FORMALIZED MATHEMATICS
Volume 4, Number 1, 1993
Université Catholique de Louvain

Representation Theorem for Heyting Lattices

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MML Identifier: OPENLATT.

The articles [11], [4], [5], [3], [9], [10], [7], [12], [13], [8], [1], [2], and [6] provide the notation and terminology for this paper.

One can check that every lower bound lattice which is Heyting is also implicative and every lattice which is implicative is also upper-bounded.

In the sequel T will denote a topological space and A, B, C will denote subsets of the carrier of T.

We now state two propositions:

(1) $A \cap \mathrm{Int}(A^{\mathrm{c}} \cup B) \subseteq B$.

(2) If C is open and $A \cap C \subseteq B$, then $C \subseteq \mathrm{Int}(A^{\mathrm{c}} \cup B)$.

Let us consider T. The functor Topology(T) yields a non empty family of subsets of the carrier of T and is defined as follows:

(Def.1) Topology(T) = the topology of T.

In the sequel P, Q denote elements of Topology(T).

The following proposition is true

(3) A is open iff $A \in$ Topology(T).

Let us consider T, P, Q. Then $P \cup Q$ is an element of Topology(T).

Let us consider T, P, Q. Then $P \cap Q$ is an element of Topology(T).

Let us consider T. The functor TopUnion(T) yields a binary operation on Topology(T) and is defined by:

(Def.2) (TopUnion(T))(P, Q) = $P \cup Q$.

Let us consider T. The functor TopMeet(T) yielding a binary operation on Topology(T) is defined as follows:

(Def.3) (TopMeet(T))(P, Q) = $P \cap Q$.

The following proposition is true

ISSN 0777–4028

(4) For every topological space T holds $\langle$Topology(T), TopUnion(T), TopMeet(T)$\rangle$ is a lattice.

Let us consider T. The functor OpenSetLatt(T) yields a lattice and is defined by:

(Def.4) $\text{OpenSetLatt}(T) = \langle \text{Topology}(T), \text{TopUnion}(T), \text{TopMeet}(T) \rangle$.

Next we state the proposition

(5) The carrier of OpenSetLatt(T) = Topology(T).

In the sequel p, q will denote elements of the carrier of OpenSetLatt(T).

Next we state several propositions:

(6) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(7) $p \sqsubseteq q$ iff $p \subseteq q$.

(8) For all elements p', q' of Topology(T) such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ iff $p' \subseteq q'$.

(9) OpenSetLatt(T) is implicative.

(10) OpenSetLatt(T) is lower-bounded and $\perp_{\text{OpenSetLatt}(T)} = \emptyset$.

(11) $\top_{\text{OpenSetLatt}(T)} =$ the carrier of T.

Let us consider T. Then OpenSetLatt(T) is a Heyting lattice.

For simplicity we adopt the following convention: L will denote a distributive lattice, F will denote a filter of L, a, b will denote elements of the carrier of L, x will be arbitrary, and X_1, X_2, Y, Z will denote sets.

Let us consider L. The functor PrimeFilters(L) yielding a set is defined as follows:

(Def.5) $\text{PrimeFilters}(L) = \{F : F \neq \text{the carrier of } L \ \wedge \ F \text{ is prime}\}$.

We now state the proposition

(12) $F \in \text{PrimeFilters}(L)$ iff $F \neq$ the carrier of L and F is prime.

Let us consider L. The functor StoneH(L) yielding a function is defined by:

(Def.6) $\text{dom}\,\text{StoneH}(L) =$ the carrier of L and $(\text{StoneH}(L))(a) = \{F : F \in \text{PrimeFilters}(L) \ \wedge \ a \in F\}$.

Next we state two propositions:

(13) $F \in (\text{StoneH}(L))(a)$ iff $F \in \text{PrimeFilters}(L)$ and $a \in F$.

(14) $x \in (\text{StoneH}(L))(a)$ iff there exists F such that $F = x$ and $F \neq$ the carrier of L and F is prime and $a \in F$.

Let us consider L. The functor StoneS(L) yielding a non empty set is defined as follows:

(Def.7) $\text{StoneS}(L) = \text{rng}\,\text{StoneH}(L)$.

The following propositions are true:

(15) $x \in \text{StoneS}(L)$ iff there exists a such that $x = (\text{StoneH}(L))(a)$.

(16) $(\text{StoneH}(L))(a \sqcup b) = (\text{StoneH}(L))(a) \cup (\text{StoneH}(L))(b)$.

(17) $(\text{StoneH}(L))(a \sqcap b) = (\text{StoneH}(L))(a) \cap (\text{StoneH}(L))(b)$.

Let us consider L and let us consider a. The functor Filters(a) yields a non empty family of subsets of L and is defined by:

(Def.8) Filters(a) $= \{F : a \in F\}$.

The following propositions are true:

(18) $x \in$ Filters(a) iff x is a filter of L and $a \in x$.

(19) If $x \in$ Filters(b) $\setminus$ Filters(a), then x is a filter of L and $b \in x$ and $a \notin x$.

(20) Given Z. Suppose $Z \neq \emptyset$ and $Z \subseteq$ Filters(b) $\setminus$ Filters(a) and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. Then there exists Y such that $Y \in$ Filters(b) $\setminus$ Filters(a) and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

(21) If $b \not\sqsubseteq a$, then $[b) \in$ Filters(b) $\setminus$ Filters(a).

(22) If $b \not\sqsubseteq a$, then there exists F such that $F \in$ PrimeFilters(L) and $a \notin F$ and $b \in F$.

(23) If $a \neq b$, then there exists F such that $F \in$ PrimeFilters(L).

(24) If $a \neq b$, then (StoneH(L))(a) $\neq$ (StoneH(L))(b).

(25) StoneH(L) is one-to-one.

Let us consider L and let A, B be elements of StoneS(L). Then $A \cup B$ is an element of StoneS(L).

Let us consider L and let A, B be elements of StoneS(L). Then $A \cap B$ is an element of StoneS(L).

Let us consider L. The functor SetUnion(L) yielding a binary operation on StoneS(L) is defined as follows:

(Def.9) For all elements A, B of StoneS(L) holds (SetUnion(L))(A, B) $= A \cup B$.

Let us consider L. The functor SetMeet(L) yielding a binary operation on StoneS(L) is defined by:

(Def.10) For all elements A, B of StoneS(L) holds (SetMeet(L))(A, B) $= A \cap B$.

The following proposition is true

(26) $\langle$StoneS(L), SetUnion(L), SetMeet(L)$\rangle$ is a lattice.

Let us consider L. The functor StoneLatt(L) yields a lattice and is defined by:

(Def.11) StoneLatt(L) $= \langle$StoneS(L), SetUnion(L), SetMeet(L)$\rangle$.

In the sequel p, q are elements of the carrier of StoneLatt(L).

We now state three propositions:

(27) For every L holds the carrier of StoneLatt(L) $=$ StoneS(L).

(28) $p \sqcup q = p \cup q$ and $p \sqcap q = p \cap q$.

(29) $p \sqsubseteq q$ iff $p \subseteq q$.

Let us consider L. Then StoneH(L) is a homomorphism from L to StoneLatt(L).

One can prove the following propositions:

(30) StoneH(L) is isomorphism.

(31) StoneLatt(L) is distributive.

(32) L and StoneLatt(L) are isomorphic.

Let us note that there exists a Heyting lattice which is non trivial.

In the sequel H denotes a non trivial Heyting lattice and p', q' denote elements of the carrier of H.

The following three propositions are true:

(33) $(\mathrm{StoneH}(H))(\top_H) = \mathrm{PrimeFilters}(H)$.

(34) $(\mathrm{StoneH}(H))(\bot_H) = \emptyset$.

(35) $\mathrm{StoneS}(H) \subseteq 2^{\mathrm{PrimeFilters}(H)}$.

Let us consider H. Then PrimeFilters(H) is a non empty set.

Let us consider H. The functor HTopSpace(H) yielding a strict topological space is defined as follows:

(Def.12) The carrier of HTopSpace(H) = PrimeFilters(H) and the topology of HTopSpace(H) = $\{\bigcup A : A$ ranges over subsets of StoneS(H), $\}$.

One can prove the following propositions:

(36) The carrier of OpenSetLatt(HTopSpace(H)) = $\{\bigcup A : A$ ranges over subsets of StoneS(H), $\}$.

(37) StoneS(H) $\subseteq$ the carrier of OpenSetLatt(HTopSpace(H)).

Let us consider H. Then StoneH(H) is a homomorphism from H to OpenSetLatt(HTopSpace(H)).

The following propositions are true:

(38) StoneH(H) is monomorphism.

(39) $(\mathrm{StoneH}(H))(p' \Rightarrow q') = (\mathrm{StoneH}(H))(p') \Rightarrow (\mathrm{StoneH}(H))(q')$.

(40) StoneH(H) preserves implication.

(41) StoneH(H) preserves top.

(42) StoneH(H) preserves bottom.

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Received July 14, 1993