FORMALIZED MATHEMATICS
Volume 3, Number 2, 1992
Université Catholique de Louvain

Coherent Space

Jarosław Kotowicz
Warsaw University
Białystok

Konrad Raczkowski
Warsaw University
Białystok

Summary. Coherent Space web of coherent space and two categories: category of coherent spaces and category of tolerances on same fixed set.

MML Identifier: COH_SP.

The articles [8], [10], [11], [1], [5], [9], [6], [2], [7], [4], and [3] provide the notation and terminology for this paper. We follow a convention: x, y will be arbitrary and a, b, X, A will be sets. Let F be a non-empty set of functions. We see that the element of F is a function.

1. Coherent Space and Web of Coherent Space

We now define three new constructions. A set is down-closed if:

(Def.1) for all a, b such that $a \in$ it and $b \subseteq a$ holds $b \in$ it.

A set is binary complete if:

(Def.2) for every A such that $A \subseteq$ it and for all a, b such that $a \in A$ and $b \in A$ holds $a \cup b \in$ it holds $\bigcup A \in$ it.

Let us observe that there exists a down-closed binary complete non-empty set.

A coherent space is a down-closed binary complete non-empty set.

In the sequel C, D are coherent spaces. Next we state four propositions:

(1) $\emptyset \in C$.

(2) 2^X is a coherent space.

(3) $\{\emptyset\}$ is a coherent space.

(4) If $x \in \bigcup C$, then $\{x\} \in C$.

Let C be a coherent space. The functor Web(C) yields a tolerance of $\bigcup C$ and is defined by:

ISSN 0777–4028

(Def.3) for all x, y holds $\langle x, y\rangle \in \mathrm{Web}(C)$ if and only if there exists X such that $X \in C$ and $x \in X$ and $y \in X$.

In the sequel T is a tolerance of $\bigcup C$. One can prove the following propositions:

(5) $T = \mathrm{Web}(C)$ if and only if for all x, y holds $\langle x, y\rangle \in T$ if and only if $\{x, y\} \in C$.

(6) $a \in C$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\{x, y\} \in C$.

(7) $a \in C$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in \mathrm{Web}(C)$.

(8) If for all x, y such that $x \in a$ and $y \in a$ holds $\{x, y\} \in C$, then $a \subseteq \bigcup C$.

(9) If $\mathrm{Web}(C) = \mathrm{Web}(D)$, then $C = D$.

(10) If $\bigcup C \in C$, then $C = 2^{\bigcup C}$.

(11) If $C = 2^{\bigcup C}$, then $\mathrm{Web}(C) = \nabla_{\bigcup C}$.

Let X be a set, and let E be a tolerance of X. The functor $\mathrm{CohSp}(E)$ yielding a coherent space is defined by:

(Def.4) for every a holds $a \in \mathrm{CohSp}(E)$ if and only if for all x, y such that $x \in a$ and $y \in a$ holds $\langle x, y\rangle \in E$.

In the sequel E denotes a tolerance of X. Next we state four propositions:

(12) $\mathrm{Web}(\mathrm{CohSp}(E)) = E$.

(13) $\mathrm{CohSp}(\mathrm{Web}(C)) = C$.

(14) $a \in \mathrm{CohSp}(E)$ if and only if a is a set of mutually elements w.r.t. E.

(15) $\mathrm{CohSp}(E) = \mathrm{TolSets}\, E$.

2. Category of Coherent Spaces

Let us consider X. The functor $\mathrm{CSp}(X)$ yielding a non-empty set is defined as follows:

(Def.5) $\mathrm{CSp}(X) = \{x : x \text{ is a coherent space}\}$, where x ranges over subsets of 2^X.

In the sequel C, C_1, C_2 denote elements of $\mathrm{CSp}(X)$. Let us consider X, C. The functor ${}^{@}C$ yielding a coherent space is defined as follows:

(Def.6) ${}^{@}C = C$.

The following proposition is true

(16) If $\{x, y\} \in C$, then $x \in \bigcup C$ and $y \in \bigcup C$.

Let us consider X. The functor $\mathrm{Funcs}_{\mathrm{C}} X$ yielding a non-empty set of functions is defined by:

(Def.7) $\mathrm{Funcs}_{\mathrm{C}} X = \bigcup\{(\bigcup y)^{\bigcup x}\}$, where x ranges over elements of $\mathrm{CSp}(X)$, and y ranges over elements of $\mathrm{CSp}(X)$.

In the sequel g is an element of $\mathrm{Funcs}_{\mathrm{C}}X$. The following proposition is true

(17) $x \in \mathrm{Funcs}_{\mathrm{C}}X$ if and only if there exist C_1, C_2 such that if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and also x is a function from $\bigcup C_1$ into $\bigcup C_2$.

Let us consider X. The functor $\mathrm{Maps}_{\mathrm{C}}X$ yielding a non-empty set is defined by:

(Def.8) $\mathrm{Maps}_{\mathrm{C}}X = \{\langle\langle C, C_3\rangle, f\rangle : (\bigcup C_3 = \emptyset \Rightarrow \bigcup C = \emptyset) \wedge f$ is a function from $\bigcup C$ into $\bigcup C_3 \wedge \bigwedge_{x,y}[\{x, y\} \in C \Rightarrow \{f(x), f(y)\} \in C_3]\}$, where C ranges over elements of $\mathrm{CSp}(X)$, and C_3 ranges over elements of $\mathrm{CSp}(X)$, and f ranges over elements of $\mathrm{Funcs}_{\mathrm{C}}X$.

In the sequel l, l_1, l_2, l_3 will be elements of $\mathrm{Maps}_{\mathrm{C}}X$. The following two propositions are true:

(18) There exist g, C_1, C_2 such that $l = \langle\langle C_1, C_2\rangle, g\rangle$ and also if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and g is a function from $\bigcup C_1$ into $\bigcup C_2$ and for all x, y such that $\{x, y\} \in C_1$ holds $\{g(x), g(y)\} \in C_2$.

(19) For every function f from $\bigcup C_1$ into $\bigcup C_2$ such that if $\bigcup C_2 = \emptyset$, then $\bigcup C_1 = \emptyset$ and also for all x, y such that $\{x, y\} \in C_1$ holds $\{f(x), f(y)\} \in C_2$ holds $\langle\langle C_1, C_2\rangle, f\rangle \in \mathrm{Maps}_{\mathrm{C}}X$.

We now define three new functors. Let us consider X, l. The functor $\mathrm{graph}(l)$ yields a function and is defined by:

(Def.9) $\mathrm{graph}(l) = l_{\mathbf{2}}$.

The functor $\mathrm{dom}\, l$ yielding an element of $\mathrm{CSp}(X)$ is defined by:

(Def.10) $\mathrm{dom}\, l = (l_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, l$ yielding an element of $\mathrm{CSp}(X)$ is defined by:

(Def.11) $\mathrm{cod}\, l = (l_{\mathbf{1}})_{\mathbf{2}}$.

Next we state the proposition

(20) $l = \langle\langle \mathrm{dom}\, l, \mathrm{cod}\, l\rangle, \mathrm{graph}(l)\rangle$.

Let us consider X, C. The functor $\mathrm{id}(C)$ yields an element of $\mathrm{Maps}_{\mathrm{C}}X$ and is defined by:

(Def.12) $\mathrm{id}(C) = \langle\langle C, C\rangle, \mathrm{id}_{\bigcup C}\rangle$.

One can prove the following proposition

(21) $\bigcup \mathrm{cod}\, l \neq \emptyset$ or $\bigcup \mathrm{dom}\, l = \emptyset$ and also $\mathrm{graph}(l)$ is a function from $\bigcup \mathrm{dom}\, l$ into $\bigcup \mathrm{cod}\, l$ and for all x, y such that $\{x, y\} \in \mathrm{dom}\, l$ holds $\{(\mathrm{graph}(l))(x), (\mathrm{graph}(l))(y)\} \in \mathrm{cod}\, l$.

Let us consider X, l_1, l_2. Let us assume that $\mathrm{cod}\, l_1 = \mathrm{dom}\, l_2$. The functor $l_2 \cdot l_1$ yielding an element of $\mathrm{Maps}_{\mathrm{C}}X$ is defined as follows:

(Def.13) $l_2 \cdot l_1 = \langle\langle \mathrm{dom}\, l_1, \mathrm{cod}\, l_2\rangle, \mathrm{graph}(l_2) \cdot \mathrm{graph}(l_1)\rangle$.

We now state four propositions:

(22) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$, then $\mathrm{graph}((l_2 \cdot l_1)) = \mathrm{graph}(l_2) \cdot \mathrm{graph}(l_1)$ and $\mathrm{dom}(l_2 \cdot l_1) = \mathrm{dom}\, l_1$ and $\mathrm{cod}(l_2 \cdot l_1) = \mathrm{cod}\, l_2$.

(23) If $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and $\mathrm{dom}\, l_3 = \mathrm{cod}\, l_2$, then $l_3 \cdot (l_2 \cdot l_1) = (l_3 \cdot l_2) \cdot l_1$.

(24) $\mathrm{graph}(\mathrm{id}(C)) = \mathrm{id}_{\bigcup C}$ and $\mathrm{dom}\,\mathrm{id}(C) = C$ and $\mathrm{cod}\,\mathrm{id}(C) = C$.

(25) $l \cdot \mathrm{id}(\mathrm{dom}\, l) = l$ and $\mathrm{id}(\mathrm{cod}\, l) \cdot l = l$.

We now define four new functors. Let us consider X. The functor $\mathrm{Dom}_{\mathrm{CSp}}\, X$ yields a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ and is defined as follows:

(Def.14) for every l holds $(\mathrm{Dom}_{\mathrm{CSp}}\, X)(l) = \mathrm{dom}\, l$.

The functor $\mathrm{Cod}_{\mathrm{CSp}}\, X$ yielding a function from $\mathrm{Maps}_{\mathrm{C}} X$ into $\mathrm{CSp}(X)$ is defined by:

(Def.15) for every l holds $(\mathrm{Cod}_{\mathrm{CSp}}\, X)(l) = \mathrm{cod}\, l$.

The functor $\cdot_{\mathrm{CSp}}\, X$ yielding a partial function from $[\!:\mathrm{Maps}_{\mathrm{C}} X, \mathrm{Maps}_{\mathrm{C}} X:\!]$ to $\mathrm{Maps}_{\mathrm{C}} X$ is defined by:

(Def.16) for all l_2, l_1 holds $\langle l_2, l_1 \rangle \in \mathrm{dom} \cdot_{\mathrm{CSp}} X$ if and only if $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ and for all l_2, l_1 such that $\mathrm{dom}\, l_2 = \mathrm{cod}\, l_1$ holds $(\cdot_{\mathrm{CSp}}\, X)(\langle l_2, l_1 \rangle) = l_2 \cdot l_1$.

The functor $\mathrm{Id}_{\mathrm{CSp}}\, X$ yielding a function from $\mathrm{CSp}(X)$ into $\mathrm{Maps}_{\mathrm{C}} X$ is defined by:

(Def.17) for every C holds $(\mathrm{Id}_{\mathrm{CSp}}\, X)(C) = \mathrm{id}(C)$.

Next we state the proposition

(26) $\langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}} X, \mathrm{Dom}_{\mathrm{CSp}}\, X, \mathrm{Cod}_{\mathrm{CSp}}\, X, \cdot_{\mathrm{CSp}}\, X, \mathrm{Id}_{\mathrm{CSp}}\, X \rangle$ is a category.

Let us consider X. The X-coherent space category yields a category and is defined by:

(Def.18) the X-coherent space category $= \langle \mathrm{CSp}(X), \mathrm{Maps}_{\mathrm{C}} X, \mathrm{Dom}_{\mathrm{CSp}}\, X, \mathrm{Cod}_{\mathrm{CSp}}\, X, \cdot_{\mathrm{CSp}}\, X, \mathrm{Id}_{\mathrm{CSp}}\, X \rangle$.

3. Category of Tolerances

We now define two new functors. Let X be a set. The tolerances on X constitute a non-empty set defined by:

(Def.19) the tolerances on X is the set of all tolerances of X.

Let X be a set. The tolerances on subsets of X constitute a non-empty set defined as follows:

(Def.20) the tolerances on subsets of $X = \bigcup\{$the tolerances on $Y\}$, where Y ranges over subsets of X.

In the sequel t denotes an element of the tolerances on subsets of X. The following propositions are true:

(27) $x \in$ the tolerances on subsets of X if and only if there exists A such that $A \subseteq X$ and x is a tolerance of A.

(28) $\nabla_a \in$ the tolerances on a.

(29) $\triangle_a \in$ the tolerances on a.

(30) $\varnothing \in$ the tolerances on subsets of X.

(31) If $a \subseteq X$, then $\nabla_a \in$ the tolerances on subsets of X.

(32) If $a \subseteq X$, then $\triangle_a \in$ the tolerances on subsets of X.

(33) $\nabla_X \in$ the tolerances on subsets of X.

(34) $\triangle_X \in$ the tolerances on subsets of X.

Let us consider X. The functor $\mathrm{TOL}(X)$ yields a non-empty set and is defined by:

(Def.21) $\mathrm{TOL}(X) = \{\langle t, Y\rangle : t \text{ is a tolerance of } Y\}$, where t ranges over elements of the tolerances on subsets of X, and Y ranges over elements of 2^X.

In the sequel T, T_1, T_2 will denote elements of $\mathrm{TOL}(X)$. Next we state several propositions:

(35) $\langle \varnothing, \emptyset\rangle \in \mathrm{TOL}(X)$.

(36) If $a \subseteq X$, then $\langle \triangle_a, a\rangle \in \mathrm{TOL}(X)$.

(37) If $a \subseteq X$, then $\langle \nabla_a, a\rangle \in \mathrm{TOL}(X)$.

(38) $\langle \triangle_X, X\rangle \in \mathrm{TOL}(X)$.

(39) $\langle \nabla_X, X\rangle \in \mathrm{TOL}(X)$.

Let us consider X, T. Then $T_{\mathbf{2}}$ is an element of 2^X. Then $T_{\mathbf{1}}$ is a tolerance of $T_{\mathbf{2}}$. Let us consider X. The functor $\mathrm{Funcs}_{\mathrm{T}}X$ yielding a non-empty set of functions is defined as follows:

(Def.22) $\mathrm{Funcs}_{\mathrm{T}}X = \bigcup\{(T_{3\mathbf{2}})^{T_{\mathbf{2}}}\}$, where T ranges over elements of $\mathrm{TOL}(X)$, and T_3 ranges over elements of $\mathrm{TOL}(X)$.

In the sequel f denotes an element of $\mathrm{Funcs}_{\mathrm{T}}X$. We now state the proposition

(40) $x \in \mathrm{Funcs}_{\mathrm{T}}X$ if and only if there exist T_1, T_2 such that if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and also x is a function from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$.

Let us consider X. The functor $\mathrm{Maps}_{\mathrm{T}}X$ yielding a non-empty set is defined by:

(Def.23) $\mathrm{Maps}_{\mathrm{T}}X = \{\langle\langle T, T_3\rangle, f\rangle : (T_{3\mathbf{2}} = \emptyset \Rightarrow T_{\mathbf{2}} = \emptyset) \wedge f \text{ is a function from } T_{\mathbf{2}} \text{ into } T_{3\mathbf{2}} \wedge \bigwedge_{x,y}[\langle x, y\rangle \in T_{\mathbf{1}} \Rightarrow \langle f(x), f(y)\rangle \in T_{3\mathbf{1}}]\}$, where T ranges over elements of $\mathrm{TOL}(X)$, and T_3 ranges over elements of $\mathrm{TOL}(X)$, and f ranges over elements of $\mathrm{Funcs}_{\mathrm{T}}X$.

In the sequel m, m_1, m_2, m_3 denote elements of $\mathrm{Maps}_{\mathrm{T}}X$. One can prove the following two propositions:

(41) There exist f, T_1, T_2 such that $m = \langle\langle T_1, T_2\rangle, f\rangle$ and also if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and f is a function from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$ and for all x, y such that $\langle x, y\rangle \in T_{1\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in T_{2\mathbf{1}}$.

(42) For every function f from $T_{1\mathbf{2}}$ into $T_{2\mathbf{2}}$ such that if $T_{2\mathbf{2}} = \emptyset$, then $T_{1\mathbf{2}} = \emptyset$ and also for all x, y such that $\langle x, y\rangle \in T_{1\mathbf{1}}$ holds $\langle f(x), f(y)\rangle \in T_{2\mathbf{1}}$ holds $\langle\langle T_1, T_2\rangle, f\rangle \in \mathrm{Maps}_{\mathrm{T}}X$.

We now define three new functors. Let us consider X, m. The functor $\mathrm{graph}(m)$ yielding a function is defined by:

(Def.24) $\mathrm{graph}(m) = m_{\mathbf{2}}$.

The functor $\mathrm{dom}\, m$ yields an element of $\mathrm{TOL}(X)$ and is defined by:

(Def.25) $\operatorname{dom} m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\operatorname{cod} m$ yields an element of $\mathrm{TOL}(X)$ and is defined by:

(Def.26) $\operatorname{cod} m = (m_{\mathbf{1}})_{\mathbf{2}}$.

One can prove the following proposition

(43) $m = \langle\langle \operatorname{dom} m, \operatorname{cod} m\rangle, \operatorname{graph}(m)\rangle$.

Let us consider X, T. The functor $\operatorname{id}(T)$ yields an element of $\mathrm{Maps}_{\mathrm{T}} X$ and is defined by:

(Def.27) $\operatorname{id}(T) = \langle\langle T, T\rangle, \operatorname{id}_{(T_{\mathbf{2}})}\rangle$.

One can prove the following proposition

(44) $(\operatorname{cod} m)_{\mathbf{2}} \neq \emptyset$ or $(\operatorname{dom} m)_{\mathbf{2}} = \emptyset$ and also $\operatorname{graph}(m)$ is a function from $(\operatorname{dom} m)_{\mathbf{2}}$ into $(\operatorname{cod} m)_{\mathbf{2}}$ and for all x, y such that $\langle x, y\rangle \in (\operatorname{dom} m)_{\mathbf{1}}$ holds $\langle(\operatorname{graph}(m))(x), (\operatorname{graph}(m))(y)\rangle \in (\operatorname{cod} m)_{\mathbf{1}}$.

Let us consider X, m_1, m_2. Let us assume that $\operatorname{cod} m_1 = \operatorname{dom} m_2$. The functor $m_2 \cdot m_1$ yielding an element of $\mathrm{Maps}_{\mathrm{T}} X$ is defined by:

(Def.28) $m_2 \cdot m_1 = \langle\langle \operatorname{dom} m_1, \operatorname{cod} m_2\rangle, \operatorname{graph}(m_2) \cdot \operatorname{graph}(m_1)\rangle$.

The following propositions are true:

(45) If $\operatorname{dom} m_2 = \operatorname{cod} m_1$, then $\operatorname{graph}((m_2 \cdot m_1)) = \operatorname{graph}(m_2) \cdot \operatorname{graph}(m_1)$ and $\operatorname{dom}(m_2 \cdot m_1) = \operatorname{dom} m_1$ and $\operatorname{cod}(m_2 \cdot m_1) = \operatorname{cod} m_2$.

(46) If $\operatorname{dom} m_2 = \operatorname{cod} m_1$ and $\operatorname{dom} m_3 = \operatorname{cod} m_2$, then $m_3 \cdot (m_2 \cdot m_1) = (m_3 \cdot m_2) \cdot m_1$.

(47) $\operatorname{graph}(\operatorname{id}(T)) = \operatorname{id}_{(T_{\mathbf{2}})}$ and $\operatorname{dom} \operatorname{id}(T) = T$ and $\operatorname{cod} \operatorname{id}(T) = T$.

(48) $m \cdot \operatorname{id}(\operatorname{dom} m) = m$ and $\operatorname{id}(\operatorname{cod} m) \cdot m = m$.

We now define four new functors. Let us consider X. The functor Dom_X yields a function from $\mathrm{Maps}_{\mathrm{T}} X$ into $\mathrm{TOL}(X)$ and is defined by:

(Def.29) for every m holds $\mathrm{Dom}_X(m) = \operatorname{dom} m$.

The functor Cod_X yields a function from $\mathrm{Maps}_{\mathrm{T}} X$ into $\mathrm{TOL}(X)$ and is defined as follows:

(Def.30) for every m holds $\mathrm{Cod}_X(m) = \operatorname{cod} m$.

The functor $\cdot_X$ yields a partial function from $[\!:\mathrm{Maps}_{\mathrm{T}} X, \mathrm{Maps}_{\mathrm{T}} X\,:\!]$ to $\mathrm{Maps}_{\mathrm{T}} X$ and is defined as follows:

(Def.31) for all m_2, m_1 holds $\langle m_2, m_1\rangle \in \operatorname{dom}(\cdot_X)$ if and only if $\operatorname{dom} m_2 = \operatorname{cod} m_1$ and for all m_2, m_1 such that $\operatorname{dom} m_2 = \operatorname{cod} m_1$ holds $\cdot_X(\langle m_2, m_1\rangle) = m_2 \cdot m_1$.

The functor Id_X yields a function from $\mathrm{TOL}(X)$ into $\mathrm{Maps}_{\mathrm{T}} X$ and is defined by:

(Def.32) for every T holds $\mathrm{Id}_X(T) = \operatorname{id}(T)$.

Next we state the proposition

(49) $\langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}} X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X\rangle$ is a category.

Let us consider X. The X-tolerance category is a category defined by:

(Def.33) the X-tolerance category $= \langle \mathrm{TOL}(X), \mathrm{Maps}_{\mathrm{T}} X, \mathrm{Dom}_X, \mathrm{Cod}_X, \cdot_X, \mathrm{Id}_X\rangle$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[5] Krzysztof Hryniewiecki. Relations of tolerance. *Formalized Mathematics*, 2(**1**):105–109, 1991.

[6] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[7] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[9] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[11] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received December 29, 1992