FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Submodules

Michał Muzalewski
Warsaw University
Białystok

Summary. This article contains the notions of trivial and nontrivial leftmodules and rings, cyclic submodules and inclusion of submodules. A few basic theorems related to these notions are proved.

MML Identifier: `LMOD_6`.

The notation and terminology used here are introduced in the following papers: [15], [16], [3], [4], [2], [1], [5], [6], [7], [14], [9], [13], [12], [10], [11], and [8].

1. Preliminaries

For simplicity we adopt the following rules: x is arbitrary, K denotes an associative ring, r denotes a scalar of K, V, M, M_1, M_2, N denote left modules over K, a denotes a vector of V, m, m_1, m_2 denote vectors of M, n, n_1, n_2 denote vectors of N, A denotes a subset of V, l denotes a linear combination of A, and W, W_1, W_2, W_3 denote submodules of V. Next we state four propositions:

(1) If $M_1 =$ the left module structure of M_2, then $x \in M_1$ if and only if $x \in M_2$.

(2) For every vector v of the left module structure of V such that $a = v$ holds $r \cdot a = r \cdot v$.

(3) The left module structure of V is a strict submodule of V.

(4) V is a submodule of Ω_V.

ISSN 0777–4028

2. Trivial and non-trivial modules and rings

We now define two new predicates. Let us consider K, V. We say that V is non-trivial if and only if:

(Def.1) there exists a vector a of V such that $a \neq \Theta_V$.

Let us consider K. We say that K is non-trivial if and only if:

(Def.2) $0_K \neq 1_K$.

We now state three propositions:

(5) If K is trivial, then for every r holds $r = 0_K$ and for every a holds $a = \Theta_V$.

(6) If K is trivial, then V is trivial.

(7) V is trivial if and only if the left module structure of $V = \mathbf{0}_V$.

3. Submodules and subsets

We now define two new functors. Let us consider K, V, and let W be a strict submodule of V. The functor ë(W) yields an element of $\mathrm{Sub}(V)$ and is defined by:

(Def.3) ë(W) $= W$.

The functor ç(V) yields a non-empty subset of V and is defined as follows:

(Def.4) ç(V) = the carrier of V.

The following two propositions are true:

(8) For all sets X, Y, A such that $X \subseteq Y$ and A is a subset of X holds A is a subset of Y.

(9) Every subset of W is a subset of V.

Let us consider K, V, W, and let A be a subset of W. The functor ï(A) yields a subset of V and is defined by:

(Def.5) ï(A) $= A$.

Let A be a non-empty subset of W. Then ï(A) is a non-empty subset of V.

The following propositions are true:

(10) $x \in$ ç(V) if and only if $x \in V$.

(11) $x \in$ ï(ç(W)) if and only if $x \in W$.

(12) $A \subseteq$ ç(Lin(A)).

(13) If $A \neq \emptyset$ and A is linearly closed, then $\sum l \in A$.

(14) If $\Theta_V \in A$ and A is linearly closed, then $\sum l \in A$.

(15) If $\Theta_V \in A$ and A is linearly closed, then $A =$ ç(Lin(A)).

4. Cyclic submodules

Let us consider K, V, a. Then $\{a\}$ is a non-empty subset of V. The functor $\prod^* a$ yielding a strict submodule of V is defined by:

(Def.6) $\prod^* a = \mathrm{Lin}(\{a\})$.

5. Inclusion of left R-modules

Let us consider K, M, N. The predicate $M \subseteq N$ is defined as follows:

(Def.7) M is a submodule of N.

We now state a number of propositions:

(16) If $M \subseteq N$, then if $x \in M$, then $x \in N$ but if x is a vector of M, then x is a vector of N.

(17) Suppose $M \subseteq N$. Then
- (i) $\Theta_M = \Theta_N$,
- (ii) if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 + m_2 = n_1 + n_2$,
- (iii) if $m = n$, then $r \cdot m = r \cdot n$,
- (iv) if $m = n$, then $-n = -m$,
- (v) if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 - m_2 = n_1 - n_2$,
- (vi) $\Theta_N \in M$,
- (vii) $\Theta_M \in N$,
- (viii) if $n_1 \in M$ and $n_2 \in M$, then $n_1 + n_2 \in M$,
- (ix) if $n \in M$, then $r \cdot n \in M$,
- (x) if $n \in M$, then $-n \in M$,
- (xi) if $n_1 \in M$ and $n_2 \in M$, then $n_1 - n_2 \in M$.

(18) Suppose $M_1 \subseteq N$ and $M_2 \subseteq N$. Then
- (i) $\Theta_{M_1} = \Theta_{M_2}$,
- (ii) $\Theta_{M_1} \in M_2$,
- (iii) if the carrier of $M_1 \subseteq$ the carrier of M_2, then $M_1 \subseteq M_2$,
- (iv) if for every n such that $n \in M_1$ holds $n \in M_2$, then $M_1 \subseteq M_2$,
- (v) if the carrier of $M_1 =$ the carrier of M_2 and M_1 is strict and M_2 is strict, then $M_1 = M_2$,
- (vi) $\mathbf{0}_{M_1} \subseteq M_2$.

(19) $W_1 + W_2 \subseteq V$ and $W_1 \cap W_2 \subseteq V$.

(20) $N \subseteq N$.

(21) For all strict left modules V, M over K such that $V \subseteq M$ and $M \subseteq V$ holds $V = M$.

(22) If $V \subseteq M$ and $M \subseteq N$, then $V \subseteq N$.

(23) If $M \subseteq N$, then $\mathbf{0}_M \subseteq N$.

(24) If $M \subseteq N$, then $\mathbf{0}_N \subseteq M$.

(25) If $M \subseteq N$, then $M \subseteq \Omega_N$.

(26) $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

(27) $W_1 \cap W_2 \subseteq W_1$ and $W_1 \cap W_2 \subseteq W_2$.

(28) If $W_1 \subseteq W_2$, then $W_1 \cap W_3 \subseteq W_2 \cap W_3$.

(29) If $W_1 \subseteq W_3$, then $W_1 \cap W_2 \subseteq W_3$.

(30) If $W_1 \subseteq W_2$ and $W_1 \subseteq W_3$, then $W_1 \subseteq W_2 \cap W_3$.

(31) $W_1 \cap W_2 \subseteq W_1 + W_2$.

(32) $W_1 \cap W_2 + W_2 \cap W_3 \subseteq W_2 \cap (W_1 + W_3)$.

(33) If $W_1 \subseteq W_2$, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3 \subseteq (W_1 + W_2) \cap (W_2 + W_3)$.

(35) If $W_1 \subseteq W_2$, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If $W_1 \subseteq W_2$, then $W_1 \subseteq W_2 + W_3$.

(37) If $W_1 \subseteq W_3$ and $W_2 \subseteq W_3$, then $W_1 + W_2 \subseteq W_3$.

(38) For all subsets A, B of V such that $A \subseteq B$ holds $\mathrm{Lin}(A) \subseteq \mathrm{Lin}(B)$.

(39) For all subsets A, B of V holds $\mathrm{Lin}(A \cap B) \subseteq \mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(40) If $M_1 \subseteq M_2$, then ç$(M_1) \subseteq$ ç(M_2).

(41) $W_1 \subseteq W_2$ if and only if for every a such that $a \in W_1$ holds $a \in W_2$.

(42) $W_1 \subseteq W_2$ if and only if ç$(W_1) \subseteq$ ç(W_2).

(43) $W_1 \subseteq W_2$ if and only if ï(ç(W_1)) $\subseteq$ ï(ç(W_2)).

(44) $\mathbf{0}_W \subseteq V$ and $\mathbf{0}_V \subseteq W$ and $\mathbf{0}_{W_1} \subseteq W_2$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[7] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[8] Michał Muzalewski. Free modules. *Formalized Mathematics*, 2(**4**):587–589, 1991.

[9] Michał Muzalewski and Wojciech Skaba. Finite sums of vectors in left module over associative ring. *Formalized Mathematics*, 2(**2**):279–282, 1991.

[10] Michał Muzalewski and Wojciech Skaba. Linear combinations in left module over associative ring. *Formalized Mathematics*, 2(**2**):295–300, 1991.

[11] Michał Muzalewski and Wojciech Skaba. Linear independence in left module over domain. *Formalized Mathematics*, 2(**2**):301–303, 1991.

[12] Michał Muzalewski and Wojciech Skaba. Operations on submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):289–293, 1991.

[13] Michał Muzalewski and Wojciech Skaba. Submodules and cosets of submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):283–287, 1991.

[14] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received June 19, 1992