g_2 form morphism between S_2 and S_3. Then $f_2 \cdot f_1$ and $g_2 \cdot g_1$ form morphism between S_1 and S_3.

Let S_1, S_2 be many sorted signatures. We say that S_1 is rougher than S_2 if and only if the condition (Def.14) is satisfied.

(Def.14) There exist functions f, g such that f and g form morphism between S_2 and S_1 and $\operatorname{rng} f =$ the carrier of S_1 and $\operatorname{rng} g =$ the operation symbols of S_1.

Let S_1, S_2 be non void non empty many sorted signatures. Let us observe that the predicate defined above is reflexive.

One can prove the following proposition

(31) For all many sorted signatures S_1, S_2, S_3 such that S_1 is rougher than S_2 and S_2 is rougher than S_3 holds S_1 is rougher than S_3.

6. Many Sorted Signature of Partial Algebra

Let A be a partial non-empty universal algebra structure and let P be a partition of A. The functor $\mathrm{MSSign}(A, P)$ yields a strict many sorted signature and is defined by the conditions (Def.15).

(Def.15) (i) The carrier of $\mathrm{MSSign}(A, P) = P$,

(ii) the operation symbols of $\mathrm{MSSign}(A, P) = \{\langle o, p\rangle : o$ ranges over operation symbols of A, p ranges over elements of P^*, $\prod p$ meets $\mathrm{dom}\,\mathrm{Den}(o, A)\}$, and

(iii) for every operation symbol o of A and for every element p of P^* such that $\prod p$ meets $\mathrm{dom}\,\mathrm{Den}(o, A)$ holds (the arity of $\mathrm{MSSign}(A, P))(\langle o, p\rangle) = p$ and $(\mathrm{Den}(o, A))^\circ \prod p \subseteq$ (the result sort of $\mathrm{MSSign}(A, P))(\langle o, p\rangle)$.

Let A be a partial non-empty universal algebra structure and let P be a partition of A. One can verify that $\mathrm{MSSign}(A, P)$ is non empty and non void.

Let A be a partial non-empty universal algebra structure, let P be a partition of A, and let o be an operation symbol of $\mathrm{MSSign}(A, P)$. Then $o_{\mathbf{1}}$ is an operation symbol of A. Then $o_{\mathbf{2}}$ is an element of P^*.

Let A be a partial non-empty universal algebra structure, let S be a non void non empty many sorted signature, let G be an algebra over S, and let P be an indexed partition of the operation symbols of S. We say that A can be characterized by S, G, and P if and only if the conditions (Def.16) are satisfied.

(Def.16) (i) The sorts of G is an indexed partition of A,

(ii) $\mathrm{dom}\,P = \mathrm{dom}$ (the characteristic of A), and

(iii) for every operation symbol o of A holds (the characteristics of $G) \restriction P(o)$ is an indexed partition of $\mathrm{Den}(o, A)$.

Let A be a partial non-empty universal algebra structure and let S be a non void non empty many sorted signature. We say that A can be characterized by S if and only if the condition (Def.17) is satisfied.

(Def.17) There exists an algebra G over S and there exists an indexed partition P of the operation symbols of S such that A can be characterized by S, G, and P.

One can prove the following propositions:

(32) Let A be a partial non-empty universal algebra structure and let P be a partition of A. Then A can be characterized by $\mathrm{MSSign}(A, P)$.

(33) Let A be a partial non-empty universal algebra structure, and let S be a non void non empty many sorted signature, and let G be an algebra over S, and let Q be an indexed partition of the operation symbols of S. Suppose A can be characterized by S, G, and Q. Let o be an operation symbol of A and let r be a finite sequence of elements of rng (the sorts of

G). Suppose $\prod r \subseteq \operatorname{dom} \operatorname{Den}(o, A)$. Then there exists an operation symbol s of S such that (the sorts of G) $\cdot$ Arity$(s) = r$ and $s \in Q(o)$.

(34) Let A be a partial non-empty universal algebra structure and let P be a partition of A. Suppose $P = \operatorname{Classes} \operatorname{LimDomRel}(A)$. Let S be a non void non empty many sorted signature. If A can be characterized by S, then $\operatorname{MSSign}(A, P)$ is rougher than S.

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The Subformula Tree of a Formula of the First Order Language

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Summary. A continuation of [12]. The notions of list of immediate constituents of a formula and subformula tree of a formula are introduced. The some propositions related to these notions are proved.

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The terminology and notation used in this paper are introduced in the following articles: [15], [18], [3], [11], [19], [9], [10], [13], [8], [17], [1], [4], [6], [5], [7], [14], [2], and [16].

1. Preliminaries

The following propositions are true:

(1) For all real numbers x, y, z such that $x \leq y$ and $y < z$ holds $x < z$.

(2) For all natural numbers m, k holds $m + 1 \leq k$ iff $m < k$.

(3) For every finite sequence r holds $r = r \restriction \mathrm{Seg}\,\mathrm{len}\,r$.

(4) For every natural number n and for every finite sequence r there exists a finite sequence q such that $q = r \restriction \mathrm{Seg}\,n$ and $q \preceq r$.

(5) For all finite sequences p, q, r such that $q \preceq r$ holds $p \frown q \preceq p \frown r$.

(6) Let D be a non empty set, and let r be a finite sequence of elements of D, and let r_1, r_2 be finite sequences, and let k be a natural number. Suppose $k + 1 \leq \mathrm{len}\,r$ and $r_1 = r \restriction \mathrm{Seg}(k+1)$ and $r_2 = r \restriction \mathrm{Seg}\,k$. Then there exists an element x of D such that $r_1 = r_2 \frown \langle x \rangle$.

(7) Let D be a non empty set, and let r be a finite sequence of elements of D, and let r_1 be a finite sequence. If $1 \leq \mathrm{len}\,r$ and $r_1 = r \restriction \mathrm{Seg}\,1$, then there exists an element x of D such that $r_1 = \langle x \rangle$.

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Let D be a non empty set and let T be a tree decorated with elements of D. Observe that every element of $\operatorname{dom} T$ is function-like and relation-like.

Let D be a non empty set and let T be a tree decorated with elements of D. One can verify that every element of $\operatorname{dom} T$ is finite sequence-like.

Let D be a non empty set. One can check that there exists a tree decorated with elements of D which is finite.

In the sequel T will be a decorated tree and p will be a finite sequence of elements of $\mathbb{N}$.

Next we state the proposition

(8) If $p \in \operatorname{dom} T$, then $T(p) = (T \restriction p)(\varepsilon)$.

In the sequel T is a finite-branching decorated tree, t is an element of $\operatorname{dom} T$, x is a finite sequence, and n is a natural number.

The following propositions are true:

(9) $\operatorname{succ}(T, t) = T \cdot \operatorname{Succ} t$.

(10) $\operatorname{dom}(T \cdot \operatorname{Succ} t) = \operatorname{dom} \operatorname{Succ} t$.

(11) $\operatorname{dom} \operatorname{succ}(T, t) = \operatorname{dom} \operatorname{Succ} t$.

(12) $t \frown \langle n \rangle \in \operatorname{dom} T$ iff $n + 1 \in \operatorname{dom} \operatorname{Succ} t$.

(13) For all T, x, n such that $x \frown \langle n \rangle \in \operatorname{dom} T$ holds $T(x \frown \langle n \rangle) = (\operatorname{succ}(T, x))(n + 1)$.

In the sequel x, x' will be elements of $\operatorname{dom} T$ and y' will be arbitrary.

One can prove the following two propositions:

(14) If $x' \in \operatorname{succ} x$, then $T(x') \in \operatorname{rng} \operatorname{succ}(T, x)$.

(15) If $y' \in \operatorname{rng} \operatorname{succ}(T, x)$, then there exists x' such that $y' = T(x')$ and $x' \in \operatorname{succ} x$.

In the sequel n, k, m will denote natural numbers.

The scheme *ExDecTrees* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a finite sequence of elements of $\mathcal{A}$, and states that:

> There exists a finite-branching tree T decorated with elements of $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\operatorname{dom} T$ and for every element w of $\mathcal{A}$ such that $w = T(t)$ holds $\operatorname{succ}(T, t) = \mathcal{F}(w)$

for all values of the parameters.

The following propositions are true:

(16) For every tree T and for every element t of T holds $\operatorname{Seg}_{\preceq}(t)$ is a finite chain of T.

(17) For every tree T holds T-level$(0) = \{\varepsilon\}$.

(18) For every tree T holds T-level$(n + 1) = \bigcup\{\operatorname{succ} w : w$ ranges over elements of T, $\operatorname{len} w = n\}$.

(19) For every finite-branching tree T and for every natural number n holds T-level(n) is finite.

(20) For every finite-branching tree T holds T is finite iff there exists a natural number n such that T-level$(n) = \emptyset$.

(21) For every finite-branching tree T such that T is not finite holds there exists chain of T which is not finite.

(22) For every finite-branching tree T such that T is not finite holds there exists branch of T which is not finite.

(23) Let T be a tree, and let C be a chain of T, and let t be an element of T. If $t \in C$ and C is not finite, then there exists an element t' of T such that $t' \in C$ and $t \prec t'$.

(24) Let T be a tree, and let B be a branch of T, and let t be an element of T. Suppose $t \in B$ and B is not finite. Then there exists an element t' of T such that $t' \in B$ and $t' \in \operatorname{succ} t$.

(25) Let f be a function from $\mathbb{N}$ into $\mathbb{N}$. Suppose that for every n holds $f(n+1)$ **qua** natural number $\leq f(n)$ **qua** natural number. Then there exists m such that for every n such that $m \leq n$ holds $f(n) = f(m)$.

The scheme *FinDecTree* concerns a non empty set $\mathcal{A}$, a finite-branching tree $\mathcal{B}$ decorated with elements of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a natural number, and states that:

$\mathcal{B}$ is finite

provided the parameters meet the following requirement:

- For all elements t, t' of $\operatorname{dom} \mathcal{B}$ and for every element d of $\mathcal{A}$ such that $t' \in \operatorname{succ} t$ and $d = \mathcal{B}(t')$ holds $\mathcal{F}(d) < \mathcal{F}(\mathcal{B}(t))$.

In the sequel D will denote a non empty set and T will denote a tree decorated with elements of D.

Next we state two propositions:

(26) For arbitrary y such that $y \in \operatorname{rng} T$ holds y is an element of D.

(27) For arbitrary x such that $x \in \operatorname{dom} T$ holds $T(x)$ is an element of D.

2. Subformula tree

In the sequel F, G, H will denote elements of WFF.

One can prove the following propositions:

(28) If F is a subformula of G, then $\operatorname{len}(^{@}F) \leq \operatorname{len}(^{@}G)$.

(29) If F is a subformula of G and $\operatorname{len}(^{@}F) = \operatorname{len}(^{@}G)$, then $F = G$.

Let p be an element of WFF. The list of immediate constituents of p yields a finite sequence of elements of WFF and is defined by:

(Def.1) (i) The list of immediate constituents of $p = \varepsilon_{\mathrm{WFF}}$ if $p =$ VERUM or p is atomic,

(ii) the list of immediate constituents of $p = \langle \operatorname{Arg}(p) \rangle$ if p is negative,

(iii) the list of immediate constituents of $p = \langle \operatorname{LeftArg}(p), \operatorname{RightArg}(p) \rangle$ if p is conjunctive,

(iv) the list of immediate constituents of $p = \langle \operatorname{Scope}(p) \rangle$, otherwise.

Next we state two propositions:

(30) Suppose $k \in \operatorname{dom}$(the list of immediate constituents of F) and $G =$ (the list of immediate constituents of F)(k). Then G is an immediate constituent of F.

(31) rng (the list of immediate constituents of F) $= \{G : G$ ranges over elements of WFF, G is an immediate constituent of $F\}$.

Let p be an element of WFF. The tree of subformulae of p yields a finite tree decorated with elements of WFF and is defined by the conditions (Def.2).

(Def.2) (i) (The tree of subformulae of p)(ε) $= p$, and

(ii) for every element x of dom (the tree of subformulae of p) holds succ(the tree of subformulae of p, x) $=$ the list of immediate constituents of (the tree of subformulae of p)(x).

In the sequel t, t' will be elements of dom (the tree of subformulae of F).

One can prove the following propositions:

(32) (The tree of subformulae of F)(ε) $= F$.

(33) succ(the tree of subformulae of F, t) $=$ the list of immediate constituents of (the tree of subformulae of F)(t).

(34) $F \in \operatorname{rng}$ (the tree of subformulae of F).

(35) Suppose $t \frown \langle n \rangle \in \operatorname{dom}$ (the tree of subformulae of F). Then there exists G such that

(i) $G =$ (the tree of subformulae of F)($t \frown \langle n \rangle$), and

(ii) G is an immediate constituent of (the tree of subformulae of F)(t).

(36) The following statements are equivalent

(i) H is an immediate constituent of (the tree of subformulae of F)(t),

(ii) there exists n such that $t \frown \langle n \rangle \in \operatorname{dom}$ (the tree of subformulae of F) and $H =$ (the tree of subformulae of F)($t \frown \langle n \rangle$).

(37) Suppose $G \in \operatorname{rng}$ (the tree of subformulae of F) and H is an immediate constituent of G. Then $H \in \operatorname{rng}$ (the tree of subformulae of F).

(38) If $G \in \operatorname{rng}$ (the tree of subformulae of F) and H is a subformula of G, then $H \in \operatorname{rng}$ (the tree of subformulae of F).

(39) $G \in \operatorname{rng}$ (the tree of subformulae of F) iff G is a subformula of F.

(40) rng (the tree of subformulae of F) $=$ Subformulae F.

(41) Suppose $t' \in \operatorname{succ} t$. Then (the tree of subformulae of F)(t') is an immediate constituent of (the tree of subformulae of F)(t).

(42) If $t \preceq t'$, then (the tree of subformulae of F)(t') is a subformula of (the tree of subformulae of F)(t).

(43) If $t \prec t'$, then len($^{@}$(the tree of subformulae of F)(t')) $<$ len($^{@}$(the tree of subformulae of F)(t)).

(44) If $t \prec t'$, then (the tree of subformulae of F)(t') $\neq$ (the tree of subformulae of F)(t).

(45) If $t \prec t'$, then (the tree of subformulae of F)(t') is a proper subformula of (the tree of subformulae of F)(t).

(46) (The tree of subformulae of F)(t) $= F$ iff $t = \varepsilon$.

(47) Suppose $t \neq t'$ and (the tree of subformulae of F)(t) = (the tree of subformulae of F)(t'). Then t and t' are not comparable.

Let F, G be elements of WFF. The F-entry points in subformula tree of G yields an antichain of prefixes of dom (the tree of subformulae of F) and is defined by the condition (Def.3).

(Def.3) Let t be an element of dom (the tree of subformulae of F). Then $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of F)$(t) = G$.

We now state several propositions:

(48) $t \in$ the F-entry points in subformula tree of G iff (the tree of subformulae of F)$(t) = G$.

(49) The F-entry points in subformula tree of $G = \{t : t$ ranges over elements of dom (the tree of subformulae of F), (the tree of subformulae of F)$(t) = G\}$.

(50) G is a subformula of F iff the F-entry points in subformula tree of $G \neq \emptyset$.

(51) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is negative. Then (the tree of subformulae of F)(t') = Arg((the tree of subformulae of F)(t)) and $m = 0$.

(52) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is conjunctive. Then

(i) (the tree of subformulae of F)(t') = LeftArg((the tree of subformulae of F)(t)) and $m = 0$, or

(ii) (the tree of subformulae of F)(t') = RightArg((the tree of subformulae of F)(t)) and $m = 1$.

(53) Suppose $t' = t \frown \langle m \rangle$ and (the tree of subformulae of F)(t) is universal. Then (the tree of subformulae of F)(t') = Scope((the tree of subformulae of F)(t)) and $m = 0$.

(54) Suppose (the tree of subformulae of F)(t) is negative. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F), and

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = Arg((the tree of subformulae of F)(t)).

(55) Suppose (the tree of subformulae of F)(t) is conjunctive. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F),

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = LeftArg((the tree of subformulae of F)(t)),

(iii) $t \frown \langle 1 \rangle \in$ dom (the tree of subformulae of F), and

(iv) (the tree of subformulae of F)$(t \frown \langle 1 \rangle)$ = RightArg((the tree of subformulae of F)(t)).

(56) Suppose (the tree of subformulae of F)(t) is universal. Then

(i) $t \frown \langle 0 \rangle \in$ dom (the tree of subformulae of F), and

(ii) (the tree of subformulae of F)$(t \frown \langle 0 \rangle)$ = Scope((the tree of subformulae of F)(t)).

In the sequel t will be an element of dom (the tree of subformulae of F) and s will be an element of dom (the tree of subformulae of G).

Next we state the proposition

(57) Suppose $t \in$ the F-entry points in subformula tree of G and $s \in$ the G-entry points in subformula tree of H. Then $t \frown s \in$ the F-entry points in subformula tree of H.

In the sequel t will be an element of dom (the tree of subformulae of F) and s will be a finite sequence.

Next we state several propositions:

(58) Suppose $t \in$ the F-entry points in subformula tree of G and $t \frown s \in$ the F-entry points in subformula tree of H. Then $s \in$ the G-entry points in subformula tree of H.

(59) Given F, G, H. Then $\{t \frown s : t$ ranges over elements of dom (the tree of subformulae of F), s ranges over elements of dom (the tree of subformulae of G), $t \in$ the F-entry points in subformula tree of $G \ \wedge \ s \in$ the G-entry points in subformula tree of $H\} \subseteq$ the F-entry points in subformula tree of H.

(60) (The tree of subformulae of F) $\restriction t =$ the tree of subformulae of (the tree of subformulae of F)(t).

(61) $t \in$ the F-entry points in subformula tree of G if and only if (the tree of subformulae of F) $\restriction t =$ the tree of subformulae of G.

(62) The F-entry points in subformula tree of $G = \{t : t$ ranges over elements of dom (the tree of subformulae of F), (the tree of subformulae of F) $\restriction t =$ the tree of subformulae of $G\}$.

In the sequel C is a chain of dom (the tree of subformulae of F).

Next we state the proposition

(63) Given F, G, H, C. Suppose that
 (i) $G \in \{$(the tree of subformulae of F)(t) : t ranges over elements of dom (the tree of subformulae of F), $t \in C\}$, and
 (ii) $H \in \{$(the tree of subformulae of F)(t) : t ranges over elements of dom (the tree of subformulae of F), $t \in C\}$.

 Then G is a subformula of H or H is a subformula of G.

Let F be an element of WFF. An element of WFF is said to be a subformula of F if:

(Def.4) It is a subformula of F.

Let F be an element of WFF and let G be a subformula of F. An element of dom (the tree of subformulae of F) is said to be an entry point in subformula tree of G if:

(Def.5) (The tree of subformulae of F)(it) $= G$.

In the sequel G will denote a subformula of F.

Next we state the proposition

(64) t is an entry point in subformula tree of G iff (the tree of subformulae of F)$(t) = G$.

In the sequel t, t' are entry points in subformula tree of G.

The following proposition is true

(65) If $t \neq t'$, then t and t' are not comparable.

Let F be an element of WFF and let G be a subformula of F. The entry points in subformula tree of G yields a non empty antichain of prefixes of dom (the tree of subformulae of F) and is defined as follows:

(Def.6) The entry points in subformula tree of $G =$ the F-entry points in subformula tree of G.

We now state three propositions:

(66) The entry points in subformula tree of $G =$ the F-entry points in subformula tree of G.

(67) $t \in$ the entry points in subformula tree of G.

(68) The entry points in subformula tree of $G = \{t : t$ ranges over entry points in subformula tree of G, $t = t\}$.

In the sequel G_1, G_2 will denote subformulae of F, t_1 will denote an entry point in subformula tree of G_1, and s will denote an element of dom (the tree of subformulae of G_1).

We now state the proposition

(69) If $s \in$ the G_1-entry points in subformula tree of G_2, then $t_1 ^\frown s$ is an entry point in subformula tree of G_2.

In the sequel s will be a finite sequence.

Next we state three propositions:

(70) If $t_1 {}^\frown s$ is an entry point in subformula tree of G_2, then $s \in$ the G_1-entry points in subformula tree of G_2.

(71) Given F, G_1, G_2. Then $\{t ^\frown s : t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom (the tree of subformulae of G_1), $s \in$ the G_1-entry points in subformula tree of $G_2\} = \{t ^\frown s : t$ ranges over elements of dom (the tree of subformulae of F), s ranges over elements of dom (the tree of subformulae of G_1), $t \in$ the F-entry points in subformula tree of $G_1 \ \wedge\ s \in$ the G_1-entry points in subformula tree of $G_2\}$.

(72) Given F, G_1, G_2. Then $\{t ^\frown s : t$ ranges over entry points in subformula tree of G_1, s ranges over elements of dom (the tree of subformulae of G_1), $s \in$ the G_1-entry points in subformula tree of $G_2\} \subseteq$ the entry points in subformula tree of G_2.

In the sequel G_1, G_2 will denote subformulae of F, t_1 will denote an entry point in subformula tree of G_1, and t_2 will denote an entry point in subformula tree of G_2.

The following two propositions are true:

(73) If there exist t_1, t_2 such that $t_1 \preceq t_2$, then G_2 is a subformula of G_1.

(74) If G_2 is a subformula of G_1, then for every t_1 there exists t_2 such that $t_1 \preceq t_2$.

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The Steinitz Theorem and the Dimension of a Vector Space

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Summary. The main purpose of the paper is to define the dimension of an abstract vector space. The dimension of a finite-dimensional vector space is, by the most common definition, the number of vectors in a basis. Obviously, each basis contains the same number of vectors. We prove the Steinitz Theorem together with Exchange Lemma in the second section. The Steinitz Theorem says that each linearly-independent subset of a vector space has cardinality less than any subset that generates the space, moreover it can be extended to a basis. Further we review some of the standard facts involving the dimension of a vector space. Additionally, in the last section, we introduce two notions: the family of subspaces of a fixed dimension and the pencil of subspaces. Both of them can be applied in the algebraic representation of several geometries.

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The terminology and notation used in this paper have been introduced in the following articles: [13], [23], [12], [8], [2], [6], [24], [4], [5], [22], [1], [7], [3], [17], [19], [9], [21], [15], [10], [20], [16], [18], [14], and [11].

1. Preliminaries

For simplicity we follow the rules: G_1 is a field, V is a vector space over G_1, W is a subspace of V, x is arbitrary, and n is a natural number.

Let S be a non empty 1-sorted structure. Observe that there exists a subset of S which is non empty.

One can prove the following proposition

(1) For every finite set X such that $n \leq \overline{\overline{X}}$ there exists a finite subset A of X such that $\overline{\overline{A}} = n$.

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In the sequel f, g will be functions.

We now state a number of propositions:

(2) For every f such that f is one-to-one holds if $x \in \operatorname{rng} f$, then $\overline{\overline{f^{-1}\{x\}}} = 1$.

(3) For every f such that $x \notin \operatorname{rng} f$ holds $\overline{\overline{f^{-1}\{x\}}} = 0$.

(4) For all f, g such that $\operatorname{rng} f = \operatorname{rng} g$ and f is one-to-one and g is one-to-one holds f and g are fiberwise equipotent.

(5) Let L be a linear combination of V, and let F, G be finite sequences of elements of the carrier of V, and let P be a permutation of $\operatorname{dom} F$. If $G = F \cdot P$, then $\sum(L\,F) = \sum(L\,G)$.

(6) Let L be a linear combination of V and let F be a finite sequence of elements of the carrier of V. If support L misses $\operatorname{rng} F$, then $\sum(L\,F) = 0_V$.

(7) Let F be a finite sequence of elements of the carrier of V. Suppose F is one-to-one. Let L be a linear combination of V. If support $L \subseteq \operatorname{rng} F$, then $\sum(L\,F) = \sum L$.

(8) Let L be a linear combination of V and let F be a finite sequence of elements of the carrier of V. Then there exists a linear combination K of V such that support $K = \operatorname{rng} F \cap$ support L and $L\,F = K\,F$.

(9) Let L be a linear combination of V, and let A be a subset of V, and let F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(10) Let L be a linear combination of V and let A be a subset of V. Suppose support $L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(11) Let L be a linear combination of V. Suppose support $L \subseteq$ the carrier of W. Let K be a linear combination of W. If $K = L \restriction$ (the carrier of W), then support $L =$ support K and $\sum L = \sum K$.

(12) For every linear combination K of W there exists a linear combination L of V such that support $K =$ support L and $\sum K = \sum L$.

(13) Let L be a linear combination of V. Suppose support $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that support $K =$ support L and $\sum K = \sum L$.

(14) For every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(15) Let A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(16) Let A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(17) For every basis A of W there exists a basis B of V such that $A \subseteq B$.

(18) Let A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$

holds $v \notin \mathrm{Lin}(B)$.

(19) Let I be a basis of V and let A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly-dependent.

(20) For every subset A of V such that $A \subseteq$ the carrier of W holds $\mathrm{Lin}(A)$ is a subspace of W.

(21) For every subset A of V and for every subset B of W such that $A = B$ holds $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

2. The Steinitz Theorem

The following two propositions are true:

(22) Let A, B be finite subsets of V and let v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(23) Let A, B be finite subsets of V. Suppose the vector space structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leq \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the vector space structure of $V = \mathrm{Lin}(B \cup C)$.

3. Finite-Dimensional Vector Spaces

Let G_1 be a field and let V be a vector space over G_1. Let us observe that V is finite dimensional if and only if:

(Def.1) There exists finite subset of V which is a basis of V.

Next we state several propositions:

(24) If V is finite dimensional, then every basis of V is finite.

(25) If V is finite dimensional, then for every subset A of V such that A is linearly independent holds A is finite.

(26) If V is finite dimensional, then for all bases A, B of V holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(27) $\mathbf{0}_V$ is finite dimensional.

(28) If V is finite dimensional, then W is finite dimensional.

Let G_1 be a field and let V be a vector space over G_1. Observe that there exists a subspace of V which is strict and finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. Note that every subspace of V is finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. One can check that there exists a subspace of V which is strict.

4. The Dimension of a Vector Space

Let G_1 be a field and let V be a vector space over G_1. Let us assume that V is finite dimensional. The functor $\dim(V)$ yields a natural number and is defined by:

(Def.2) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

We adopt the following rules: V denotes a finite dimensional vector space over G_1, W, W_1, W_2 denote subspaces of V, and u, v denote vectors of V.

The following propositions are true:

(29) $\dim(W) \leq \dim(V)$.

(30) For every subset A of V such that A is linearly independent holds $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(31) $\dim(V) = \dim(\Omega_V)$.

(32) $\dim(V) = \dim(W)$ iff $\Omega_V = \Omega_W$.

(33) $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(34) $\dim(V) = 1$ iff there exists v such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(35) $\dim(V) = 2$ iff there exist u, v such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(36) $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(37) $\dim(W_1 \cap W_2) \geq (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(38) If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

5. The Fixed-Dimensional Subspace Family and the Pencil of Subspaces

One can prove the following proposition

(39) $n \leq \dim(V)$ iff there exists a strict subspace W of V such that $\dim(W) = n$.

Let G_1 be a field, let V be a finite dimensional vector space over G_1, and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def.3) $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

We now state three propositions:

(40) If $n \leq \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(41) If $\dim(V) < n$, then $\mathrm{Sub}_n(V) = \emptyset$.

(42) $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

Let G_1 be a field, let V be a finite dimensional vector space over G_1, let W_2 be a subspace of V, and let W_1 be a strict subspace of W_2. Let us assume that $\dim(W_2) = \dim(W_1) + 2$. The functor $\mathbf{p}(W_1, W_2)$ yields a non empty set and is defined by:

(Def.4) $x \in \mathbf{p}(W_1, W_2)$ iff there exists a strict subspace W of W_2 such that $W = x$ and $\dim(W) = \dim(W_1) + 1$ and W_1 is a subspace of W.

We now state two propositions:

(43) Let W_1 be a strict subspace of W_2. Suppose $\dim(W_2) = \dim(W_1) + 2$. Then $x \in \mathbf{p}(W_1, W_2)$ if and only if there exists a strict subspace W of V such that $W = x$ and $\dim(W) = \dim(W_1) + 1$ and W_1 is a subspace of W and W is a subspace of W_2.

(44) For every strict subspace W_1 of W_2 such that $\dim(W_2) = \dim(W_1) + 2$ holds $\mathbf{p}(W_1, W_2) \subseteq \mathrm{Sub}_{\dim(W_1)+1}(V)$.

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On the Go-Board of a Standard Special Circular Sequence

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The articles [21], [24], [5], [23], [9], [2], [19], [17], [1], [4], [3], [7], [22], [10], [11], [18], [25], [6], [8], [12], [13], [15], [20], [16], and [14] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity we adopt the following convention: f will denote a standard special circular sequence, i, j, k, n, i_1, i_2, j_1, j_2 will denote natural numbers, r, s, r_1, r_2 will denote real numbers, p, q, p_1 will denote points of $\mathcal{E}^2_{\rm T}$, and G will denote a Go-board.

The following propositions are true:

(1) If $|r_1 - r_2| > s$, then $r_1 + s < r_2$ or $r_2 + s < r_1$.

(2) $|r - s| = 0$ iff $r = s$.

(3) For all points p, p_1, q of $\mathcal{E}^n_{\rm T}$ such that $p + p_1 = q + p_1$ holds $p = q$.

(4) For all points p, p_1, q of $\mathcal{E}^n_{\rm T}$ such that $p_1 + p = p_1 + q$ holds $p = q$.

(5) If $p_1 \in \mathcal{L}(p, q)$ and $p_{\mathbf{1}} = q_{\mathbf{1}}$, then $(p_1)_{\mathbf{1}} = q_{\mathbf{1}}$.

(6) If $p_1 \in \mathcal{L}(p, q)$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $(p_1)_{\mathbf{2}} = q_{\mathbf{2}}$.

(7) $\frac{1}{2} \cdot (p + q) \in \mathcal{L}(p, q)$.

(8) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $q_{\mathbf{1}} = (p_1)_{\mathbf{1}}$ and $p_{\mathbf{2}} \leq q_{\mathbf{2}}$ and $q_{\mathbf{2}} \leq (p_1)_{\mathbf{2}}$, then $q \in \mathcal{L}(p, p_1)$.

(9) If $p_{\mathbf{1}} \leq q_{\mathbf{1}}$ and $q_{\mathbf{1}} \leq (p_1)_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $q_{\mathbf{2}} = (p_1)_{\mathbf{2}}$, then $q \in \mathcal{L}(p, p_1)$.

(10) Let D be a non empty set, and given i, j, and let M be a matrix over D. If $1 \leq i$ and $i \leq \operatorname{len} M$ and $1 \leq j$ and $j \leq \operatorname{width} M$, then $\langle i, j \rangle \in$ the indices of M.

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(11) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\frac{1}{2} \cdot (G_{i,j} + G_{i+1,j+1}) = \frac{1}{2} \cdot (G_{i,j+1} + G_{i+1,j})$.

(12) Suppose $\mathcal{L}(f,k)$ is horizontal. Then there exists j such that $1 \leq j$ and $j \leq \operatorname{width}$ the Go-board of f and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{2}} = ((\text{the Go-board of } f)_{1,j})_{\mathbf{2}}$.

(13) Suppose $\mathcal{L}(f,k)$ is vertical. Then there exists i such that $1 \leq i$ and $i \leq \operatorname{len}$ the Go-board of f and for every p such that $p \in \mathcal{L}(f,k)$ holds $p_{\mathbf{1}} = ((\text{the Go-board of } f)_{i,1})_{\mathbf{1}}$.

(14) If $i \leq \operatorname{len}$ the Go-board of f and $j \leq \operatorname{width}$ the Go-board of f, then $\operatorname{Int}\operatorname{cell}(\text{the Go-board of } f, i, j)$ misses $\widetilde{\mathcal{L}}(f)$.

2. Segments on a Go-Board

Next we state a number of propositions:

(15) If $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j+2 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i,j+1}) \cap \mathcal{L}(G_{i,j+1}, G_{i,j+2}) = \{G_{i,j+1}\}$.

(16) If $1 \leq i$ and $i+2 \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \cap \mathcal{L}(G_{i+1,j}, G_{i+2,j}) = \{G_{i+1,j}\}$.

(17) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i,j+1}) \cap \mathcal{L}(G_{i,j+1}, G_{i+1,j+1}) = \{G_{i,j+1}\}$.

(18) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j+1}, G_{i+1,j+1}) \cap \mathcal{L}(G_{i+1,j}, G_{i+1,j+1}) = \{G_{i+1,j+1}\}$.

(19) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \cap \mathcal{L}(G_{i,j}, G_{i,j+1}) = \{G_{i,j}\}$.

(20) If $1 \leq i$ and $i+1 \leq \operatorname{len} G$ and $1 \leq j$ and $j+1 \leq \operatorname{width} G$, then $\mathcal{L}(G_{i,j}, G_{i+1,j}) \cap \mathcal{L}(G_{i+1,j}, G_{i+1,j+1}) = \{G_{i+1,j}\}$.

(21) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1+1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2+1 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1})$. Then $i_1 = i_2$ and $|j_1 - j_2| \leq 1$.

(22) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1+1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2+1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1+1,j_1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$. Then $j_1 = j_2$ and $|i_1 - i_2| \leq 1$.

(23) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1+1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2+1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$. Then $i_1 = i_2$ or $i_1 = i_2+1$ but $j_1 = j_2$ or $j_1+1 = j_2$.

(24) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1+1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2+1 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1})$. Then

(i) $j_1 = j_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1}) = \mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1})$, or
(ii) $j_1 = j_2 + 1$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1}) = \{G_{i_1,j_1}\}$, or
(iii) $j_1 + 1 = j_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1}) = \{G_{i_2,j_2}\}$.

(25) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1+1,j_1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$. Then
(i) $i_1 = i_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1+1,j_1}) = \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$, or
(ii) $i_1 = i_2 + 1$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1+1,j_1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2}) = \{G_{i_1,j_1}\}$, or
(iii) $i_1 + 1 = i_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1+1,j_1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2}) = \{G_{i_2,j_2}\}$.

(26) Let i_1, j_1, i_2, j_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 + 1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1})$ meets $\mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$. Then $j_1 = j_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2}) = \{G_{i_1,j_1}\}$ or $j_1 + 1 = j_2$ and $\mathcal{L}(G_{i_1,j_1}, G_{i_1,j_1+1}) \cap \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2}) = \{G_{i_1,j_1+1}\}$.

(27) Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 + 1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 + 1 \leq \operatorname{width} G$ and $\frac{1}{2} \cdot (G_{i_1,j_1} + G_{i_1,j_1+1}) \in \mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1})$. Then $i_1 = i_2$ and $j_1 = j_2$.

(28) Suppose $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 \leq \operatorname{width} G$ and $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\frac{1}{2} \cdot (G_{i_1,j_1} + G_{i_1+1,j_1}) \in \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$. Then $i_1 = i_2$ and $j_1 = j_2$.

(29) Suppose $1 \leq i_1$ and $i_1 + 1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 \leq \operatorname{width} G$. Then it is not true that there exist i_2, j_2 such that $1 \leq i_2$ and $i_2 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 + 1 \leq \operatorname{width} G$ and $\frac{1}{2} \cdot (G_{i_1,j_1} + G_{i_1+1,j_1}) \in \mathcal{L}(G_{i_2,j_2}, G_{i_2,j_2+1})$.

(30) Suppose $1 \leq i_1$ and $i_1 \leq \operatorname{len} G$ and $1 \leq j_1$ and $j_1 + 1 \leq \operatorname{width} G$. Then it is not true that there exist i_2, j_2 such that $1 \leq i_2$ and $i_2 + 1 \leq \operatorname{len} G$ and $1 \leq j_2$ and $j_2 \leq \operatorname{width} G$ and $\frac{1}{2} \cdot (G_{i_1,j_1} + G_{i_1,j_1+1}) \in \mathcal{L}(G_{i_2,j_2}, G_{i_2+1,j_2})$.

3. Standard Special Circular Sequences

In the sequel f will be a non constant standard special circular sequence.

The following propositions are true:

(31) For every standard non empty finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $i \in \operatorname{dom} f$ and $i + 1 \in \operatorname{dom} f$ holds $\pi_i f \neq \pi_{i+1} f$.

(32) There exists i such that $i \in \operatorname{dom} f$ and $(\pi_i f)_{\mathbf{1}} \neq (\pi_1 f)_{\mathbf{1}}$.

(33) There exists i such that $i \in \operatorname{dom} f$ and $(\pi_i f)_{\mathbf{2}} \neq (\pi_1 f)_{\mathbf{2}}$.

(34) len the Go-board of $f > 1$.

(35) width the Go-board of $f > 1$.

(36) $\operatorname{len} f > 4$.

(37) Let f be a circular s.c.c. finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\operatorname{len} f > 4$. Let i, j be natural numbers. If $1 \leq i$ and $i < j$ and $j < \operatorname{len} f$, then $\pi_i f \neq \pi_j f$.

(38) For all natural numbers i, j such that $1 \leq i$ and $i < j$ and $j < \operatorname{len} f$ holds $\pi_i f \neq \pi_j f$.

(39) For all natural numbers i, j such that $1 < i$ and $i < j$ and $j \leq \operatorname{len} f$ holds $\pi_i f \neq \pi_j f$.

(40) For every natural number i such that $1 < i$ and $i \leq \operatorname{len} f$ and $\pi_i f = \pi_1 f$ holds $i = \operatorname{len} f$.

(41) Suppose that
(i) $1 \leq i$,
(ii) $i \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j + 1 \leq \operatorname{width}$ the Go-board of f, and
(v) $\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i,j+1}) \in \widetilde{\mathcal{L}}(f)$.

Then there exists k such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ and $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) = \mathcal{L}(f, k)$.

(42) Suppose that
(i) $1 \leq i$,
(ii) $i + 1 \leq \operatorname{len}$ the Go-board of f
(iii) $1 \leq j$,
(iv) $j \leq \operatorname{width}$ the Go-board of f and
(v) $\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i+1,j}) \in \widetilde{\mathcal{L}}(f)$.

Then there exists k such that $1 \leq k$ and $k+1 \leq \operatorname{len} f$ and $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) = \mathcal{L}(f, k)$.

(43) Suppose that
(i) $1 \leq i$,
(ii) $i + 1 \leq \operatorname{len}$ the Go-board of f
(iii) $1 \leq j$,
(iv) $j + 1 \leq \operatorname{width}$ the Go-board of f
(v) $1 \leq k$,
(vi) $k + 1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f, k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f, k + 1)$.

Then $\pi_k f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j}$.

(44) Suppose that
(i) $1 \leq i$,
(ii) $i \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j + 1 < \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k + 1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i,j+2}) = \mathcal{L}(f, k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) = \mathcal{L}(f, k + 1)$.

Then $\pi_k f = (\text{the Go-board of } f)_{i,j+2}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(45) Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) = \mathcal{L}(f,k+1)$.
Then $\pi_k f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(46) Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f,k+1)$.
Then $\pi_k f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j+1}$.

(47) Suppose that
(i) $1 \leq i$,
(ii) $i+1 < \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+2,j}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) = \mathcal{L}(f,k+1)$.
Then $\pi_k f = (\text{the Go-board of } f)_{i+2,j}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(48) Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) = \mathcal{L}(f,k+1)$.
Then $\pi_k f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(49) Suppose that
- (i) $1 \leq i$,
- (ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
- (iii) $1 \leq j$,
- (iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
- (v) $1 \leq k$,
- (vi) $k+1 < \operatorname{len} f$,
- (vii) $\mathcal{L}(($the Go-board of $f)_{i+1,j},($the Go-board of $f)_{i+1,j+1}) = \mathcal{L}(f,k)$, and
- (viii) $\mathcal{L}(($the Go-board of $f)_{i,j+1},($the Go-board of $f)_{i+1,j+1}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i,j+1}$.

(50) Suppose that
- (i) $1 \leq i$,
- (ii) $i \leq \operatorname{len}$ the Go-board of f,
- (iii) $1 \leq j$,
- (iv) $j+1 < \operatorname{width}$ the Go-board of f,
- (v) $1 \leq k$,
- (vi) $k+1 < \operatorname{len} f$,
- (vii) $\mathcal{L}(($the Go-board of $f)_{i,j},($the Go-board of $f)_{i,j+1}) = \mathcal{L}(f,k)$, and
- (viii) $\mathcal{L}(($the Go-board of $f)_{i,j+1},($the Go-board of $f)_{i,j+2}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f =$ (the Go-board of $f)_{i,j}$ and $\pi_{k+1} f =$ (the Go-board of $f)_{i,j+1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i,j+2}$.

(51) Suppose that
- (i) $1 \leq i$,
- (ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
- (iii) $1 \leq j$,
- (iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
- (v) $1 \leq k$,
- (vi) $k+1 < \operatorname{len} f$,
- (vii) $\mathcal{L}(($the Go-board of $f)_{i,j},($the Go-board of $f)_{i,j+1}) = \mathcal{L}(f,k)$, and
- (viii) $\mathcal{L}(($the Go-board of $f)_{i,j+1},($the Go-board of $f)_{i+1,j+1}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f =$ (the Go-board of $f)_{i,j}$ and $\pi_{k+1} f =$ (the Go-board of $f)_{i,j+1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,j+1}$.

(52) Suppose that
- (i) $1 \leq i$,
- (ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
- (iii) $1 \leq j$,
- (iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
- (v) $1 \leq k$,
- (vi) $k+1 < \operatorname{len} f$,
- (vii) $\mathcal{L}(($the Go-board of $f)_{i,j+1},($the Go-board of $f)_{i+1,j+1}) = \mathcal{L}(f,k)$, and
- (viii) $\mathcal{L}(($the Go-board of $f)_{i+1,j},($the Go-board of $f)_{i+1,j+1}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f =$ (the Go-board of $f)_{i,j+1}$ and $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,j}$.

(53) Suppose that
(i) $1 \leq i$,
(ii) $i+1 < \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+2,j}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+2,j}$.

(54) Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
(v) $1 \leq k$,
(vi) $k+1 < \operatorname{len} f$,
(vii) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) = \mathcal{L}(f,k)$, and
(viii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) = \mathcal{L}(f,k+1)$.

Then $\pi_k f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j+1}$.

(55) Suppose that
(i) $1 \leq i$,
(ii) $i \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 < \operatorname{width}$ the Go-board of f,
(v) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$, and
(vi) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i,j+2}) \subseteq \widetilde{\mathcal{L}}(f)$.

Then
(vii) $\pi_1 f = (\text{the Go-board of } f)_{i,j+1}$ but $\pi_2 f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i,j+2}$ or $\pi_2 f = (\text{the Go-board of } f)_{i,j+2}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i,j}$, or
(viii) there exists k such that $1 \leq k$ and $k+1 < \operatorname{len} f$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_k f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j+2}$ or $\pi_k f = (\text{the Go-board of } f)_{i,j+2}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(56) Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+1 \leq \operatorname{width}$ the Go-board of f,
(v) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$, and
(vi) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$.

Then

(vii) $\pi_1 f$ = (the Go-board of $f)_{i,j+1}$ but $\pi_2 f$ = (the Go-board of $f)_{i,j}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i+1,j+1}$ or $\pi_2 f$ = (the Go-board of $f)_{i+1,j+1}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i,j}$, or

(viii) there exists k such that $1 \leq k$ and $k+1 < \mathrm{len}\, f$ and $\pi_{k+1} f$ = (the Go-board of $f)_{i,j+1}$ and $\pi_k f$ = (the Go-board of $f)_{i,j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+1,j+1}$ or $\pi_k f$ = (the Go-board of $f)_{i+1,j+1}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i,j}$.

(57) Suppose that

(i) $1 \leq i$,

(ii) $i+1 \leq$ len the Go-board of f,

(iii) $1 \leq j$,

(iv) $j+1 \leq$ width the Go-board of f,

(v) $\mathcal{L}$((the Go-board of $f)_{i,j+1}$, (the Go-board of $f)_{i+1,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\mathcal{L}$((the Go-board of $f)_{i+1,j+1}$, (the Go-board of $f)_{i+1,j}) \subseteq \widetilde{\mathcal{L}}(f)$.

Then

(vii) $\pi_1 f$ = (the Go-board of $f)_{i+1,j+1}$ but $\pi_2 f$ = (the Go-board of $f)_{i,j+1}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i+1,j}$ or $\pi_2 f$ = (the Go-board of $f)_{i+1,j}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i,j+1}$, or

(viii) there exists k such that $1 \leq k$ and $k+1 < \mathrm{len}\, f$ and $\pi_{k+1} f$ = (the Go-board of $f)_{i+1,j+1}$ and $\pi_k f$ = (the Go-board of $f)_{i,j+1}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+1,j}$ or $\pi_k f$ = (the Go-board of $f)_{i+1,j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i,j+1}$.

(58) Suppose that

(i) $1 \leq i$,

(ii) $i+1 <$ len the Go-board of f,

(iii) $1 \leq j$,

(iv) $j \leq$ width the Go-board of f,

(v) $\mathcal{L}$((the Go-board of $f)_{i,j}$, (the Go-board of $f)_{i+1,j}) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\mathcal{L}$((the Go-board of $f)_{i+1,j}$, (the Go-board of $f)_{i+2,j}) \subseteq \widetilde{\mathcal{L}}(f)$.

Then

(vii) $\pi_1 f$ = (the Go-board of $f)_{i+1,j}$ but $\pi_2 f$ = (the Go-board of $f)_{i,j}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i+2,j}$ or $\pi_2 f$ = (the Go-board of $f)_{i+2,j}$ and $\pi_{\mathrm{len}\, f -' 1} f$ = (the Go-board of $f)_{i,j}$, or

(viii) there exists k such that $1 \leq k$ and $k+1 < \mathrm{len}\, f$ and $\pi_{k+1} f$ = (the Go-board of $f)_{i+1,j}$ and $\pi_k f$ = (the Go-board of $f)_{i,j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+2,j}$ or $\pi_k f$ = (the Go-board of $f)_{i+2,j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i,j}$.

(59) Suppose that

(i) $1 \leq i$,

(ii) $i+1 \leq$ len the Go-board of f,

(iii) $1 \leq j$,

(iv) $j+1 \leq$ width the Go-board of f,

(v) $\mathcal{L}$((the Go-board of $f)_{i,j}$, (the Go-board of $f)_{i+1,j}) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$.
Then
(vii) $\pi_1 f = (\text{the Go-board of } f)_{i+1,j}$ but $\pi_2 f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i+1,j+1}$ or $\pi_2 f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i,j}$, or
(viii) there exists k such that $1 \leq k$ and $k+1 < \operatorname{len} f$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_k f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j+1}$ or $\pi_k f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$.

(60) Suppose that
(i) $1 \leq i$,
(ii) $i + 1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j + 1 \leq \operatorname{width}$ the Go-board of f,
(v) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$, and
(vi) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j+1}, (\text{the Go-board of } f)_{i,j+1}) \subseteq \widetilde{\mathcal{L}}(f)$.
Then
(vii) $\pi_1 f = (\text{the Go-board of } f)_{i+1,j+1}$ but $\pi_2 f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i,j+1}$ or $\pi_2 f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{\operatorname{len} f -' 1} f = (\text{the Go-board of } f)_{i+1,j}$, or
(viii) there exists k such that $1 \leq k$ and $k+1 < \operatorname{len} f$ and $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$ and $\pi_k f = (\text{the Go-board of } f)_{i+1,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j+1}$ or $\pi_k f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j}$.

(61) Suppose $1 \leq i$ and $i < \operatorname{len}$ the Go-board of f and $1 \leq j$ and $j + 1 < \operatorname{width}$ the Go-board of f. Then
(i) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(ii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i,j+2}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(iii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$.

(62) Suppose $1 \leq i$ and $i < \operatorname{len}$ the Go-board of f and $1 \leq j$ and $j + 1 < \operatorname{width}$ the Go-board of f. Then
(i) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(ii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j+1}, (\text{the Go-board of } f)_{i+1,j+2}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(iii) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$.

(63) Suppose $1 \leq j$ and $j < \operatorname{width}$ the Go-board of f and $1 \leq i$ and $i + 1 < \operatorname{len}$ the Go-board of f. Then
(i) $\mathcal{L}((\text{the Go-board of } f)_{i,j}, (\text{the Go-board of } f)_{i+1,j}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(ii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+2,j}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(iii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$.

(64) Suppose $1 \leq j$ and $j < \operatorname{width}$ the Go-board of f and $1 \leq i$ and $i + 1 < \operatorname{len}$ the Go-board of f. Then
(i) $\mathcal{L}((\text{the Go-board of } f)_{i,j+1}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or
(ii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j+1}, (\text{the Go-board of } f)_{i+2,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$, or

(iii) $\mathcal{L}((\text{the Go-board of } f)_{i+1,j}, (\text{the Go-board of } f)_{i+1,j+1}) \not\subseteq \widetilde{\mathcal{L}}(f)$.

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On the Monoid of Endomorphisms of Universal Algebra and Many Sorted Algebra

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MML Identifier: `ENDALG`.

The articles [17], [7], [18], [5], [6], [4], [14], [16], [1], [12], [3], [10], [11], [8], [9], [2], [13], and [15] provide the terminology and notation for this paper.

In this paper U_1 is a universal algebra and f is a function from U_1 into U_1.

Let us consider U_1. The functor $\mathrm{end}(U_1)$ yields a non empty set of functions from the carrier of U_1 to the carrier of U_1 and is defined as follows:

(Def.1) For every function h from U_1 into U_1 holds $h \in \mathrm{end}(U_1)$ iff h is a homomorphism of U_1 into U_1.

Next we state four propositions:

(1) $\mathrm{end}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

(2) For every f holds $f \in \mathrm{end}(U_1)$ iff f is a homomorphism of U_1 into U_1.

(3) $\mathrm{id}_{(\text{the carrier of } U_1)} \in \mathrm{end}(U_1)$.

(4) For all elements f_1, f_2 of $\mathrm{end}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{end}(U_1)$.

Let us consider U_1. The functor $\mathrm{Comp}(U_1)$ yielding a binary operation on $\mathrm{end}(U_1)$ is defined as follows:

(Def.2) For all elements x, y of $\mathrm{end}(U_1)$ holds $(\mathrm{Comp}(U_1))(x, y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{End}(U_1)$ yields a strict multiplicative loop structure and is defined by:

(Def.3) The carrier of $\mathrm{End}(U_1) = \mathrm{end}(U_1)$ and the multiplication of $\mathrm{End}(U_1) = \mathrm{Comp}(U_1)$ and the unity of $\mathrm{End}(U_1) = \mathrm{id}_{(\text{the carrier of } U_1)}$.

Let us consider U_1. One can check that $\mathrm{End}(U_1)$ is non empty.

Let us consider U_1. One can verify that $\mathrm{End}(U_1)$ is left unital well unital and associative.

Next we state two propositions:

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(5) Let x, y be elements of the carrier of $\mathrm{End}(U_1)$ and let f, g be elements of $\mathrm{end}(U_1)$. If $x = f$ and $y = g$, then $x \cdot y = g \cdot f$.

(6) $\mathrm{id}_{(\text{the carrier of } U_1)} = 1_{\mathrm{End}(U_1)}$.

In the sequel S will be a non void non empty many sorted signature and U_2 will be a non-empty algebra over S.

Let us consider S, U_2. The functor $\mathrm{end}(U_2)$ yields a set of many sorted functions from the sorts of U_2 into the sorts of U_2 and is defined by the conditions (Def.4).

(Def.4) (i) Every element of $\mathrm{end}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{end}(U_2)$ iff h is a homomorphism of U_2 into U_2.

One can prove the following propositions:

(7) For every many sorted function F from U_2 into U_2 holds $F \in \mathrm{end}(U_2)$ iff F is a homomorphism of U_2 into U_2.

(8) For every element f of $\mathrm{end}(U_2)$ holds $f \in \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(9) $\mathrm{end}(U_2) \subseteq \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(10) $\mathrm{id}_{(\text{the sorts of } U_2)} \in \mathrm{end}(U_2)$.

(11) For all elements f_1, f_2 of $\mathrm{end}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{end}(U_2)$.

(12) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{end}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{end}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{Comp}(U_2)$ yielding a binary operation on $\mathrm{end}(U_2)$ is defined as follows:

(Def.5) For all elements x, y of $\mathrm{end}(U_2)$ holds $(\mathrm{Comp}(U_2))(x, y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{End}(U_2)$ yields a strict multiplicative loop structure and is defined by:

(Def.6) The carrier of $\mathrm{End}(U_2) = \mathrm{end}(U_2)$ and the multiplication of $\mathrm{End}(U_2) = \mathrm{Comp}(U_2)$ and the unity of $\mathrm{End}(U_2) = \mathrm{id}_{(\text{the sorts of } U_2)}$.

Let us consider S, U_2. Note that $\mathrm{End}(U_2)$ is non empty.

Let us consider S, U_2. Note that $\mathrm{End}(U_2)$ is left unital well unital and associative.

The following four propositions are true:

(13) Let x, y be elements of the carrier of $\mathrm{End}(U_2)$ and let f, g be elements of $\mathrm{end}(U_2)$. If $x = f$ and $y = g$, then $x \cdot y = g \circ f$.

(14) $\mathrm{id}_{(\text{the sorts of } U_2)} = 1_{\mathrm{End}(U_2)}$.

(15) Let U_3, U_4 be universal algebras. Suppose U_3 and U_4 are similar. Let F be a many sorted function from $\mathrm{MSAlg}(U_3)$ into $(\mathrm{MSAlg}(U_4) \operatorname{over} \mathrm{MSSign}(U_3))$. Then $F(0)$ is a function from U_3 into U_4.

(16) For every element f of $\mathrm{end}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

Let G, H be multiplicative loop structures.

(Def.7) A function from the carrier of G into the carrier of H is called a map from G into H.

Let G, H be non empty multiplicative loop structures. A map from G into H is multiplicative if:

(Def.8) For all elements x, y of the carrier of G holds $\mathrm{it}(x \cdot y) = \mathrm{it}(x) \cdot \mathrm{it}(y)$.

A map from G into H is unity-preserving if:

(Def.9) $\mathrm{It}(1_G) = 1_H$.

Let us mention that there exists a non empty multiplicative loop structure which is left unital.

Let G, H be left unital non empty multiplicative loop structures. Observe that there exists a map from G into H which is multiplicative and unity-preserving.

Let G, H be left unital non empty multiplicative loop structures. A homomorphism from G to H is a multiplicative unity-preserving map from G into H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is a monomorphism if and only if:

(Def.10) h is one-to-one.

We say that h is an epimorphism if and only if:

(Def.11) $\mathrm{rng}\, h =$ the carrier of H.

Let G, H be left unital non empty multiplicative loop structures and let h be a map from G into H. We say that h is an isomorphism if and only if:

(Def.12) h is an epimorphism and a monomorphism.

We now state the proposition

(17) Let G be a left unital non empty multiplicative loop structure. Then $\mathrm{id}_{(\text{the carrier of } G)}$ is a homomorphism from G to G.

Let G, H be left unital non empty multiplicative loop structures. We say that G and H are isomorphic if and only if:

(Def.13) There exists homomorphism from G to H which is an isomorphism.

Let us observe that this predicate is reflexive.

One can prove the following propositions:

(18) Let h be a function. Suppose $\mathrm{dom}\, h = \mathrm{end}(U_1)$ and for arbitrary x such that $x \in \mathrm{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\mathrm{End}(U_1)$ to $\mathrm{End}(\mathrm{MSAlg}(U_1))$.

(19) Let h be a homomorphism from $\mathrm{End}(U_1)$ to $\mathrm{End}(\mathrm{MSAlg}(U_1))$. Suppose that for arbitrary x such that $x \in \mathrm{end}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is an isomorphism.

(20) $\mathrm{End}(U_1)$ and $\mathrm{End}(\mathrm{MSAlg}(U_1))$ are isomorphic.

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More on Segments on a Go-Board

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Summary. We continue the preparatory work for the Jordan Curve Theorem.

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The terminology and notation used here are introduced in the following articles: [20], [23], [22], [8], [2], [18], [16], [1], [4], [3], [6], [21], [9], [10], [17], [24], [5], [7], [11], [12], [14], [19], [15], and [13].

We adopt the following rules: i, j, k will be natural numbers, p will be a point of $\mathcal{E}^2_{\rm T}$, and f will be a non constant standard special circular sequence.

One can prove the following propositions:

(1) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that
(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq$ width the Go-board of f,
(v) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_k f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,j+2}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,j+2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j} +$ (the Go-board of $f)_{i+1,j+1})$, $\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j+1} +$ (the Go-board of $f)_{i+1,j+2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(2) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that
(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq$ width the Go-board of f,
(v) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,j+1}$, and

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(vi) $\pi_k f$ = (the Go-board of $f)_{i+2,j+1}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+1,j+2}$ or $\pi_{k+2} f$ = (the Go-board of $f)_{i+2,j+1}$ and $\pi_k f$ = (the Go-board of $f)_{i+1,j+2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j}$ + (the Go-board of $f)_{i+1,j+1}$), $\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j+1}$ + (the Go-board of $f)_{i+1,j+2}$)) misses $\widetilde{\mathcal{L}}(f)$.

(3) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that

(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq \operatorname{width}$ the Go-board of f,
(v) $\pi_{k+1} f$ = (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_k f$ = (the Go-board of $f)_{i+2,j+1}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+1,j}$ or $\pi_{k+2} f$ = (the Go-board of $f)_{i+2,j+1}$ and $\pi_k f$ = (the Go-board of $f)_{i+1,j}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j}$ + (the Go-board of $f)_{i+1,j+1}$), $\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j+1}$ + (the Go-board of $f)_{i+1,j+2}$)) misses $\widetilde{\mathcal{L}}(f)$.

(4) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that

(i) $1 \leq i$,
(ii) $i+1 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq \operatorname{width}$ the Go-board of f,
(v) $\pi_{k+1} f$ = (the Go-board of $f)_{i,j+1}$, and
(vi) $\pi_k f$ = (the Go-board of $f)_{i,j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i,j+2}$ or $\pi_{k+2} f$ = (the Go-board of $f)_{i,j}$ and $\pi_k f$ = (the Go-board of $f)_{i,j+2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j}$ + (the Go-board of $f)_{i+1,j+1}$), $\frac{1}{2} \cdot ($(the Go-board of $f)_{i,j+1}$ + (the Go-board of $f)_{i+1,j+2}$)) misses $\widetilde{\mathcal{L}}(f)$.

(5) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that

(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq \operatorname{width}$ the Go-board of f,
(v) $\pi_{k+1} f$ = (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_k f$ = (the Go-board of $f)_{i,j+1}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{i+1,j+2}$ or $\pi_{k+2} f$ = (the Go-board of $f)_{i,j+1}$ and $\pi_k f$ = (the Go-board of $f)_{i+1,j+2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ($(the Go-board of $f)_{i+1,j}$ + (the Go-board of $f)_{i+2,j+1}$), $\frac{1}{2} \cdot ($(the Go-board of $f)_{i+1,j+1}$ + (the Go-board of $f)_{i+2,j+2}$)) misses $\widetilde{\mathcal{L}}(f)$.

(6) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that

(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $1 \leq j$,
(iv) $j+2 \leq \operatorname{width}$ the Go-board of f,
(v) $\pi_{k+1} f$ = (the Go-board of $f)_{i+1,j+1}$, and

(vi) $\pi_k f =$ (the Go-board of $f)_{i,j+1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,j}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i,j+1}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,j}$.

Then $\mathcal{L}(\frac{1}{2} \cdot (($the Go-board of $f)_{i+1,j} + ($the Go-board of $f)_{i+2,j+1}), \frac{1}{2} \cdot (($the Go-board of $f)_{i+1,j+1} + ($the Go-board of $f)_{i+2,j+2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(7) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i. Suppose that
(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,1}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{i+2,1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,2}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i+2,1}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot (($the Go-board of $f)_{i,1} + ($the Go-board of $f)_{i+1,1}) - [0,1], \frac{1}{2} \cdot (($the Go-board of $f)_{i,1} + ($the Go-board of $f)_{i+1,2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(8) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i. Suppose that
(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,1}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{i,1}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,2}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i,1}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,2}$.

Then $\mathcal{L}(\frac{1}{2} \cdot (($the Go-board of $f)_{i+1,1} + ($the Go-board of $f)_{i+2,1}) - [0, 1], \frac{1}{2} \cdot (($the Go-board of $f)_{i+1,1} + ($the Go-board of $f)_{i+2,2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(9) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i. Suppose that
(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{i+2,\text{width the Go-board of } f}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f -' 1}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i+2,\text{width the Go-board of } f}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f -' 1}$.

Then $\mathcal{L}(\frac{1}{2} \cdot (($the Go-board of $f)_{i,\text{width the Go-board of } f -' 1} + ($the Go-board of $f)_{i+1,\text{width the Go-board of } f}), \frac{1}{2} \cdot (($the Go-board of $f)_{i,\text{width the Go-board of } f} + ($the Go-board of $f)_{i+1,\text{width the Go-board of } f}) + [0,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(10) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i. Suppose that
(i) $1 \leq i$,
(ii) $i+2 \leq \operatorname{len}$ the Go-board of f,
(iii) $\pi_{k+1} f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{i,\text{width the Go-board of } f}$ and $\pi_{k+2} f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f -' 1}$ or $\pi_{k+2} f =$ (the Go-board of $f)_{i,\text{width the Go-board of } f}$ and $\pi_k f =$ (the Go-board of $f)_{i+1,\text{width the Go-board of } f -' 1}$.

Then $\mathcal{L}(\frac{1}{2} \cdot (($the Go-board of $f)_{i+1,\text{width the Go-board of } f -' 1} + ($the Go-board of $f)_{i+2,\text{width the Go-board of } f}), \frac{1}{2} \cdot (($the Go-board of $f)_{i+1,\text{width the Go-board of } f} + ($the Go-board of $f)_{i+2,\text{width the Go-board of } f}) + [0,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(11) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given i, j. Suppose that

(i) $1 \leq j$,
(ii) $j+1 \leq \operatorname{width}$ the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq \operatorname{len}$ the Go-board of f,
(v) $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$, and
(vi) $\pi_k f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+2,j+1}$ or $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j+1}$ and $\pi_k f = (\text{the Go-board of } f)_{i+2,j+1}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i+1,j+1}), \frac{1}{2} \cdot ((\text{the Go-board of } f)_{i+1,j} + (\text{the Go-board of } f)_{i+2,j+1}))$ misses $\widetilde{\mathcal{L}}(f)$.

(12) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j, i. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq \operatorname{width}$ the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq \operatorname{len}$ the Go-board of f,
(v) $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$, and
(vi) $\pi_k f = (\text{the Go-board of } f)_{i+1,j+2}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+2,j+1}$ or $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j+2}$ and $\pi_k f = (\text{the Go-board of } f)_{i+2,j+1}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i+1,j+1}), \frac{1}{2} \cdot ((\text{the Go-board of } f)_{i+1,j} + (\text{the Go-board of } f)_{i+2,j+1}))$ misses $\widetilde{\mathcal{L}}(f)$.

(13) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j, i. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq \operatorname{width}$ the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq \operatorname{len}$ the Go-board of f,
(v) $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j+1}$, and
(vi) $\pi_k f = (\text{the Go-board of } f)_{i+1,j+2}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j+1}$ or $\pi_{k+2} f = (\text{the Go-board of } f)_{i+1,j+2}$ and $\pi_k f = (\text{the Go-board of } f)_{i,j+1}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i+1,j+1}), \frac{1}{2} \cdot ((\text{the Go-board of } f)_{i+1,j} + (\text{the Go-board of } f)_{i+2,j+1}))$ misses $\widetilde{\mathcal{L}}(f)$.

(14) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j, i. Suppose that
(i) $1 \leq j$,
(ii) $j+1 \leq \operatorname{width}$ the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq \operatorname{len}$ the Go-board of f,
(v) $\pi_{k+1} f = (\text{the Go-board of } f)_{i+1,j}$, and
(vi) $\pi_k f = (\text{the Go-board of } f)_{i,j}$ and $\pi_{k+2} f = (\text{the Go-board of } f)_{i+2,j}$ or $\pi_{k+2} f = (\text{the Go-board of } f)_{i,j}$ and $\pi_k f = (\text{the Go-board of } f)_{i+2,j}$.

Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{i,j} + (\text{the Go-board of } f)_{i+1,j+1}), \frac{1}{2} \cdot ((\text{the Go-board of } f)_{i+1,j} + (\text{the Go-board of } f)_{i+2,j+1}))$ misses $\widetilde{\mathcal{L}}(f)$.

(15) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j, i. Suppose that

(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq$ len the Go-board of f,
(v) $\pi_{k+1}f =$ (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_k f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_{k+2}f =$ (the Go-board of $f)_{i+2,j+1}$ or $\pi_{k+2}f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_k f =$ (the Go-board of $f)_{i+2,j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{i,j+1}+($the Go-board of $f)_{i+1,j+2}), \frac{1}{2}\cdot(($the Go-board of $f)_{i+1,j+1}+($the Go-board of $f)_{i+2,j+2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(16) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j, i. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $1 \leq i$,
(iv) $i+2 \leq$ len the Go-board of f,
(v) $\pi_{k+1}f =$ (the Go-board of $f)_{i+1,j+1}$, and
(vi) $\pi_k f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_{k+2}f =$ (the Go-board of $f)_{i,j+1}$ or $\pi_{k+2}f =$ (the Go-board of $f)_{i+1,j}$ and $\pi_k f =$ (the Go-board of $f)_{i,j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{i,j+1}+($the Go-board of $f)_{i+1,j+2}), \frac{1}{2}\cdot(($the Go-board of $f)_{i+1,j+1}+($the Go-board of $f)_{i+2,j+2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(17) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $\pi_{k+1}f =$ (the Go-board of $f)_{1,j+1}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{1,j+2}$ and $\pi_{k+2}f =$ (the Go-board of $f)_{2,j+1}$ or $\pi_{k+2}f =$ (the Go-board of $f)_{1,j+2}$ and $\pi_k f =$ (the Go-board of $f)_{2,j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{1,j}+($the Go-board of $f)_{1,j+1})-[1,0], \frac{1}{2}\cdot(($the Go-board of $f)_{1,j}+($the Go-board of $f)_{2,j+1}))$ misses $\widetilde{\mathcal{L}}(f)$.

(18) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $\pi_{k+1}f =$ (the Go-board of $f)_{1,j+1}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{1,j}$ and $\pi_{k+2}f =$ (the Go-board of $f)_{2,j+1}$ or $\pi_{k+2}f =$ (the Go-board of $f)_{1,j}$ and $\pi_k f =$ (the Go-board of $f)_{2,j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{1,j+1}+($the Go-board of $f)_{1,j+2})-[1,0], \frac{1}{2}\cdot(($the Go-board of $f)_{1,j+1}+($the Go-board of $f)_{2,j+2}))$ misses $\widetilde{\mathcal{L}}(f)$.

(19) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $\pi_{k+1}f =$ (the Go-board of $f)_{\text{len the Go-board of } f,\, j+1}$, and
(iv) $\pi_k f =$ (the Go-board of $f)_{\text{len the Go-board of } f,\, j+2}$ and $\pi_{k+2}f =$ (the Go-board of $f)_{\text{len the Go-board of } f-'1,j+1}$ or $\pi_{k+2}f =$ (the Go-

board of $f)_{\text{len the Go-board of } f,\, j+2}$ and $\pi_k f$ = (the Go-board of $f)_{\text{len the Go-board of } f-'1, j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{\text{len the Go-board of } f-'1,j}$ + (the Go-board of $f)_{\text{len the Go-board of } f,\, j+1}), \frac{1}{2}\cdot(($the Go-board of $f)_{\text{len the Go-board of } f,\, j}$ + (the Go-board of $f)_{\text{len the Go-board of } f,\, j+1}) + [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

(20) Given k. Suppose $1 \leq k$ and $k+2 \leq \operatorname{len} f$. Given j. Suppose that
(i) $1 \leq j$,
(ii) $j+2 \leq$ width the Go-board of f,
(iii) $\pi_{k+1} f =$ (the Go-board of $f)_{\text{len the Go-board of } f,\, j+1}$, and
(iv) $\pi_k f$ = (the Go-board of $f)_{\text{len the Go-board of } f,\, j}$ and $\pi_{k+2} f$ = (the Go-board of $f)_{\text{len the Go-board of } f-'1,j+1}$ or $\pi_{k+2} f$ = (the Go-board of $f)_{\text{len the Go-board of } f,\, j}$ and $\pi_k f$ = (the Go-board of $f)_{\text{len the Go-board of } f-'1,j+1}$.
Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{\text{len the Go-board of } f-'1,j+1}$ + (the Go-board of $f)_{\text{len the Go-board of } f,\, j+2}), \frac{1}{2}\cdot(($the Go-board of $f)_{\text{len the Go-board of } f,\, j+1}$ + (the Go-board of $f)_{\text{len the Go-board of } f,\, j+2}) + [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

In the sequel P will be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$.

We now state a number of propositions:

(21) If for every p such that $p \in P$ holds $p_{\mathbf{1}} < (($the Go-board of $f)_{1,1})_{\mathbf{1}}$, then P misses $\widetilde{\mathcal{L}}(f)$.

(22) If for every p such that $p \in P$ holds $p_{\mathbf{1}} > (($the Go-board of $f)_{\text{len the Go-board of } f,\, 1})_{\mathbf{1}}$, then P misses $\widetilde{\mathcal{L}}(f)$.

(23) If for every p such that $p \in P$ holds $p_{\mathbf{2}} < (($the Go-board of $f)_{1,1})_{\mathbf{2}}$, then P misses $\widetilde{\mathcal{L}}(f)$.

(24) If for every p such that $p \in P$ holds $p_{\mathbf{2}} > (($the Go-board of $f)_{1,\text{width the Go-board of } f})_{\mathbf{2}}$, then P misses $\widetilde{\mathcal{L}}(f)$.

(25) Given i. Suppose $1 \leq i$ and $i+2 \leq$ len the Go-board of f. Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{i,1}$ + (the Go-board of $f)_{i+1,1}) - [0,1], \frac{1}{2}\cdot(($the Go-board of $f)_{i+1,1}$ + (the Go-board of $f)_{i+2,1}) - [0,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(26) $\mathcal{L}(($the Go-board of $f)_{1,1} - [1,1], \frac{1}{2}\cdot(($the Go-board of $f)_{1,1}$ + (the Go-board of $f)_{2,1}) - [0,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(27) $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{\text{len the Go-board of } f-'1,1}$ + (the Go-board of $f)_{\text{len the Go-board of } f,\, 1}) - [0,1]$, (the Go-board of $f)_{\text{len the Go-board of } f,\, 1} + [1,-1])$ misses $\widetilde{\mathcal{L}}(f)$.

(28) Given i. Suppose $1 \leq i$ and $i+2 \leq$ len the Go-board of f. Then $\mathcal{L}(\frac{1}{2}\cdot(($the Go-board of $f)_{i,\text{width the Go-board of } f}$ + (the Go-board of $f)_{i+1,\text{width the Go-board of } f}) + [0,1], \frac{1}{2}\cdot(($the Go-board of $f)_{i+1,\text{width the Go-board of } f}$ + (the Go-board of $f)_{i+2,\text{width the Go-board of } f}) + [0,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(29) $\mathcal{L}(($the Go-board of $f)_{1,\text{width the Go-board of } f} + [-1,1], \frac{1}{2}\cdot(($the Go-board of $f)_{1,\text{width the Go-board of } f}$ + (the Go-board of $f)_{2,\text{width the Go-board of } f}) + [0,

1]) misses $\widetilde{\mathcal{L}}(f)$.

(30) $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{\text{len the Go-board of } f -' 1, \text{width the Go-board of } f} + (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f}) + [0,1], (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f} + [1,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(31) Given j. Suppose $1 \leq j$ and $j+2 \leq$ width the Go-board of f. Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{1,j} + (\text{the Go-board of } f)_{1,j+1}) - [1,0], \frac{1}{2} \cdot ((\text{the Go-board of } f)_{1,j+1} + (\text{the Go-board of } f)_{1,j+2}) - [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

(32) $\mathcal{L}((\text{the Go-board of } f)_{1,1} - [1,1], \frac{1}{2} \cdot ((\text{the Go-board of } f)_{1,1} + (\text{the Go-board of } f)_{1,2}) - [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

(33) $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{1, \text{width the Go-board of } f -' 1} + (\text{the Go-board of } f)_{1, \text{width the Go-board of } f}) - [1,0], (\text{the Go-board of } f)_{1, \text{width the Go-board of } f} + [-1,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(34) Given j. Suppose $1 \leq j$ and $j+2 \leq$ width the Go-board of f. Then $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j} + (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j+1}) + [1,0], \frac{1}{2} \cdot ((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j+1} + (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ j+2}) + [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

(35) $\mathcal{L}((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ 1} + [1,-1], \frac{1}{2} \cdot ((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ 1} + (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ 2}) + [1,0])$ misses $\widetilde{\mathcal{L}}(f)$.

(36) $\mathcal{L}(\frac{1}{2} \cdot ((\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f -' 1} + (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f}) + [1,0], (\text{the Go-board of } f)_{\text{len the Go-board of } f,\ \text{width the Go-board of } f} + [1,1])$ misses $\widetilde{\mathcal{L}}(f)$.

(37) If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then Int leftcell$(f,k)$ misses $\widetilde{\mathcal{L}}(f)$.

(38) If $1 \leq k$ and $k+1 \leq \operatorname{len} f$, then Int rightcell$(f,k)$ misses $\widetilde{\mathcal{L}}(f)$.

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Certain Facts about Families of Subsets of Many Sorted Sets

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The terminology and notation used in this paper are introduced in the following papers: [22], [23], [6], [19], [16], [24], [3], [4], [2], [7], [18], [5], [21], [20], [1], [12], [13], [14], [10], [15], [9], [17], [8], and [11].

1. Preliminaries

For simplicity we follow the rules: I, G, H will denote sets, i will be arbitrary, A, B, M will denote many sorted sets indexed by I, s_1, s_2, s_3 will denote families of subsets of I, v, w will denote subsets of I, and F will denote a many sorted function of I.

The scheme *MSFExFunc* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, a many sorted set $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a many sorted function F from $\mathcal{B}$ into $\mathcal{C}$ such that for arbitrary i if $i \in \mathcal{A}$, then there exists a function f from $\mathcal{B}(i)$ into $\mathcal{C}(i)$ such that $f = F(i)$ and for arbitrary x such that $x \in \mathcal{B}(i)$ holds $\mathcal{P}[f(x), x, i]$

provided the following condition is satisfied:

- Let i be arbitrary. Suppose $i \in \mathcal{A}$. Let x be arbitrary. If $x \in \mathcal{B}(i)$, then there exists arbitrary y such that $y \in \mathcal{C}(i)$ and $\mathcal{P}[y, x, i]$.

We now state a number of propositions:

(1) If $s_1 \neq \emptyset$, then $\mathrm{Intersect}(s_1) \subseteq \bigcup s_1$.

(2) If $G \in s_1$, then $\mathrm{Intersect}(s_1) \subseteq G$.

(3) If $\emptyset \in s_1$, then $\mathrm{Intersect}(s_1) = \emptyset$.

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(4) For every subset Z of I such that for arbitrary Z_1 such that $Z_1 \in s_1$ holds $Z \subseteq Z_1$ holds $Z \subseteq \mathrm{Intersect}(s_1)$.

(5) If $s_1 \neq \emptyset$ and for every set Z_1 such that $Z_1 \in s_1$ holds $G \subseteq Z_1$, then $G \subseteq \mathrm{Intersect}(s_1)$.

(6) If $G \in s_1$ and $G \subseteq H$, then $\mathrm{Intersect}(s_1) \subseteq H$.

(7) If $G \in s_1$ and $G \cap H = \emptyset$, then $\mathrm{Intersect}(s_1) \cap H = \emptyset$.

(8) If $s_3 = s_1 \cup s_2$, then $\mathrm{Intersect}(s_3) = \mathrm{Intersect}(s_1) \cap \mathrm{Intersect}(s_2)$.

(9) If $s_1 = \{v\}$, then $\mathrm{Intersect}(s_1) = v$.

(10) If $s_1 = \{v, w\}$, then $\mathrm{Intersect}(s_1) = v \cap w$.

(11) If $A \in B$, then A is an element of B.

(12) For every non-empty many sorted set B indexed by I such that A is an element of B holds $A \in B$.

(13) For every function f such that $i \in I$ and $f = F(i)$ holds $(\mathrm{rng}_\kappa F(\kappa))(i) = \mathrm{rng}\, f$.

(14) For every function f such that $i \in I$ and $f = F(i)$ holds $(\mathrm{dom}_\kappa F(\kappa))(i) = \mathrm{dom}\, f$.

(15) For all many sorted functions F, G of I holds $G \circ F$ is a many sorted function of I.

(16) Let A be a non-empty many sorted set indexed by I and let F be a many sorted function from A into $\emptyset_I$. Then $F = \emptyset_I$.

(17) If A is transformable to B and F is a many sorted function from A into B, then $\mathrm{dom}_\kappa F(\kappa) = A$ and $\mathrm{rng}_\kappa F(\kappa) \subseteq B$.

2. Finite Many Sorted Sets

Let us consider I. Note that every many sorted set indexed by I which is empty yielding is also locally-finite.

Let us consider I. Note that $\emptyset_I$ is empty yielding and locally-finite.

Let us consider I, A. Note that there exists a many sorted subset of A which is empty yielding and locally-finite.

Next we state the proposition

(18) If $A \subseteq B$ and B is locally-finite, then A is locally-finite.

Let us consider I and let A be a locally-finite many sorted set indexed by I. One can check that every many sorted subset of A is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Note that $A \cup B$ is locally-finite.

Let us consider I, A and let B be a locally-finite many sorted set indexed by I. Note that $A \cap B$ is locally-finite.

Let us consider I, B and let A be a locally-finite many sorted set indexed by I. Observe that $A \cap B$ is locally-finite.

Let us consider I, B and let A be a locally-finite many sorted set indexed by I. Note that $A \setminus B$ is locally-finite.

Let us consider I, F and let A be a locally-finite many sorted set indexed by I. Observe that $F \circ A$ is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Observe that $[\![A, B]\!]$ is locally-finite.

The following propositions are true:

(19) If B is non-empty and $[\![A, B]\!]$ is locally-finite, then A is locally-finite.

(20) If A is non-empty and $[\![A, B]\!]$ is locally-finite, then B is locally-finite.

(21) A is locally-finite iff 2^A is locally-finite.

Let us consider I and let M be a locally-finite many sorted set indexed by I. Observe that 2^M is locally-finite.

The following propositions are true:

(22) Let A be a non-empty many sorted set indexed by I. Suppose A is locally-finite and for every many sorted set M indexed by I such that $M \in A$ holds M is locally-finite. Then $\bigcup A$ is locally-finite.

(23) If $\bigcup A$ is locally-finite, then A is locally-finite and for every M such that $M \in A$ holds M is locally-finite.

(24) If $\mathrm{dom}_\kappa F(\kappa)$ is locally-finite, then $\mathrm{rng}_\kappa F(\kappa)$ is locally-finite.

(25) Suppose $A \subseteq \mathrm{rng}_\kappa F(\kappa)$ and for arbitrary i and for every function f such that $i \in I$ and $f = F(i)$ holds $f^{-1} A(i)$ is finite. Then A is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Observe that $\mathrm{MSFuncs}(A, B)$ is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Note that $A \dot{-} B$ is locally-finite.

In the sequel X, Y, Z denote many sorted sets indexed by I.

One can prove the following propositions:

(26) Suppose X is locally-finite and $X \subseteq [\![Y, Z]\!]$. Then there exist A, B such that A is locally-finite and $A \subseteq Y$ and B is locally-finite and $B \subseteq Z$ and $X \subseteq [\![A, B]\!]$.

(27) Suppose X is locally-finite and Z is locally-finite and $X \subseteq [\![Y, Z]\!]$. Then there exists A such that A is locally-finite and $A \subseteq Y$ and $X \subseteq [\![A, Z]\!]$.

(28) Let M be a non-empty locally-finite many sorted set indexed by I. Suppose that for all many sorted sets A, B indexed by I such that $A \in M$ and $B \in M$ holds $A \subseteq B$ or $B \subseteq A$. Then there exists a many sorted set m indexed by I such that $m \in M$ and for every many sorted set K indexed by I such that $K \in M$ holds $m \subseteq K$.

(29) Let M be a non-empty locally-finite many sorted set indexed by I. Suppose that for all many sorted sets A, B indexed by I such that $A \in M$ and $B \in M$ holds $A \subseteq B$ or $B \subseteq A$. Then there exists a many sorted set m indexed by I such that $m \in M$ and for every many sorted set K indexed by I such that $K \in M$ holds $K \subseteq m$.

(30) If Z is locally-finite and $Z \subseteq \mathrm{rng}_\kappa F(\kappa)$, then there exists Y such that $Y \subseteq \mathrm{dom}_\kappa F(\kappa)$ and Y is locally-finite and $F \circ Y = Z$.

3. A Family of Subsets of Many Sorted Sets

Let us consider I, M.

(Def.1) A many sorted subset of 2^M is said to be a subset family of M.

Let us consider I, M. Note that there exists a subset family of M which is non-empty.

Let us consider I, M. Then 2^M is a subset family of M.

Let us consider I, M. One can check that there exists a subset family of M which is empty yielding and locally-finite.

One can prove the following proposition

(31) $\emptyset_I$ is an empty yielding locally-finite subset family of M.

Let us consider I and let M be a locally-finite many sorted set indexed by I. Note that there exists a subset family of M which is non-empty and locally-finite.

We follow the rules: S_1, S_2, S_3 will be subset families of M, S_4 will be a non-empty subset family of M, and V, W will be many sorted subsets of M.

Let I be a non empty set, let M be a many sorted set indexed by I, let S_1 be a subset family of M, and let i be an element of I. Then $S_1(i)$ is a family of subsets of $M(i)$.

The following propositions are true:

(32) If $i \in I$, then $S_1(i)$ is a family of subsets of $M(i)$.

(33) If $A \in S_1$, then A is a many sorted subset of M.

(34) $S_1 \cup S_2$ is a subset family of M.

(35) $S_1 \cap S_2$ is a subset family of M.

(36) $S_1 \setminus A$ is a subset family of M.

(37) $S_1 \dot{-} S_2$ is a subset family of M.

(38) If $A \subseteq M$, then $\{A\}$ is a subset family of M.

(39) If $A \subseteq M$ and $B \subseteq M$, then $\{A, B\}$ is a subset family of M.

(40) $\bigcup S_1 \subseteq M$.

4. Intersection of a Family of Many Sorted Sets

Let us consider I, M, S_1. The functor $\bigcap S_1$ yields a many sorted set indexed by I and is defined by:

(Def.2) For arbitrary i such that $i \in I$ there exists a family Q of subsets of $M(i)$ such that $Q = S_1(i)$ and $(\bigcap S_1)(i) = \mathrm{Intersect}(Q)$.

Let us consider I, M, S_1. Then $\bigcap S_1$ is a many sorted subset of M.

We now state a number of propositions:

(41) If $S_1 = \emptyset_I$, then $\bigcap S_1 = M$.

(42) $\bigcap S_4 \subseteq \bigcup S_4$.

(43) If $A \in S_1$, then $\bigcap S_1 \subseteq A$.

(44) If $\emptyset_I \in S_1$, then $\bigcap S_1 = \emptyset_I$.

(45) Let Z, M be many sorted sets indexed by I and let S_1 be a non-empty subset family of M. Suppose that for every many sorted set Z_1 indexed by I such that $Z_1 \in S_1$ holds $Z \subseteq Z_1$. Then $Z \subseteq \bigcap S_1$.

(46) If $S_1 \subseteq S_2$, then $\bigcap S_2 \subseteq \bigcap S_1$.

(47) If $A \in S_1$ and $A \subseteq B$, then $\bigcap S_1 \subseteq B$.

(48) If $A \in S_1$ and $A \cap B = \emptyset_I$, then $\bigcap S_1 \cap B = \emptyset_I$.

(49) If $S_3 = S_1 \cup S_2$, then $\bigcap S_3 = \bigcap S_1 \cap \bigcap S_2$.

(50) If $S_1 = \{V\}$, then $\bigcap S_1 = V$.

(51) If $S_1 = \{V, W\}$, then $\bigcap S_1 = V \cap W$.

(52) If $A \in \bigcap S_1$, then for every B such that $B \in S_1$ holds $A \in B$.

(53) Let A, M be many sorted sets indexed by I and let S_1 be a non-empty subset family of M. Suppose $A \in M$ and for every many sorted set B indexed by I such that $B \in S_1$ holds $A \in B$. Then $A \in \bigcap S_1$.

Let us consider I, M. A subset family of M is additive if:

(Def.3) For all A, B such that $A \in$ it and $B \in$ it holds $A \cup B \in$ it.

A subset family of M is absolutely-additive if:

(Def.4) For every subset family F of M such that $F \subseteq$ it holds $\bigcup F \in$ it.

A subset family of M is multiplicative if:

(Def.5) For all A, B such that $A \in$ it and $B \in$ it holds $A \cap B \in$ it.

A subset family of M is absolutely-multiplicative if:

(Def.6) For every subset family F of M such that $F \subseteq$ it holds $\bigcap F \in$ it.

A subset family of M is properly-upper-bound if:

(Def.7) $M \in$ it.

A subset family of M is properly-lower-bound if:

(Def.8) $\emptyset_I \in$ it.

Let us consider I, M. Observe that there exists a subset family of M which is non-empty additive absolutely-additive multiplicative absolutely-multiplicative properly-upper-bound and properly-lower-bound.

Let us consider I, M. Then 2^M is an additive absolutely-additive multiplicative absolutely-multiplicative properly-upper-bound properly-lower-bound subset family of M.

Let us consider I, M. Note that every subset family of M which is absolutely-additive is also additive.

Let us consider I, M. Note that every subset family of M which is absolutely-multiplicative is also multiplicative.

Let us consider I, M. One can check that every subset family of M which is absolutely-multiplicative is also properly-upper-bound.

Let us consider I, M. Observe that every subset family of M which is properly-upper-bound is also non-empty.

Let us consider I, M. Note that every subset family of M which is absolutely-additive is also properly-lower-bound.

Let us consider I, M. Note that every subset family of M which is properly-lower-bound is also non-empty.

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On the Concept of the Triangulation

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The terminology and notation used in this paper have been introduced in the following articles: [22], [28], [11], [29], [31], [30], [27], [14], [2], [9], [10], [6], [19], [13], [5], [8], [25], [23], [3], [4], [12], [26], [15], [17], [18], [1], [21], [20], [24], [16], and [7].

1. Introduction

In this paper A will be a set and k, m, n will be natural numbers.

The scheme *Regr1* concerns a natural number $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every k such that $k \leq \mathcal{A}$ holds $\mathcal{P}[k]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{A}]$,
- For every k such that $k < \mathcal{A}$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k]$.

Let n be a natural number. Observe that $\mathrm{Seg}(n+1)$ is non empty.

Let X be a non empty set and let R be an order in X. Note that $\langle X, R \rangle$ is non empty.

One can prove the following proposition

(1) $\emptyset \mid^2 A = \emptyset$.

Let X be a set. Note that there exists a subset of $\mathrm{Fin}\, X$ which is non empty.

Let X be a non empty set. Note that there exists a subset of $\mathrm{Fin}\, X$ which is non empty and has non empty elements.

Let X be a non empty set and let F be a non empty subset of $\mathrm{Fin}\, X$ with non empty elements. Observe that there exists an element of F which is non empty.

A set has a non-empty element if:

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(Def.1) There exists a non empty set X such that $X \in$ it.

Let us mention that there exists a set which has a non-empty element.

Let X be a set with a non-empty element. Note that there exists an element of X which is non empty.

One can check that every set which has a non-empty element is non empty.

Let X be a non empty set. Note that there exists a subset of $\mathrm{Fin}\, X$ which has a non-empty element.

Let X be a non empty set, let F be a subset of $\mathrm{Fin}\, X$ with a non-empty element, let R be an order in X, and let A be an element of F. Then $R\,|^2\, A$ is an order in A.

The scheme *SubFinite* concerns a set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are satisfied:

- $\mathcal{B}$ is finite,
- $\mathcal{P}[\emptyset_{\mathcal{A}}]$,
- For every element x of $\mathcal{A}$ and for every subset B of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $\mathcal{P}[B]$ holds $\mathcal{P}[B \cup \{x\}]$.

We now state the proposition

(2) Let F be a non empty poset and let A be a subset of F. Suppose A is finite and $A \neq \emptyset$ and for all elements B, C of F such that $B \in A$ and $C \in A$ holds $B \leq C$ or $C \leq B$. Then there exists an element m of F such that $m \in A$ and for every element C of F such that $C \in A$ holds $m \leq C$.

Let X be a non empty set and let F be a subset of $\mathrm{Fin}\, X$ with a non-empty element. Observe that there exists an element of F which is finite and non empty.

Let A be a non empty poset and let a_1, a_2 be elements of A. We introduce $a_2 \geq a_1$ as a synonym of $a_1 \leq a_2$ We introduce $a_2 > a_1$ as a synonym of $a_1 < a_2$.

Let P be a non empty poset. Note that there exists a subset of P which is non empty and finite.

Let P be a non empty poset, let A be a non empty finite subset of P, and let x be an element of P. One can check that $\mathrm{InitSegm}(A, x)$ is finite.

The following proposition is true

(3) For all finite sets A, B such that $A \subseteq B$ and $\mathrm{card}\, A = \mathrm{card}\, B$ holds $A = B$.

Let A, B be non empty sets, let f be a function from A into B, and let x be an element of A. Then $f(x)$ is an element of B.

Let F be a non empty poset and let A be a non empty subset of F. We see that the element of A is an element of F.

Let X be a non empty set, let F be a subset of $\mathrm{Fin}\, X$ with a non-empty element, let A be a non empty element of F, and let R be an order in X. Let us assume that R linearly orders A. The functor $\mathrm{SgmX}(R, A)$ yields a finite sequence of elements of the carrier of $\langle A, R\,|^2\, A\rangle$ and is defined by the conditions (Def.2).

(Def.2) (i) $\operatorname{rng}\mathrm{SgmX}(R, A) = A$, and

(ii) for all natural numbers n, m and for all elements p, q of $\langle A, R \mid^2 A\rangle$ such that $n \in \operatorname{dom}\mathrm{SgmX}(R, A)$ and $m \in \operatorname{dom}\mathrm{SgmX}(R, A)$ and $n < m$ and $p = \pi_n \mathrm{SgmX}(R, A)$ and $q = \pi_m \mathrm{SgmX}(R, A)$ holds $p > q$.

Next we state the proposition

(4) Let X be a non empty set, and let F be a subset of $\mathrm{Fin}\,X$ with a non-empty element, and let A be a non empty element of F, and let R be an order in X, and let f be a finite sequence of elements of the carrier of $\langle X, R\rangle$. Suppose that

(i) $\operatorname{rng} f = A$, and

(ii) for all natural numbers n, m and for all elements p, q of $\langle X, R\rangle$ such that $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $n < m$ and $p = \pi_n f$ and $q = \pi_m f$ holds $p > q$.

Then $f = \mathrm{SgmX}(R, A)$.

2. Abstract Complexes

Let C be a non empty poset. The functor symplexes(C) yields a subset of Fin (the carrier of C) and is defined by:

(Def.3) symplexes(C) $= \{A : A$ ranges over elements of Fin (the carrier of C), the internal relation of C linearly orders $A\}$.

Let C be a non empty poset. Note that symplexes(C) has a non-empty element.

In the sequel C denotes a non empty poset.

Next we state three propositions:

(5) For every element x of C holds $\{x\} \in$ symplexes(C).

(6) $\emptyset \in$ symplexes(C).

(7) For arbitrary x, s such that $x \subseteq s$ and $s \in$ symplexes(C) holds $x \in$ symplexes(C).

Let us consider C. Observe that every element of symplexes(C) is finite.

One can prove the following propositions:

(8) For every non empty poset C and for every non empty element A of symplexes(C) holds SgmX(the internal relation of C, A) is one-to-one.

(9) Let C be a non empty poset and let A be a non empty element of symplexes(C). If $\overline{\overline{A}} = n$, then len SgmX(the internal relation of C, A) $=$ n.

(10) Let C be a non empty poset and let A be a non empty element of symplexes(C). If $\overline{\overline{A}} = n$, then dom SgmX(the internal relation of C, A) $= \mathrm{Seg}\, n$.

Let C be a non empty poset. One can verify that there exists an element of symplexes(C) which is non empty.

3. Triangulations

A set sequence is a many sorted set indexed by $\mathbb{N}$.

A set sequence is lower non-empty if:

(Def.4) For every n such that it(n) is non empty and for every m such that $m < n$ holds it(m) is non empty.

Let us observe that there exists a set sequence which is lower non-empty.

Let X be a set sequence. The functor FuncsSeq(X) yields a set sequence and is defined by:

(Def.5) For every natural number n holds $(\mathrm{FuncsSeq}(X))(n) = X(n)^{X(n+1)}$.

Let X be a lower non-empty set sequence and let n be a natural number. Observe that $(\mathrm{FuncsSeq}(X))(n)$ is non empty.

Let us consider n and let f be an element of $(\mathrm{Seg}(n+1))^{\mathrm{Seg}\,n}$. The functor ${}^{@}f$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.6) ${}^{@}f = f$.

The set sequence NatEmbSeq is defined by:

(Def.7) For every natural number n holds $(\mathrm{NatEmbSeq})(n) = \{f : f$ ranges over elements of $(\mathrm{Seg}(n+1))^{\mathrm{Seg}\,n}$, ${}^{@}f$ is increasing$\}$.

Let us consider n. Observe that $(\mathrm{NatEmbSeq})(n)$ is non empty.

Let n be a natural number. Note that every element of NatEmbSeq(n) is function-like and relation-like.

Let X be a set sequence.

(Def.8) A many sorted function from NatEmbSeq into FuncsSeq(X) is called a triangulation of X.

We consider triangulation structures as systems

$\langle$ a skeleton sequence, a faces assignment $\rangle$,

where the skeleton sequence is a set sequence and the faces assignment is a many sorted function from NatEmbSeq into FuncsSeq(the skeleton sequence).

Let T be a triangulation structure. We say that T is lower non-empty if and only if:

(Def.9) The skeleton sequence of T is lower non-empty.

Let us note that there exists a triangulation structure which is lower non-empty and strict.

Let T be a lower non-empty triangulation structure. Note that the skeleton sequence of T is lower non-empty.

Let S be a lower non-empty set sequence and let F be a many sorted function from NatEmbSeq into FuncsSeq(S). Note that $\langle S, F\rangle$ is lower non-empty.

4. Relationship Between Abstract Complexes and Triangulations

Let T be a triangulation structure and let n be a natural number. A symplex of T and n is an element of (the skeleton sequence of T)(n).

Let n be a natural number. A face of n is an element of (NatEmbSeq)(n).

Let T be a lower non-empty triangulation structure, let n be a natural number, let x be a symplex of T and $n+1$, and let f be a face of n. Let us assume that (the skeleton sequence of T)($n+1$) $\neq \emptyset$. The functor face(x, f) yields a symplex of T and n and is defined by:

(Def.10) For all functions F, G such that $F =$ (the faces assignment of T)(n) and $G = F(f)$ holds face(x, f) $= G(x)$.

Let C be a non empty poset. The functor Triang(C) yielding a lower non-empty strict triangulation structure is defined by the conditions (Def.11).

(Def.11) (i) (The skeleton sequence of Triang(C))(0) $= \{\emptyset\}$,

(ii) for every natural number n such that $n > 0$ holds (the skeleton sequence of Triang(C))(n) $=$ {SgmX(the internal relation of C, A) : A ranges over non empty elements of symplexes(C), $\overline{\overline{A}} = n$}, and

(iii) for every natural number n and for every face f of n and for every element s of (the skeleton sequence of Triang(C))($n+1$) such that $s \in$ (the skeleton sequence of Triang(C))($n+1$) and for every non empty element A of symplexes(C) such that SgmX(the internal relation of C, A) $= s$ holds face(s, f) $=$ SgmX(the internal relation of C, A) $\cdot f$.

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