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Sets and Functions of Trees and Joining Operations of Trees

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Summary. In the article we deal with sets of trees and functions yielding trees. So, we introduce the sets of all trees, all finite trees and of all trees decorated by elements from some set. Next, the functions and the finite sequences yielding (finite, decorated) trees are introduced. There are shown some convenient but technical lemmas and clusters concerning with those concepts. In the fourth section we deal with trees decorated by Cartesian product and we introduce the concept of a tree called a substitution of structure of some finite tree. Finally, we introduce the operations of joining trees, i.e. for the finite sequence of trees we define the tree which is made by joining the trees from the sequence by common root. For one and two trees there are introduced the same operations.

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The notation and terminology used here are introduced in the following papers: [17], [13], [3], [9], [18], [6], [11], [16], [15], [19], [1], [10], [14], [7], [8], [4], [5], [12], and [2].

1. Finite sets

For simplicity we adopt the following rules: x, y will be arbitrary, i, n will be natural numbers, p, q will be finite sequences, X, Y will be sets, and f will be a function. Let X be a set. Observe that there exists a finite subset of X and every finite sequence-like function is finite.

Let X be a non-empty set. One can check that there exists a finite non-empty subset of X.

Let X be a finite set. Observe that every subset of X is finite.

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Let us consider x. Then $\{x\}$ is a finite non-empty set. Let us consider y. Then $\{x, y\}$ is a finite non-empty set. Let us consider n. Then $\operatorname{Seg} n$ is a finite set of natural numbers. Then the elementary tree of n is a finite tree.

2. Sets of trees

We now define five new constructions. The non-empty set Trees is defined by:

(Def.1) Trees is the set of all trees.

The non-empty subset FinTrees of Trees is defined as follows:

(Def.2) FinTrees is the set of all finite trees.

A set is constituted of trees if:

(Def.3) for every x such that $x \in$ it holds x is a tree.

A set is constituted of finite trees if:

(Def.4) for every x such that $x \in$ it holds x is a finite tree.

A set is constituted of decorated trees if:

(Def.5) for every x such that $x \in$ it holds x is a decorated tree.

Next we state a number of propositions:

(1) X is constituted of trees if and only if $X \subseteq$ Trees.

(2) X is constituted of finite trees if and only if $X \subseteq$ FinTrees.

(3) X is constituted of trees and Y is constituted of trees if and only if $X \cup Y$ is constituted of trees.

(4) If X is constituted of trees and Y is constituted of trees, then $X \dot{-} Y$ is constituted of trees.

(5) If X is constituted of trees, then $X \cap Y$ is constituted of trees and $Y \cap X$ is constituted of trees and $X \setminus Y$ is constituted of trees.

(6) X is constituted of finite trees and Y is constituted of finite trees if and only if $X \cup Y$ is constituted of finite trees.

(7) If X is constituted of finite trees and Y is constituted of finite trees, then $X \dot{-} Y$ is constituted of finite trees.

(8) If X is constituted of finite trees, then $X \cap Y$ is constituted of finite trees and $Y \cap X$ is constituted of finite trees and $X \setminus Y$ is constituted of finite trees.

(9) X is constituted of decorated trees and Y is constituted of decorated trees if and only if $X \cup Y$ is constituted of decorated trees.

(10) If X is constituted of decorated trees and Y is constituted of decorated trees, then $X \dot{-} Y$ is constituted of decorated trees.

(11) If X is constituted of decorated trees, then $X \cap Y$ is constituted of decorated trees and $Y \cap X$ is constituted of decorated trees and $X \setminus Y$ is constituted of decorated trees.

(12) $\emptyset$ is constituted of trees, constituted of finite trees and constituted of decorated trees.

(13) $\{x\}$ is constituted of trees if and only if x is a tree.

(14) $\{x\}$ is constituted of finite trees if and only if x is a finite tree.

(15) $\{x\}$ is constituted of decorated trees if and only if x is a decorated tree.

(16) $\{x, y\}$ is constituted of trees if and only if x is a tree and y is a tree.

(17) $\{x, y\}$ is constituted of finite trees if and only if x is a finite tree and y is a finite tree.

(18) $\{x, y\}$ is constituted of decorated trees if and only if x is a decorated tree and y is a decorated tree.

(19) If X is constituted of trees and $Y \subseteq X$, then Y is constituted of trees.

(20) If X is constituted of finite trees and $Y \subseteq X$, then Y is constituted of finite trees.

(21) If X is constituted of decorated trees and $Y \subseteq X$, then Y is constituted of decorated trees.

We now define three new constructions. One can verify the following observations:

* there exists a finite constituted of trees constituted of finite trees non-empty set,
* there exists a finite constituted of decorated trees non-empty set, and
* every constituted of finite trees set is constituted of trees.

Let X be a constituted of trees set. One can check that every subset of X is constituted of trees.

Let X be a constituted of finite trees set. One can check that every subset of X is constituted of finite trees.

Let X be a constituted of decorated trees set. Note that every subset of X is constituted of decorated trees.

Let D be a constituted of trees non-empty set. We see that the element of D is a tree. Let D be a constituted of finite trees non-empty set. We see that the element of D is a finite tree. Let D be a constituted of decorated trees non-empty set. We see that the element of D is a decorated tree. Let us note that it makes sense to consider the following constant. Then Trees is a constituted of trees non-empty set. Let us observe that there exists a constituted of finite trees non-empty subset of Trees.

Let us note that it makes sense to consider the following constant. Then FinTrees is a constituted of finite trees non-empty subset of Trees. Let D be a non-empty set. A set is called a set of trees decorated by D if:

(Def.6) for every x such that $x \in$ it holds x is a tree decorated by D.

Let D be a non-empty set. Note that every set of trees decorated by D is constituted of decorated trees.

Let D be a non-empty set. Note that there exists a set of trees decorated by D which is finite and non-empty.

Let D be a non-empty set, and let E be a non-empty set of trees decorated by D. We see that the element of E is a tree decorated by D. Let T be a tree, and let D be a non-empty set. Then D^T is a non-empty set of trees decorated by D. We see that the function from T into D is a tree decorated by D. Let D be a non-empty set. The functor $\mathrm{Trees}(D)$ yielding a non-empty set of trees decorated by D is defined as follows:

(Def.7) for every tree T decorated by D holds $T \in \mathrm{Trees}(D)$.

Let D be a non-empty set. The functor $\mathrm{FinTrees}(D)$ yielding a non-empty set of trees decorated by D is defined as follows:

(Def.8) for every tree T decorated by D holds $\mathrm{dom}\, T$ is finite if and only if $T \in \mathrm{FinTrees}(D)$.

The following proposition is true

(22) For every non-empty set D holds $\mathrm{FinTrees}(D) \subseteq \mathrm{Trees}(D)$.

3. Functions yielding trees

We now define three new attributes. A function is tree yielding if:

(Def.9) rng it is constituted of trees.

A function is finite tree yielding if:

(Def.10) rng it is constituted of finite trees.

A function is decorated tree yielding if:

(Def.11) rng it is constituted of decorated trees.

One can prove the following propositions:

(23) ε is tree yielding, finite tree yielding and decorated tree yielding.

(24) f is tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a tree.

(25) f is finite tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a finite tree.

(26) f is decorated tree yielding if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a decorated tree.

(27) p is tree yielding and q is tree yielding if and only if $p ^\frown q$ is tree yielding.

(28) p is finite tree yielding and q is finite tree yielding if and only if $p ^\frown q$ is finite tree yielding.

(29) p is decorated tree yielding and q is decorated tree yielding if and only if $p ^\frown q$ is decorated tree yielding.

(30) $\langle x \rangle$ is tree yielding if and only if x is a tree.

(31) $\langle x \rangle$ is finite tree yielding if and only if x is a finite tree.

(32) $\langle x \rangle$ is decorated tree yielding if and only if x is a decorated tree.

(33) $\langle x, y \rangle$ is tree yielding if and only if x is a tree and y is a tree.

(34) $\langle x, y\rangle$ is finite tree yielding if and only if x is a finite tree and y is a finite tree.

(35) $\langle x, y\rangle$ is decorated tree yielding if and only if x is a decorated tree and y is a decorated tree.

(36) If $i \neq 0$, then $i \longmapsto x$ is tree yielding if and only if x is a tree.

(37) If $i \neq 0$, then $i \longmapsto x$ is finite tree yielding if and only if x is a finite tree.

(38) If $i \neq 0$, then $i \longmapsto x$ is decorated tree yielding if and only if x is a decorated tree.

One can verify the following observations:

* there exists a tree yielding finite tree yielding non-empty finite sequence,
* there exists a decorated tree yielding non-empty finite sequence,
* there exists a tree yielding finite tree yielding non-empty function,
* there exists a decorated tree yielding non-empty function, and
* every function which is finite tree yielding is also tree yielding.

Let D be a constituted of trees non-empty set. Observe that every finite sequence of elements of D is tree yielding.

Let p, q be tree yielding finite sequences. Then $p \frown q$ is a tree yielding finite sequence. Let D be a constituted of finite trees non-empty set. Note that every finite sequence of elements of D is finite tree yielding.

Let p, q be finite tree yielding finite sequences. Then $p \frown q$ is a finite tree yielding finite sequence. Let D be a constituted of decorated trees non-empty set. One can check that every finite sequence of elements of D is decorated tree yielding.

Let p, q be decorated tree yielding finite sequences. Then $p \frown q$ is a decorated tree yielding finite sequence. Let T be a tree. Then $\langle T\rangle$ is a tree yielding non-empty finite sequence. Let S be a tree. Then $\langle T, S\rangle$ is a tree yielding non-empty finite sequence. Let n be a natural number, and let T be a tree. Then $n \longmapsto T$ is a tree yielding finite sequence. Let T be a finite tree. Then $\langle T\rangle$ is a finite tree yielding tree yielding non-empty finite sequence. Let S be a finite tree. Then $\langle T, S\rangle$ is a finite tree yielding non-empty tree yielding finite sequence. Let n be a natural number, and let T be a finite tree. Then $n \longmapsto T$ is a finite tree yielding finite sequence. Let T be a decorated tree. Then $\langle T\rangle$ is a decorated tree yielding non-empty finite sequence. Let S be a decorated tree. Then $\langle T, S\rangle$ is a decorated tree yielding non-empty finite sequence. Let n be a natural number, and let T be a decorated tree. Then $n \longmapsto T$ is a decorated tree yielding finite sequence.

The following proposition is true

(39) For every decorated tree yielding function f holds $\mathrm{dom}(\mathrm{dom}_\kappa f(\kappa)) = \mathrm{dom}\, f$ and $\mathrm{dom}_\kappa f(\kappa)$ is tree yielding.

Let p be a decorated tree yielding finite sequence. Then $\mathrm{dom}_\kappa p(\kappa)$ is a tree yielding finite sequence.

One can prove the following proposition

(40) For every decorated tree yielding finite sequence p holds $\operatorname{len}(\operatorname{dom}_\kappa p(\kappa)) = \operatorname{len} p$.

4. Trees decorated by Cartesian product and structure of substitution

We now define four new constructions. Let D, E be non-empty sets. A tree decorated by D and E is a tree decorated by $[\!:D, E:\!]$.

A set of trees decorated by D and E is a set of trees decorated by $[\!:D, E:\!]$.

Let T_1, T_2 be decorated trees. Then $\langle T_1, T_2\rangle$ is a decorated tree. Let D_1, D_2 be non-empty sets, and let T_1 be a tree decorated by D_1, and let T_2 be a tree decorated by D_2. Then $\langle T_1, T_2\rangle$ is a tree decorated by D_1 and D_2. Let D, E be non-empty sets, and let T be a tree decorated by D, and let f be a function from D into E. Then $f \cdot T$ is a tree decorated by E. Let D_1, D_2 be non-empty sets. Then $\pi_1(D_1 \times D_2)$ is a function from $[\!:D_1, D_2:\!]$ into D_1. Then $\pi_2(D_1 \times D_2)$ is a function from $[\!:D_1, D_2:\!]$ into D_2. Let D_1, D_2 be non-empty sets, and let T be a tree decorated by D_1 and D_2. The functor $T_{\mathbf{1}}$ yielding a tree decorated by D_1 is defined by:

(Def.12) $T_{\mathbf{1}} = \pi_1(D_1 \times D_2) \cdot T$.

The functor $T_{\mathbf{2}}$ yielding a tree decorated by D_2 is defined by:

(Def.13) $T_{\mathbf{2}} = \pi_2(D_1 \times D_2) \cdot T$.

The following propositions are true:

(41) For all non-empty sets D_1, D_2 and for every tree T decorated by D_1 and D_2 and for every element t of $\operatorname{dom} T$ holds $T(t)_{\mathbf{1}} = T_{\mathbf{1}}(t)$ and $T_{\mathbf{2}}(t) = T(t)_{\mathbf{2}}$.

(42) For all non-empty sets D_1, D_2 and for every tree T decorated by D_1 and D_2 holds $\langle T_{\mathbf{1}}, T_{\mathbf{2}}\rangle = T$.

We now define two new modes. Let T be a finite tree. Then $\operatorname{Leaves} T$ is a finite non-empty subset of T. Let T be a tree, and let S be a non-empty subset of T. We see that the element of S is an element of T. Let T be a finite tree. We see that the leaf of T is an element of $\operatorname{Leaves} T$. Let T be a finite tree. A tree is called a substitution of structure of T if:

(Def.14) for every element t of it holds $t \in T$ or there exists a leaf l of T such that $l \prec t$.

Let T be a finite tree, and let t be a leaf of T, and let S be a tree. Then $T(t/S)$ is a substitution of structure of T. Let T be a finite tree. Observe that there exists a finite substitution of structure of T.

Let us consider n. A substitution of structure of n is a substitution of structure of the elementary tree of n.

We now state two propositions:

(43) Every tree is a substitution of structure of 0.

(44) For all trees T_1, T_2 such that T_1-level(1) $\subseteq T_2$-level(1) and for every n such that $\langle n\rangle \in T_1$ holds $T_1 \restriction \langle n\rangle = T_2 \restriction \langle n\rangle$ holds $T_1 \subseteq T_2$.

5. Joining of trees

Next we state several propositions:

(45) For all trees T, T' and for every element p of T holds $p \in T(p/T')$.

(46) For all trees T, T' and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in \mathrm{Leaves}\, T$ holds $T \subseteq T(p/T')$.

(47) For all decorated trees T, T' and for every element p of $\mathrm{dom}\, T$ holds $T(p/T')(p) = T'(\varepsilon)$.

(48) For all decorated trees T, T' and for all elements p, q of $\mathrm{dom}\, T$ such that $p \not\preceq q$ holds $T(p/T')(q) = T(q)$.

(49) For all decorated trees T, T' and for every element p of $\mathrm{dom}\, T$ and for every element q of $\mathrm{dom}\, T'$ holds $T(p/T')(p \frown q) = T'(q)$.

Let T_1, T_2 be trees. Then $T_1 \cup T_2$ is a tree.

One can prove the following proposition

(50) Let T_1, T_2 be trees. Let p be an element of $T_1 \cup T_2$. Then

(i) if $p \in T_1$ and $p \in T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p \cup T_2 \restriction p$,

(ii) if $p \notin T_1$, then $(T_1 \cup T_2) \restriction p = T_2 \restriction p$,

(iii) if $p \notin T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p$.

We now define three new functors. Let us consider p satisfying the condition: p is tree yielding. The functor $\overbrace{p}$ yielding a tree is defined as follows:

(Def.15) $x \in \overbrace{p}$ if and only if $x = \varepsilon$ or there exist n, q such that $n < \mathrm{len}\, p$ and $q \in p(n+1)$ and $x = \langle n\rangle \frown q$.

Let T be a tree. The functor $\overbrace{T}$ yields a tree and is defined by:

(Def.16) $\overbrace{T} = \overbrace{\langle T\rangle}$.

Let T_1, T_2 be trees. The functor $\overbrace{T_1, T_2}$ yields a tree and is defined by:

(Def.17) $\overbrace{T_1, T_2} = \overbrace{\langle T_1, T_2\rangle}$.

One can prove the following propositions:

(51) If p is tree yielding, then $\langle n\rangle \frown q \in \overbrace{p}$ if and only if $n < \mathrm{len}\, p$ and $q \in p(n+1)$.

(52) If p is tree yielding, then $\overbrace{p}$-level(1) $= \{\langle n\rangle : n < \mathrm{len}\, p\}$ and for every n such that $n < \mathrm{len}\, p$ holds $\overbrace{p} \restriction \langle n\rangle = p(n+1)$.

(53) For all tree yielding finite sequences p, q such that $\overbrace{p} = \overbrace{q}$ holds $p = q$.

(54) For all tree yielding finite sequences p_1, p_2 and for every tree T holds $p \in T$ if and only if $\langle \operatorname{len} p_1 \rangle \frown p \in \overbrace{p_1 \frown \langle T \rangle \frown p_2}$.

(55) $\overbrace{\varepsilon}$ = the elementary tree of 0.

(56) If p is tree yielding, then the elementary tree of $\operatorname{len} p \subseteq \overbrace{p}$.

(57) The elementary tree of $i = \overbrace{i \longmapsto \text{the elementary tree of } 0}$.

(58) For every tree T and for every tree yielding finite sequence p holds $\overbrace{p \frown \langle T \rangle} = (\overbrace{p} \cup \text{the elementary tree of } \operatorname{len} p + 1)(\langle \operatorname{len} p \rangle / T)$.

(59) For every tree yielding finite sequence p holds $\overbrace{p \frown \langle \text{the elementary tree of } 0 \rangle} = \overbrace{p} \cup$ the elementary tree of $\operatorname{len} p + 1$.

(60) For all tree yielding finite sequences p, q and for all trees T_1, T_2 holds $\overbrace{p \frown \langle T_1 \rangle \frown q} = \overbrace{p \frown \langle T_2 \rangle \frown q}(\langle \operatorname{len} p \rangle / T_1)$.

(61) For every tree T holds $\overbrace{T}$ = (the elementary tree of 1)$(\langle 0 \rangle / T)$.

(62) For all trees T_1, T_2 holds $\overbrace{T_1, T_2}$ = (the elementary tree of 2)$(\langle 0 \rangle / T_1)(\langle 1 \rangle / T_2)$.

Let p be a finite tree yielding finite sequence. Then $\overbrace{p}$ is a finite tree. Let T be a finite tree. Then $\overbrace{T}$ is a finite tree. Let T_1, T_2 be finite trees. Then $\overbrace{T_1, T_2}$ is a finite tree.

One can prove the following propositions:

(63) For every tree T and for an arbitrary x holds $x \in \overbrace{T}$ if and only if $x = \varepsilon$ or there exists p such that $p \in T$ and $x = \langle 0 \rangle \frown p$.

(64) For every tree T and for every finite sequence p holds $p \in T$ if and only if $\langle 0 \rangle \frown p \in \overbrace{T}$.

(65) For every tree T holds the elementary tree of $1 \subseteq \overbrace{T}$.

(66) For all trees T_1, T_2 such that $T_1 \subseteq T_2$ holds $\overbrace{T_1} \subseteq \overbrace{T_2}$.

(67) For all trees T_1, T_2 such that $\overbrace{T_1} = \overbrace{T_2}$ holds $T_1 = T_2$.

(68) For every tree T holds $\overbrace{T} {\restriction} \langle 0 \rangle = T$.

(69) For all trees T_1, T_2 holds $\overbrace{T_1}(\langle 0 \rangle / T_2) = \overbrace{T_2}$.

(70) $\overbrace{\text{the elementary tree of } 0}$ = the elementary tree of 1.

(71) For all trees T_1, T_2 and for an arbitrary x holds $x \in \overbrace{T_1, T_2}$ if and only if $x = \varepsilon$ or there exists p such that $p \in T_1$ and $x = \langle 0 \rangle \frown p$ or $p \in T_2$ and $x = \langle 1 \rangle \frown p$.

(72) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_1$ if and only if $\langle 0 \rangle \frown p \in \overbrace{T_1, T_2}$.

(73) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_2$ if and only if $\langle 1 \rangle \frown p \in \overbrace{T_1, T_2}$.

(74) For all trees T_1, T_2 holds the elementary tree of $2 \subseteq \overbrace{T_1, T_2}$.

(75) For all trees T_1, T_2, W_1, W_2 such that $T_1 \subseteq W_1$ and $T_2 \subseteq W_2$ holds $\overbrace{T_1, T_2} \subseteq \overbrace{W_1, W_2}$.

(76) For all trees T_1, T_2, W_1, W_2 such that $\overbrace{T_1, T_2} = \overbrace{W_1, W_2}$ holds $T_1 = W_1$ and $T_2 = W_2$.

(77) For all trees T_1, T_2 holds $\overbrace{T_1, T_2} \restriction \langle 0 \rangle = T_1$ and $\overbrace{T_1, T_2} \restriction \langle 1 \rangle = T_2$.

(78) For all trees T, T_1, T_2 holds $\overbrace{T_1, T_2}(\langle 0 \rangle / T) = \overbrace{T, T_2}$ and $\overbrace{T_1, T_2}(\langle 1 \rangle / T) = \overbrace{T_1, T}$.

(79) $\overbrace{\text{the elementary tree of } 0, \text{the elementary tree of } 0} =$ the elementary tree of 2.

In the sequel w is a finite tree yielding finite sequence. One can prove the following propositions:

(80) For every w if for every finite tree t such that $t \in \operatorname{rng} w$ holds height $t \leq n$, then height $\widehat{w} \leq n + 1$.

(81) For every finite tree t such that $t \in \operatorname{rng} w$ holds height $\widehat{w} >$ height t.

(82) For every finite tree t such that $t \in \operatorname{rng} w$ and for every finite tree t' such that $t' \in \operatorname{rng} w$ holds height $t' \leq$ height t holds height $\widehat{w} =$ height $t + 1$.

(83) For every finite tree T holds height $\widehat{T} =$ height $T + 1$.

(84) For all finite trees T_1, T_2 holds height $\overbrace{T_1, T_2} = \max(\text{height } T_1, \text{height } T_2) + 1$.

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