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Basic Properties of Connecting Points with Line Segments in $\mathcal{E}^2_{\rm T}$

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Summary. Some properties of line segments in 2-dimensional Euclidean space and some relations between line segments and balls are proved.

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The terminology and notation used in this paper have been introduced in the following papers: [17], [13], [1], [7], [2], [8], [4], [15], [16], [18], [6], [14], [5], [9], [10], [3], [11], and [12].

1. Real Numbers Preliminaries

For simplicity we follow the rules: p, p_1, p_2, p_3, q will denote points of $\mathcal{E}^2_{\rm T}$, f, h will denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r, r_1, r_2, s, s_1, s_2 will denote real numbers, u, u_1, u_2 will denote points of $\mathcal{E}^2$, n, m, i, j, k will denote natural numbers, and x, y, z will be arbitrary. One can prove the following propositions:

(1) $3-2=1$ and $3-1=2$ and $\frac{1}{2}=1-\frac{1}{2}$.

(2) $0 \leq \frac{1}{2}$ and $\frac{1}{2} \leq 1$.

(3) If $r < s$, then $r < \frac{r+s}{2}$ and $\frac{r+s}{2} < s$ and $r < \frac{s+r}{2}$ and $\frac{s+r}{2} < s$.

(4) If $r \neq s$, then $r \neq \frac{r+s}{2}$ and $\frac{r+s}{2} \neq s$.

(5) If $r_1 > s_1$ and $r_2 \geq s_2$ or $r_1 \geq s_1$ and $r_2 > s_2$, then $r_1+r_2 > s_1+s_2$.

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2. Properties of Line Segments

We now state a number of propositions:

(6) $1 \in \mathrm{Seg}\,\mathrm{len}\langle x, y, z\rangle$ and $2 \in \mathrm{Seg}\,\mathrm{len}\langle x, y, z\rangle$ and $3 \in \mathrm{Seg}\,\mathrm{len}\langle x, y, z\rangle$.

(7) $(p_1 + p_2)_{\mathbf{1}} = p_{1\mathbf{1}} + p_{2\mathbf{1}}$ and $(p_1 + p_2)_{\mathbf{2}} = p_{1\mathbf{2}} + p_{2\mathbf{2}}$.

(8) $(p_1 - p_2)_{\mathbf{1}} = p_{1\mathbf{1}} - p_{2\mathbf{1}}$ and $(p_1 - p_2)_{\mathbf{2}} = p_{1\mathbf{2}} - p_{2\mathbf{2}}$.

(9) $(r \cdot p)_{\mathbf{1}} = r \cdot p_{\mathbf{1}}$ and $(r \cdot p)_{\mathbf{2}} = r \cdot p_{\mathbf{2}}$.

(10) If $p_1 = \langle r_1, s_1\rangle$ and $p_2 = \langle r_2, s_2\rangle$, then $p_1 + p_2 = \langle r_1 + r_2, s_1 + s_2\rangle$ and $p_1 - p_2 = \langle r_1 - r_2, s_1 - s_2\rangle$.

(11) $p = q$ if and only if $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$.

(12) If $u_1 = p_1$ and $u_2 = p_2$, then $\rho^2(u_1, u_2) = \sqrt{(p_{1\mathbf{1}} - p_{2\mathbf{1}})^{\mathbf{2}} + (p_{1\mathbf{2}} - p_{2\mathbf{2}})^{\mathbf{2}}}$.

(13) The carrier of $\mathcal{E}^n_{\mathrm{T}}$ = the carrier of $\mathcal{E}^n$.

(14) x is a point of $\mathcal{E}^2$ if and only if x is a point of $\mathcal{E}^2_{\mathrm{T}}$.

(15) If $r_1 < s_1$, then $\{p_1 : p_{1\mathbf{1}} = r \wedge r_1 \leq p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} \leq s_1\} = \mathcal{L}([r, r_1], [r, s_1])$.

(16) If $r_1 < s_1$, then $\{p_1 : p_{1\mathbf{2}} = r \wedge r_1 \leq p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} \leq s_1\} = \mathcal{L}([r_1, r], [s_1, r])$.

(17) If $p \in \mathcal{L}([r, r_1], [r, s_1])$, then $p_{\mathbf{1}} = r$.

(18) If $p \in \mathcal{L}([r_1, r], [s_1, r])$, then $p_{\mathbf{2}} = r$.

(19) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $[\frac{p_{\mathbf{1}} + q_{\mathbf{1}}}{2}, p_{\mathbf{2}}] \in \mathcal{L}(p, q)$.

(20) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $[p_{\mathbf{1}}, \frac{p_{\mathbf{2}} + q_{\mathbf{2}}}{2}] \in \mathcal{L}(p, q)$.

(21) If $f = \langle p, p_1, q\rangle$ and $i \neq 0$ and $j - i > 1$, then $\mathcal{L}(f, j, j + 1) = \emptyset$.

(22) If $i = 0$, then $\mathcal{L}(f, i, i + 1) = \emptyset$.

(23) If $f = \langle p_1, p_2, p_3\rangle$, then $\widetilde{\mathcal{L}}(f) = \mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, p_3)$.

(24) If $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}(f \restriction i)$ and $k \in \mathrm{dom}(f \restriction i)$, then $\mathcal{L}(f, j, k) = \mathcal{L}(f \restriction i, j, k)$.

(25) If $j \in \mathrm{dom}\, f$ and $i \in \mathrm{dom}\, f$, then $\mathcal{L}(f \frown h, j, i) = \mathcal{L}(f, j, i)$.

(26) $\mathcal{L}(f, i, i + 1) \subseteq \widetilde{\mathcal{L}}(f)$.

(27) $\widetilde{\mathcal{L}}(f \restriction i) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) For all r, p_1, p_2, u such that $r > 0$ and $p_1 \in \mathrm{Ball}(u, r)$ and $p_2 \in \mathrm{Ball}(u, r)$ holds $\mathcal{L}(p_1, p_2) \subseteq \mathrm{Ball}(u, r)$.

(29) If $u = p_1$ and $p_1 = [r_1, s_1]$ and $p_2 = [r_2, s_2]$ and $p = [r_2, s_1]$ and $p_2 \in \mathrm{Ball}(u, r)$, then $p \in \mathrm{Ball}(u, r)$.

(30) If $r_1 \neq s_1$ and $r > 0$ and $[s, r_1] \in \mathrm{Ball}(u, r)$ and $[s, s_1] \in \mathrm{Ball}(u, r)$, then $[s, \frac{r_1 + s_1}{2}] \in \mathrm{Ball}(u, r)$.

(31) If $r_1 \neq s_1$ and $r > 0$ and $[r_1, s] \in \mathrm{Ball}(u, r)$ and $[s_1, s] \in \mathrm{Ball}(u, r)$, then $[\frac{r_1 + s_1}{2}, s] \in \mathrm{Ball}(u, r)$.

(32) If $r_1 \neq s_1$ and $s_2 \neq r_2$ and $r > 0$ and $[r_1, r_2] \in \mathrm{Ball}(u, r)$ and $[s_1, s_2] \in \mathrm{Ball}(u, r)$, then $[r_1, s_2] \in \mathrm{Ball}(u, r)$ or $[s_1, r_2] \in \mathrm{Ball}(u, r)$.

(33) Suppose that

(i) $f(1) \notin \mathrm{Ball}(u, r)$,

(ii) $1 \leq m$,
(iii) $m \leq \operatorname{len} f - 1$,
(iv) $\mathcal{L}(f, m, m+1) \cap \operatorname{Ball}(u, r) \neq \emptyset$,
(v) for every i such that $1 \leq i$ and $i \leq \operatorname{len} f - 1$ and $\mathcal{L}(f, i, i+1) \cap \operatorname{Ball}(u, r) \neq \emptyset$ holds $m \leq i$.
Then $f(m) \notin \operatorname{Ball}(u, r)$.

(34) For all q, p_2, p such that $q_{\mathbf{2}} = p_{2\mathbf{2}}$ and $p_{\mathbf{2}} \neq p_{2\mathbf{2}}$ holds $(\mathcal{L}(p_2, [p_{2\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([p_{2\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \mathcal{L}(q, p_2) = \{p_2\}$.

(35) For all q, p_2, p such that $q_{\mathbf{1}} = p_{2\mathbf{1}}$ and $p_{\mathbf{1}} \neq p_{2\mathbf{1}}$ holds $(\mathcal{L}(p_2, [p_{\mathbf{1}}, p_{2\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, p_{2\mathbf{2}}], p)) \cap \mathcal{L}(q, p_2) = \{p_2\}$.

(36) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [p_{\mathbf{1}}, q_{\mathbf{2}}]) \cap \mathcal{L}([p_{\mathbf{1}}, q_{\mathbf{2}}], q) = \{[p_{\mathbf{1}}, q_{\mathbf{2}}]\}$.

One can prove the following propositions:

(37) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [q_{\mathbf{1}}, p_{\mathbf{2}}]) \cap \mathcal{L}([q_{\mathbf{1}}, p_{\mathbf{2}}], q) = \{[q_{\mathbf{1}}, p_{\mathbf{2}}]\}$.

(38) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]) \cap \mathcal{L}([p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q) = \{[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]\}$.

(39) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]) \cap \mathcal{L}([\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q) = \{[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]\}$.

(40) If $i > 2$ and $i \in \operatorname{dom} f$ and f is a special sequence, then $f \restriction i$ is a special sequence.

(41) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(42) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(43) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(44) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(45) If $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $f(i) = p$ and $f(i+1) = q$, then $\widetilde{\mathcal{L}}(f \restriction (i+1)) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(p, q)$.

(46) If $\operatorname{len} f \geq 2$ and $p \notin \widetilde{\mathcal{L}}(f)$, then for every n such that $1 \leq n$ and $n \leq \operatorname{len} f$ holds $f(n) \neq p$.

(47) If $q \neq p$ and $\mathcal{L}(q, p) \cap \widetilde{\mathcal{L}}(f) = \{q\}$, then $p \notin \widetilde{\mathcal{L}}(f)$.

(48) Suppose that
(i) f is a special sequence,
(ii) $f(1) = p$,
(iii) $f(\operatorname{len} f) = q$,
(iv) $p \notin \operatorname{Ball}(u, r)$,
(v) $q \in \operatorname{Ball}(u, r)$,
(vi) $q \in \mathcal{L}(f, m, m+1)$,
(vii) $1 \leq m$,
(viii) $m \leq \operatorname{len} f - 1$,

(ix) $\mathcal{L}(f,m,m+1)\cap \mathrm{Ball}(u,r)\neq\emptyset$.
Then $m=\mathrm{len}\,f-1$.

(49) Suppose that
(i) $r>0$,
(ii) $p_1\notin \mathrm{Ball}(u,r)$,
(iii) $q\in \mathrm{Ball}(u,r)$,
(iv) $p\in \mathrm{Ball}(u,r)$,
(v) $p\notin \mathcal{L}(p_1,q)$,
(vi) $q_{\mathbf{1}}=p_{\mathbf{1}}$ and $q_{\mathbf{2}}\neq p_{\mathbf{2}}$ or $q_{\mathbf{1}}\neq p_{\mathbf{1}}$ and $q_{\mathbf{2}}=p_{\mathbf{2}}$,
(vii) $p_{1\mathbf{1}}=q_{\mathbf{1}}$ or $p_{1\mathbf{2}}=q_{\mathbf{2}}$.
Then $\mathcal{L}(p_1,q)\cap\mathcal{L}(q,p)=\{q\}$.

(50) Suppose that
(i) $r>0$,
(ii) $p_1\notin \mathrm{Ball}(u,r)$,
(iii) $p\in \mathrm{Ball}(u,r)$,
(iv) $[p_{\mathbf{1}},q_{\mathbf{2}}]\in \mathrm{Ball}(u,r)$,
(v) $q\in \mathrm{Ball}(u,r)$,
(vi) $[p_{\mathbf{1}},q_{\mathbf{2}}]\notin\mathcal{L}(p_1,p)$,
(vii) $p_{1\mathbf{1}}=p_{\mathbf{1}}$,
(viii) $p_{\mathbf{1}}\neq q_{\mathbf{1}}$,
(ix) $p_{\mathbf{2}}\neq q_{\mathbf{2}}$.
Then $(\mathcal{L}(p,[p_{\mathbf{1}},q_{\mathbf{2}}])\cup\mathcal{L}([p_{\mathbf{1}},q_{\mathbf{2}}],q))\cap\mathcal{L}(p_1,p)=\{p\}$.

(51) Suppose that
(i) $r>0$,
(ii) $p_1\notin \mathrm{Ball}(u,r)$,
(iii) $p\in \mathrm{Ball}(u,r)$,
(iv) $[q_{\mathbf{1}},p_{\mathbf{2}}]\in \mathrm{Ball}(u,r)$,
(v) $q\in \mathrm{Ball}(u,r)$,
(vi) $[q_{\mathbf{1}},p_{\mathbf{2}}]\notin\mathcal{L}(p_1,p)$,
(vii) $p_{1\mathbf{2}}=p_{\mathbf{2}}$,
(viii) $p_{\mathbf{1}}\neq q_{\mathbf{1}}$,
(ix) $p_{\mathbf{2}}\neq q_{\mathbf{2}}$.
Then $(\mathcal{L}(p,[q_{\mathbf{1}},p_{\mathbf{2}}])\cup\mathcal{L}([q_{\mathbf{1}},p_{\mathbf{2}}],q))\cap\mathcal{L}(p_1,p)=\{p\}$.

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