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On Defining Functions on Trees [1]

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Piotr Rudnicki
University of Alberta
Department of Computing Science
Edmonton

Summary. The continuation of the sequence of articles on trees (see [3,5,7,4]) and on context-free grammars ([15]). We define the set of complete parse trees for a given context-free grammar. Next we define the scheme of induction for the set and the scheme of defining functions by induction on the set. For each symbol of a context-free grammar we define the terminal, the pretraversal, and the posttraversal languages. The introduced terminology is tested on the example of Peano naturals.

MML Identifier: DTCONSTR.

The terminology and notation used in this paper are introduced in the following articles: [18], [2], [21], [12], [13], [9], [1], [14], [8], [11], [16], [19], [6], [17], [10], [20], [15], [3], [5], [7], and [4].

1. Preliminaries

The following propositions are true:

(1) For every non empty set D holds every finite sequence of elements of $\mathrm{FinTrees}(D)$ is a finite sequence of elements of $\mathrm{Trees}(D)$.

(2) For arbitrary x, y and for every finite sequence p of elements of x such that $y \in \mathrm{dom}\, p$ or $y \in \mathrm{Seg}\,\mathrm{len}\, p$ holds $p(y) \in x$.

Let X be a set. Observe that every element of X^* is function-like.

Let X be a set. Note that every element of X^* is finite sequence-like.

Let D be a set and let p, q be elements of D^*. Then $p \frown q$ is an element of D^*.

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Let D be a non empty set and let t be an element of FinTrees(D). Then dom t is a finite tree.

Let D be a non empty set and let T be a set of trees decorated by D. One can verify that every finite sequence of elements of T is decorated tree yielding.

Let D be a non empty set, let F be a non empty set of trees decorated by D, and let T_1 be a non empty subset of F. We see that the element of T_1 is an element of F.

Let p be a finite sequence. Let us assume that p is decorated tree yielding. The roots of p constitute finite sequences and is defined by the conditions (Def.1).

(Def.1) (i) dom (the roots of p) = dom p, and

(ii) for every natural number i such that $i \in \mathrm{dom}\, p$ there exists a decorated tree T such that $T = p(i)$ and (the roots of p)$(i) = T(\varepsilon)$.

Let D be a non empty set, let T be a set of trees decorated by D, and let p be a finite sequence of elements of T. Then the roots of p is a finite sequence of elements of D.

One can prove the following propositions:

(3) The roots of $\varepsilon = \varepsilon$.

(4) For every decorated tree T holds the roots of $\langle T \rangle = \langle T(\varepsilon) \rangle$.

(5) Let D be a non empty set, and let F be a subset of FinTrees(D), and let p be a finite sequence of elements of F. Suppose len (the roots of p) = 1. Then there exists an element x of FinTrees(D) such that $p = \langle x \rangle$ and $x \in F$.

(6) For all decorated trees T_2, T_3 holds the roots of $\langle T_2, T_3 \rangle = \langle T_2(\varepsilon), T_3(\varepsilon) \rangle$.

Let f be a function. The functor pr1(f) yields a function and is defined by:

(Def.2) dom pr1(f) = dom f and for arbitrary x such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr1}(f)(x) = f(x)_1$.

The functor pr2(f) yielding a function is defined by:

(Def.3) dom pr2(f) = dom f and for arbitrary x such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr2}(f)(x) = f(x)_2$.

Let X, Y be sets and let f be a finite sequence of elements of $[: X, Y :]$. Then pr1(f) is a finite sequence of elements of X. Then pr2(f) is a finite sequence of elements of Y.

One can prove the following proposition

(7) $\mathrm{pr1}(\varepsilon) = \varepsilon$ and $\mathrm{pr2}(\varepsilon) = \varepsilon$.

The scheme *MonoSetSeq* concerns a function $\mathcal{A}$, a set $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

For all natural numbers k, s holds $\mathcal{A}(k) \subseteq \mathcal{A}(k+s)$

provided the parameters meet the following requirement:

- For every natural number n and for arbitrary x such that $x = \mathcal{A}(n)$ holds $\mathcal{A}(n+1) = x \cup \mathcal{F}(n, x)$.

2. The set of parse trees

Now we present two schemes. The scheme *DTConstrStrEx* concerns a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a strict tree construction structure G such that
(i) the carrier of $G = \mathcal{A}$, and
(ii) for every symbol x of G and for every finite sequence p of elements of the carrier of G holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$

for all values of the parameters.

The scheme *DTConstrStrUniq* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

Let G_1, G_2 be strict tree construction structure. Suppose that
(i) the carrier of $G_1 = \mathcal{A}$,
(ii) for every symbol x of G_1 and for every finite sequence p of elements of the carrier of G_1 holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$,
(iii) the carrier of $G_2 = \mathcal{A}$, and
(iv) for every symbol x of G_2 and for every finite sequence p of elements of the carrier of G_2 holds $x \Rightarrow p$ iff $\mathcal{P}[x,p]$.

Then $G_1 = G_2$

for all values of the parameters.

Next we state the proposition

(8) For every tree construction structure G holds (the terminals of G)$\cap$(the nonterminals of G) $= \emptyset$.

Now we present four schemes. The scheme *DTCMin* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X of FinTrees($[\!:$the carrier of $\mathcal{B}$, $\mathcal{C}:\!]$) such that
(i) $X = \bigcup \mathcal{A}$,
(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d)\rangle \in X$,
(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X such that $o \Rightarrow$ pr1(the roots of p) and for arbitrary s, v such that $s =$ pr1(the roots of p) and $v =$ pr2(the roots of p) holds $\langle o, \mathcal{G}(o,s,v)\rangle$-tree($p$) $\in X$, and
(iv) for every subset F of FinTrees($[\!:$the carrier of $\mathcal{B}$, $\mathcal{C}:\!]$) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d)\rangle \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ pr1(the roots of p) holds $\langle o, \mathcal{G}(o,$pr1(the roots of p), pr2(the roots of p))$\rangle$-tree(p) $\in F$ holds $X \subseteq F$

provided the following conditions are satisfied:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p))\rangle\text{-tree}(p) : o$ ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge \ o \Rightarrow \mathrm{pr1}(\text{the roots of } q)\}$.

The scheme *DTCSymbols* deals with a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset X_1 of FinTrees(the carrier of $\mathcal{B}$) such that
(i) $X_1 = \{t_1 : t$ ranges over elements of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]), $t \in \bigcup \mathcal{A}\}$,
(ii) for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in X_1$,
(iii) for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of X_1 such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in X_1$, and
(iv) for every subset F of FinTrees(the carrier of $\mathcal{B}$) such that for every symbol d of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in F$ and for every symbol o of $\mathcal{B}$ and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree$(p) \in F$ holds $X_1 \subseteq F$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the roots of } p))\rangle\text{-tree}(p) : o$ ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge \ o \Rightarrow \mathrm{pr1}(\text{the roots of } q)\}$.

The scheme *DTCHeight* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let n be a natural number and let d_1 be an element of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$:]). If $d_1 \in \bigcup \mathcal{A}$, then $d_1 \in \mathcal{A}(n)$ iff height dom $d_1 \leq n$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n+1) = x \cup \{\langle o, \mathcal{G}(o, \mathrm{pr1}(\text{the roots of } p), \mathrm{pr2}(\text{the

roots of p))⟩-tree(p) : o ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge\ o \Rightarrow$ pr1(the roots of q)}.

The scheme *DTCUniq* concerns a function $\mathcal{A}$, a tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

For all trees d_2, d_3 decorated by [: the carrier of $\mathcal{B}$, $\mathcal{C}$:] such that $d_2 \in \bigcup\mathcal{A}$ and $d_3 \in \bigcup\mathcal{A}$ and $(d_2)_1 = (d_3)_1$ holds $d_2 = d_3$

provided the following conditions are satisfied:

- dom $\mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d\rangle$: t ranges over symbols of $\mathcal{B}$, d ranges over elements of $\mathcal{C}$, $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \varepsilon \wedge d = \mathcal{G}(t, \varepsilon, \varepsilon)\}$,
- Let n be a natural number and let x be arbitrary. Suppose $x = \mathcal{A}(n)$. Then $\mathcal{A}(n + 1) = x \cup \{\langle o, \mathcal{G}(o,$ pr1(the roots of p), pr2(the roots of p))⟩-tree(p) : o ranges over symbols of $\mathcal{B}$, p ranges over elements of x^*, $\exists_q\ p = q \ \wedge\ o \Rightarrow$ pr1(the roots of q)}.

Let G be a tree construction structure. The functor TS(G) yields a subset of FinTrees(the carrier of G) and is defined by the conditions (Def.4).

(Def.4) (i) For every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in$ TS(G),

(ii) for every symbol o of G and for every finite sequence p of elements of TS(G) such that $o \Rightarrow$ the roots of p holds o-tree(p) $\in$ TS(G), and

(iii) for every subset F of FinTrees(the carrier of G) such that for every symbol d of G such that $d \in$ the terminals of G holds the root tree of $d \in F$ and for every symbol o of G and for every finite sequence p of elements of F such that $o \Rightarrow$ the roots of p holds o-tree(p) $\in F$ holds TS(G) $\subseteq F$.

Now we present three schemes. The scheme *DTConstrInd* concerns a tree construction structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every tree t decorated by the carrier of $\mathcal{A}$ such that $t \in$ TS($\mathcal{A}$) holds $\mathcal{P}[t]$

provided the parameters meet the following requirements:

- For every symbol s of $\mathcal{A}$ such that $s \in$ the terminals of $\mathcal{A}$ holds $\mathcal{P}$[the root tree of s],
- Let n_1 be a symbol of $\mathcal{A}$ and let t_1 be a finite sequence of elements of TS($\mathcal{A}$). Suppose $n_1 \Rightarrow$ the roots of t_1 and for every tree t decorated by the carrier of $\mathcal{A}$ such that $t \in$ rng t_1 holds $\mathcal{P}[t]$. Then $\mathcal{P}[n_1$-tree$(t_1)]$.

The scheme *DTConstrIndDef* concerns a tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from TS($\mathcal{A}$) into $\mathcal{B}$ such that

(i) for every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds f(the root tree of t) $= \mathcal{F}(t)$, and

(ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$

for all values of the parameters.

The scheme *DTConstrUniqDef* deals with a tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from TS($\mathcal{A}$) into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters satisfy the following conditions:

- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = \mathcal{C} \cdot t_1$ holds $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$,
- (i) For every symbol t of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of t) $= \mathcal{F}(t)$, and
  (ii) for every symbol n_1 of $\mathcal{A}$ and for every finite sequence t_1 of elements of TS($\mathcal{A}$) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathcal{B}$ such that $x = \mathcal{D} \cdot t_1$ holds $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, r_1, x)$.

3. An example: Peano naturals

The strict tree construction structure $\mathbb{N}_{\text{Peano}}$ is defined by the conditions (Def.5).

(Def.5) (i) The carrier of $\mathbb{N}_{\text{Peano}} = \{0, 1\}$, and
(ii) for every symbol x of $\mathbb{N}_{\text{Peano}}$ and for every finite sequence y of elements of the carrier of $\mathbb{N}_{\text{Peano}}$ holds $x \Rightarrow y$ iff $x = 1$ but $y = \langle 0 \rangle$ or $y = \langle 1 \rangle$.

4. Properties of parse trees

Let G be a tree construction structure. We say that G has terminals if and only if:

(Def.6) The terminals of $G \neq \emptyset$.

We say that G has nonterminals if and only if:

(Def.7) The nonterminals of $G \neq \emptyset$.

We say that G has useful nonterminals if and only if the condition (Def.8) is satisfied.

(Def.8) Let n_1 be a symbol of G. Suppose $n_1 \in$ the nonterminals of G. Then there exists a finite sequence p of elements of TS(G) such that $n_1 \Rightarrow$ the roots of p.

Let us note that there exists a tree construction structure which is strict and has terminals, nonterminals, and useful nonterminals.

Let G be a tree construction structure with terminals. Then the terminals of G is a non empty subset of the carrier of G. Then TS(G) is a non empty subset of FinTrees(the carrier of G).

Let G be a tree construction structure with useful nonterminals. Then TS(G) is a non empty subset of FinTrees(the carrier of G).

Let G be a tree construction structure with nonterminals. Then the nonterminals of G is a non empty subset of the carrier of G.

Let G be a tree construction structure with terminals. A terminal of G is an element of the terminals of G.

Let G be a tree construction structure with nonterminals. A nonterminal of G is an element of the nonterminals of G.

Let G be a tree construction structure with nonterminals and useful nonterminals and let n_1 be a nonterminal of G. A finite sequence of elements of TS(G) is called a subtree sequence joinable by n_1 if:

(Def.9) $n_1 \Rightarrow$ the roots of it.

Let G be a tree construction structure with terminals and let t be a terminal of G. Then the root tree of t is an element of TS(G).

Let G be a tree construction structure with nonterminals and useful nonterminals, let n_1 be a nonterminal of G, and let p be a subtree sequence joinable by n_1. Then n_1-tree(p) is an element of TS(G).

One can prove the following two propositions:

(9) Let G be a tree construction structure with terminals, and let t_2 be an element of TS(G), and let s be a terminal of G. If $t_2(\varepsilon) = s$, then $t_2 =$ the root tree of s.

(10) Let G be a tree construction structure with terminals and nonterminals, and let t_2 be an element of TS(G), and let n_1 be a nonterminal of G. Suppose $t_2(\varepsilon) = n_1$. Then there exists a finite sequence t_1 of elements of TS(G) such that $t_2 = n_1$-tree(t_1) and $n_1 \Rightarrow$ the roots of t_1.

5. The Example Continued

$\mathbb{N}_{\mathrm{Peano}}$ is a strict tree construction structure with terminals, nonterminals, and useful nonterminals.

Let n_1 be a nonterminal of $\mathbb{N}_{\mathrm{Peano}}$ and let t be an element of TS($\mathbb{N}_{\mathrm{Peano}}$). Then n_1-tree(t) is an element of TS($\mathbb{N}_{\mathrm{Peano}}$).

Let x be a finite sequence of elements of $\mathbb{N}$. Let us assume that $x \neq \varepsilon$. The functor $(x)(_1{+}1)$ yielding a natural number is defined as follows:

(Def.10) There exists a natural number n such that $(x)(_1{+}1) = n+1$ and $x(1) = n$.

The function $\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N}$ from $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ into $\mathbb{N}$ is defined by the conditions (Def.11).

(Def.11) (i) For every symbol t of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in$ the terminals of $\mathbb{N}_{\mathrm{Peano}}$ holds $(\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})$(the root tree of t) $= 0$, and

(ii) for every symbol n_1 of $\mathbb{N}_{\mathrm{Peano}}$ and for every finite sequence t_1 of elements of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $\mathbb{N}$ such that $x = (\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N}) \cdot t_1$ holds $(\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})(n_1\text{-tree}(t_1)) = (x)(_1{+}1)$.

Let x be an element of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$. The functor $\mathrm{succ}(x)$ yielding an element of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ is defined as follows:

(Def.12) $\mathrm{succ}(x) = 1\text{-tree}(\langle x \rangle)$.

The function $\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}}$ from $\mathbb{N}$ into $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ is defined by the conditions (Def.13).

(Def.13) (i) $(\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(0) =$ the root tree of 0, and

(ii) for every natural number n and for every element x of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ such that $x = (\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n)$ holds $(\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n+1) = \mathrm{succ}(x)$.

One can prove the following propositions:

(11) For every element p_1 of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds $p_1 = (\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})((\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})(p_1))$.

(12) For every natural number n holds $n = (\mathbb{N}_{\mathrm{Peano}} \to \mathbb{N})((\mathbb{N} \to \mathbb{N}_{\mathrm{Peano}})(n))$.

6. Tree traversals and terminal language

Let D be a set and let F be a finite sequence of elements of D^*. The functor $\mathrm{Flat}(F)$ yields an element of D^* and is defined as follows:

(Def.14) There exists a binary operation g on D^* such that for all elements p, q of D^* holds $g(p, q) = p \frown q$ and $\mathrm{Flat}(F) = g \odot F$.

Next we state the proposition

(13) For every set D and for every element d of D^* holds $\mathrm{Flat}(\langle d \rangle) = d$.

Let G be a tree construction structure and let t_2 be a tree decorated by the carrier of G. Let us assume that $t_2 \in \mathrm{TS}(G)$. The terminals of t_2 is a finite sequence of elements of the terminals of G and is defined by the condition (Def.15).

(Def.15) There exists a function f from $\mathrm{TS}(G)$ into (the terminals of G)* such that

(i) the terminals of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the terminals of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \text{Flat}(x)$.

The pretraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def.16).

(Def.16) There exists a function f from TS(G) into (the carrier of G)* such that

(i) the pretraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \langle n_1 \rangle \frown \text{Flat}(x)$.

The posttraversal string of t_2 is a finite sequence of elements of the carrier of G and is defined by the condition (Def.17).

(Def.17) There exists a function f from TS(G) into (the carrier of G)* such that

(i) the posttraversal string of $t_2 = f(t_2)$,

(ii) for every symbol t of G such that $t \in$ the terminals of G holds f(the root tree of t) $= \langle t \rangle$, and

(iii) for every symbol n_1 of G and for every finite sequence t_1 of elements of TS(G) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of (the carrier of G)* such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \text{Flat}(x) \frown \langle n_1 \rangle$.

Let G be a tree construction structure with nonterminals and let n_1 be a symbol of G. The language derivable from n_1 is a subset of (the terminals of G)* and is defined by the condition (Def.18).

(Def.18) The language derivable from n_1 = {the terminals of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1$}.

The language of pretraversals derivable from n_1 is a subset of (the carrier of G)* and is defined by the condition (Def.19).

(Def.19) The language of pretraversals derivable from n_1 = {the pretraversal string of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1$}.

The language of posttraversals derivable from n_1 is a subset of (the carrier of G)* and is defined by the condition (Def.20).

(Def.20) The language of posttraversals derivable from n_1 = {the posttraversal string of t_2: t_2 ranges over elements of FinTrees(the carrier of G), $t_2 \in$ TS(G) $\wedge$ $t_2(\varepsilon) = n_1$}.

One can prove the following propositions:

(14) For every tree t decorated by the carrier of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in \mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the terminals of $t = \langle 0 \rangle$.

(15) For every symbol n_1 of $\mathbb{N}_{\mathrm{Peano}}$ holds the language derivable from $n_1 = \{\langle 0 \rangle\}$.

(16) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the pretraversal string of $t = (\mathrm{height}\,\mathrm{dom}\, t \longmapsto 1) \frown \langle 0 \rangle$.

(17) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of pretraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of pretraversals derivable from $n_1 = \{(n \longmapsto 1) \frown \langle 0 \rangle : n$ ranges over natural numbers, $n \neq 0\}$.

(18) For every element t of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the posttraversal string of $t = \langle 0 \rangle \frown (\mathrm{height}\,\mathrm{dom}\, t \longmapsto 1)$.

(19) Let n_1 be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle \frown (n \longmapsto 1) : n$ ranges over natural numbers, $n \neq 0\}$.

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