FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Continuity of Mappings over the Union of Subspaces

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X and Y be topological spaces and let X_1 and X_2 be subspaces of X. Let $f : X_1 \cup X_2 \to Y$ be a mapping defined on the union of X_1 and X_2 such that the restriction mappings $f_{|X_1}$ and $f_{|X_2}$ are continuous. It is well known that if X_1 and X_2 are both open (closed) subspaces of X, then f is continuous (see e.g. [6, p.106]).

The aim is to show, using Mizar System, the following theorem (see Section 5): *If X_1 and X_2 are weakly separated, then f is continuous* (compare also [15, p.358] for related results). This theorem generalizes the preceding one because if X_1 and X_2 are both open (closed), then these subspaces are weakly separated (see [5]). However, the following problem remains open.

> **Problem 1.** Characterize the class of pairs of subspaces X_1 and X_2 of a topological space X such that $(*)$ for any topological space Y and for any mapping $f : X_1 \cup X_2 \to Y$, f is continuous if the restrictions $f_{|X_1}$ and $f_{|X_2}$ are continuous.

In some special case we have the following characterization: *X_1 and X_2 are separated iff X_1 misses X_2 and the condition $(*)$ is fulfilled.* In connection with this fact we hope that the following specification of the preceding problem has an affirmative answer.

> **Problem 2.** Suppose the condition $(*)$ is fulfilled. Must X_1 and X_2 be weakly separated ?

Note that in the last section the concept of the union of two mappings is introduced and studied. In particular, all results presented above are reformulated using this notion. In the remaining sections we introduce concepts needed for the formulation and the proof of theorems on properties of continuous mappings, restriction mappings and modifications of the topology.

MML Identifier: `TMAP_1`.

The articles [13], [14], [2], [3], [1], [4], [11], [8], [10], [16], [7], [9], [12], and [5] provide the notation and terminology for this paper.

ISSN 0777–4028

1. Set-Theoretic Preliminaries

In the sequel A, B will denote non-empty sets. Next we state several propositions:

(1) For every function f from A into B and for every subset A_0 of A and for every subset B_0 of B holds $f\,^\circ\,A_0 \subseteq B_0$ if and only if $A_0 \subseteq f^{-1}\,B_0$.

(2) For every function f from A into B and for every non-empty subset A_0 of A and for every function f_0 from A_0 into B such that for every element c of A such that $c \in A_0$ holds $f(c) = f_0(c)$ holds $f \restriction A_0 = f_0$.

(3) For every function f from A into B and for every non-empty subset A_0 of A and for every element c of A such that $c \in A_0$ holds $f(c) = (f \restriction A_0)(c)$.

(4) For every function f from A into B and for every non-empty subset A_0 of A and for every subset C of A such that $C \subseteq A_0$ holds $f^\circ C = (f \restriction A_0)^\circ C$.

(5) For every function f from A into B and for every non-empty subset A_0 of A and for every subset D of B such that $f^{-1}\,D \subseteq A_0$ holds $f^{-1}\,D = (f \restriction A_0)^{-1}\,D$.

Let A, B be non-empty sets, and let A_1, A_2 be non-empty subsets of A, and let f_1 be a function from A_1 into B, and let f_2 be a function from A_2 into B. Let us assume that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$. The functor $f_1 \cup f_2$ yielding a function from $A_1 \cup A_2$ into B is defined by:

(Def.1) $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

The following proposition is true

(6) Let A, B be non-empty sets. Then for all non-empty subsets A_1, A_2 of A such that A_1 misses A_2 and for every function f_1 from A_1 into B and for every function f_2 from A_2 into B holds $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $(f_1 \cup f_2) \restriction A_1 = f_1$ and $(f_1 \cup f_2) \restriction A_2 = f_2$.

We follow the rules: A, B are non-empty sets and A_1, A_2, A_3 are non-empty subsets of A. We now state four propositions:

(7) For every function g from $A_1 \cup A_2$ into B and for every function g_1 from A_1 into B and for every function g_2 from A_2 into B such that $g \restriction A_1 = g_1$ and $g \restriction A_2 = g_2$ holds $g = g_1 \cup g_2$.

(8) For every function f_1 from A_1 into B and for every function f_2 from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ holds $f_1 \cup f_2 = f_2 \cup f_1$.

(9) Let A_{12}, A_{23} be non-empty subsets of A. Suppose $A_{12} = A_1 \cup A_2$ and $A_{23} = A_2 \cup A_3$. Let f_1 be a function from A_1 into B. Let f_2 be a function from A_2 into B. Let f_3 be a function from A_3 into B. Suppose $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ and $f_2 \restriction (A_2 \cap A_3) = f_3 \restriction (A_2 \cap A_3)$ and $f_1 \restriction (A_1 \cap A_3) = f_3 \restriction (A_1 \cap A_3)$. Then for every function f_{12} from A_{12} into B and for every function f_{23} from A_{23} into B such that $f_{12} = f_1 \cup f_2$ and $f_{23} = f_2 \cup f_3$ holds $f_{12} \cup f_3 = f_1 \cup f_{23}$.

(10) For every function f_1 from A_1 into B and for every function f_2 from A_2 into B such that $f_1 \restriction (A_1 \cap A_2) = f_2 \restriction (A_1 \cap A_2)$ holds A_1 is a subset

of A_2 if and only if $f_1 \cup f_2 = f_2$ but A_2 is a subset of A_1 if and only if $f_1 \cup f_2 = f_1$.

2. Selected Properties of Subspaces of Topological Spaces

In the sequel X is a topological space. Next we state four propositions:

(11) For every subspace X_0 of X holds the topological structure of X_0 is a strict subspace of X.

(12) For all topological spaces X_1, X_2 such that $X_1 =$ the topological structure of X_2 holds X_1 is a subspace of X if and only if X_2 is a subspace of X.

(13) For all topological spaces X_1, X_2 such that $X_2 =$ the topological structure of X_1 holds X_1 is a closed subspace of X if and only if X_2 is a closed subspace of X.

(14) For all topological spaces X_1, X_2 such that $X_2 =$ the topological structure of X_1 holds X_1 is an open subspace of X if and only if X_2 is an open subspace of X.

In the sequel X_1, X_2 will denote subspaces of X. Next we state several propositions:

(15) If X_1 is a subspace of X_2, then for every point x_1 of X_1 there exists a point x_2 of X_2 such that $x_2 = x_1$.

(16) For every point x of $X_1 \cup X_2$ holds there exists a point x_1 of X_1 such that $x_1 = x$ or there exists a point x_2 of X_2 such that $x_2 = x$.

(17) If X_1 meets X_2, then for every point x of $X_1 \cap X_2$ holds there exists a point x_1 of X_1 such that $x_1 = x$ and there exists a point x_2 of X_2 such that $x_2 = x$.

(18) For every point x of $X_1 \cup X_2$ and for every subset F_1 of X_1 and for every subset F_2 of X_2 such that F_1 is closed and $x \in F_1$ and F_2 is closed and $x \in F_2$ there exists a subset H of $X_1 \cup X_2$ such that H is closed and $x \in H$ and $H \subseteq F_1 \cup F_2$.

(19) For every point x of $X_1 \cup X_2$ and for every subset U_1 of X_1 and for every subset U_2 of X_2 such that U_1 is open and $x \in U_1$ and U_2 is open and $x \in U_2$ there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq U_1 \cup U_2$.

(20) For every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x_1 = x$ and $x_2 = x$ and for every neighbourhood A_1 of x_1 and for every neighbourhood A_2 of x_2 there exists a subset V of $X_1 \cup X_2$ such that V is open and $x \in V$ and $V \subseteq A_1 \cup A_2$.

(21) For every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x_1 = x$ and $x_2 = x$ and for every neighbourhood A_1 of x_1 and for every neighbourhood A_2 of x_2 there exists a neighbourhood A of x such that $A \subseteq A_1 \cup A_2$.

In the sequel X_0, X_1, X_2, Y_1, Y_2 will be subspaces of X. One can prove the following propositions:

(22) If X_0 is a subspace of X_1, then X_0 meets X_1 and X_1 meets X_0.

(23) If X_0 is a subspace of X_1 but X_0 meets X_2 or X_2 meets X_0, then X_1 meets X_2 and X_2 meets X_1.

(24) If X_0 is a subspace of X_1 but X_1 misses X_2 or X_2 misses X_1, then X_0 misses X_2 and X_2 misses X_0.

(25) $X_0 \cup X_0 =$ the topological structure of X_0.

(26) $X_0 \cap X_0 =$ the topological structure of X_0.

(27) If Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cup Y_2$ is a subspace of $X_1 \cup X_2$.

(28) If Y_1 meets Y_2 and Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2, then $Y_1 \cap Y_2$ is a subspace of $X_1 \cap X_2$.

(29) If X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cup X_2$ is a subspace of X_0.

(30) If X_1 meets X_2 and X_1 is a subspace of X_0 and X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of X_0.

(31) (i) If X_1 misses X_0 or X_0 misses X_1 but X_2 meets X_0 or X_0 meets X_2, then $(X_1 \cup X_2) \cap X_0 = X_2 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_2$,

(ii) if X_1 meets X_0 or X_0 meets X_1 but X_2 misses X_0 or X_0 misses X_2, then $(X_1 \cup X_2) \cap X_0 = X_1 \cap X_0$ and $X_0 \cap (X_1 \cup X_2) = X_0 \cap X_1$.

(32) If X_1 meets X_2, then if X_1 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_0 \cap X_2$ but if X_2 is a subspace of X_0, then $X_1 \cap X_2$ is a subspace of $X_1 \cap X_0$.

(33) If X_1 is a subspace of X_0 but X_0 misses X_2 or X_2 misses X_0, then $X_0 \cap (X_1 \cup X_2) =$ the topological structure of X_1 and $X_0 \cap (X_2 \cup X_1) =$ the topological structure of X_1.

(34) If X_1 meets X_2, then if X_1 is a subspace of X_0, then $X_0 \cap X_2$ meets X_1 and $X_2 \cap X_0$ meets X_1 but if X_2 is a subspace of X_0, then $X_1 \cap X_0$ meets X_2 and $X_0 \cap X_1$ meets X_2.

(35) If X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$, then Y_1 misses X_2 and Y_2 misses X_1.

(36) Suppose X_1 is not a subspace of X_2 and X_2 is not a subspace of X_1 and $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ and $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2. Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(37) Suppose that

(i) X_1 meets X_2,

(ii) X_1 is not a subspace of X_2,

(iii) X_2 is not a subspace of X_1,

(iv) the topological structure of $X = Y_1 \cup Y_2 \cup X_0$,

(v) $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1,

(vi) $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2,
(vii) $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.
Then Y_1 meets $X_1 \cup X_2$ and Y_2 meets $X_1 \cup X_2$.

(38) Suppose that
(i) X_1 meets X_2,
(ii) X_1 is not a subspace of X_2,
(iii) X_2 is not a subspace of X_1,
(iv) $X_1 \cup X_2$ is not a subspace of $Y_1 \cup Y_2$,
(v) the topological structure of $X = Y_1 \cup Y_2 \cup X_0$,
(vi) $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1,
(vii) $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2,
(viii) $X_0 \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.
Then $Y_1 \cup Y_2$ meets $X_1 \cup X_2$ and X_0 meets $X_1 \cup X_2$.

(39) $X_1 \cup X_2$ meets X_0 if and only if X_1 meets X_0 or X_2 meets X_0 but X_0 meets $X_1 \cup X_2$ if and only if X_0 meets X_1 or X_0 meets X_2.

(40) $X_1 \cup X_2$ misses X_0 if and only if X_1 misses X_0 and X_2 misses X_0 but X_0 misses $X_1 \cup X_2$ if and only if X_0 misses X_1 and X_0 misses X_2.

(41) If X_1 meets X_2, then if $X_1 \cap X_2$ meets X_0, then X_1 meets X_0 and X_2 meets X_0 but if X_0 meets $X_1 \cap X_2$, then X_0 meets X_1 and X_0 meets X_2.

(42) If X_1 meets X_2, then if X_1 misses X_0 or X_2 misses X_0, then $X_1 \cap X_2$ misses X_0 but if X_0 misses X_1 or X_0 misses X_2, then X_0 misses $X_1 \cap X_2$.

(43) For every closed subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is a closed subspace of X_1.

(44) For every open subspace X_0 of X such that X_0 meets X_1 holds $X_0 \cap X_1$ is an open subspace of X_1.

(45) For every closed subspace X_0 of X such that X_1 is a subspace of X_0 and X_0 misses X_2 holds X_1 is a closed subspace of $X_1 \cup X_2$ and X_1 is a closed subspace of $X_2 \cup X_1$.

(46) For every open subspace X_0 of X such that X_1 is a subspace of X_0 and X_0 misses X_2 holds X_1 is an open subspace of $X_1 \cup X_2$ and X_1 is an open subspace of $X_2 \cup X_1$.

3. Continuity of Mappings

We now define two new constructions. Let X, Y be topological spaces. A mapping from X into Y is a function from the carrier of X into the carrier of Y.

We say that f is continuous at x if and only if:

(Def.2) for every neighbourhood G of $f(x)$ there exists a neighbourhood H of x such that $f \circ H \subseteq G$.

In the sequel X, Y denote topological spaces and f denotes a mapping from X into Y. One can prove the following propositions:

(47) For every point x of X holds f is continuous at x if and only if for every neighbourhood G of $f(x)$ holds $f^{-1}\,G$ is a neighbourhood of x.

(48) For every point x of X holds f is continuous at x if and only if for every subset G of Y such that G is open and $f(x) \in G$ there exists a subset H of X such that H is open and $x \in H$ and $f\,^\circ\,H \subseteq G$.

(49) f is continuous if and only if for every point x of X holds f is continuous at x.

(50) For all topological spaces X, Y, Z such that the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y and for every mapping f from X into Y and for every mapping g from X into Z such that $f = g$ and for every point x of X such that f is continuous at x holds g is continuous at x.

(51) Let X, Y, Z be topological spaces. Then if the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X, then for every mapping f from X into Z and for every mapping g from Y into Z such that $f = g$ and for every point x of X and for every point y of Y such that $x = y$ holds if g is continuous at y, then f is continuous at x.

Let X, Y, Z be topological spaces, and let f be a mapping from X into Y, and let g be a mapping from Y into Z. Then $g \cdot f$ is a mapping from X into Z.

We follow a convention: X, Y, Z are topological spaces, f is a mapping from X into Y, and g is a mapping from Y into Z. The following propositions are true:

(52) For every point x of X and for every point y of Y such that $y = f(x)$ holds if f is continuous at x and g is continuous at y, then $g \cdot f$ is continuous at x.

(53) For every point y of Y such that f is continuous and g is continuous at y and for every point x of X such that $x \in f^{-1}\,\{y\}$ holds $g \cdot f$ is continuous at x.

(54) For every point x of X such that f is continuous at x and g is continuous holds $g \cdot f$ is continuous at x.

Let X, Y be topological spaces. We introduce continuous mapping from X into Y as a synonym of continuous map from X into Y.

The following propositions are true:

(55) f is a continuous mapping from X into Y if and only if for every point x of X holds f is continuous at x.

(56) For all topological spaces X, Y, Z such that the carrier of Y = the carrier of Z and the topology of $Z \subseteq$ the topology of Y every continuous mapping from X into Y is a continuous mapping from X into Z.

(57) For all topological spaces X, Y, Z such that the carrier of X = the carrier of Y and the topology of $Y \subseteq$ the topology of X every continuous mapping from Y into Z is a continuous mapping from X into Z.

Let X, Y be topological spaces, and let X_0 be a subspace of X, and let f be a mapping from X into Y. The functor $f \restriction X_0$ yielding a mapping from X_0 into Y is defined by:

(Def.3) $f \restriction X_0 = f \restriction$ the carrier of X_0.

In the sequel X, Y will denote topological spaces, X_0 will denote a subspace of X, and f will denote a mapping from X into Y. The following propositions are true:

(58) For every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = (f \restriction X_0)(x)$.

(59) For every mapping f_0 from X_0 into Y such that for every point x of X such that $x \in$ the carrier of X_0 holds $f(x) = f_0(x)$ holds $f \restriction X_0 = f_0$.

(60) If the topological structure of $X_0 =$ the topological structure of X, then $f = f \restriction X_0$.

(61) For every subset A of X such that $A \subseteq$ the carrier of X_0 holds $f \circ A = (f \restriction X_0) \circ A$.

(62) For every subset B of Y such that $f^{-1} B \subseteq$ the carrier of X_0 holds $f^{-1} B = (f \restriction X_0)^{-1} B$.

(63) For every mapping g from X_0 into Y there exists a mapping h from X into Y such that $h \restriction X_0 = g$.

In the sequel f is a mapping from X into Y and X_0 is a subspace of X. Next we state several propositions:

(64) For every point x of X and for every point x_0 of X_0 such that $x = x_0$ holds if f is continuous at x, then $f \restriction X_0$ is continuous at x_0.

(65) For every subset A of X and for every point x of X and for every point x_0 of X_0 such that $A \subseteq$ the carrier of X_0 and A is a neighbourhood of x and $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(66) For every subset A of X and for every point x of X and for every point x_0 of X_0 such that A is open and $x \in A$ and $A \subseteq$ the carrier of X_0 and $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(67) For every open subspace X_0 of X and for every point x of X and for every point x_0 of X_0 such that $x = x_0$ holds f is continuous at x if and only if $f \restriction X_0$ is continuous at x_0.

(68) For every continuous mapping f from X into Y and for every subspace X_0 of X holds $f \restriction X_0$ is a continuous mapping from X_0 into Y.

(69) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every mapping f from X into Y and for every mapping g from Y into Z holds $(g \cdot f) \restriction X_0 = g \cdot (f \restriction X_0)$.

(70) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every mapping g from Y into Z and for every mapping f from X into Y such that g is continuous and $f \restriction X_0$ is continuous holds $(g \cdot f) \restriction X_0$ is continuous.

(71) For all topological spaces X, Y, Z and for every subspace X_0 of X and for every continuous mapping g from Y into Z and for every mapping f from X into Y such that $f \restriction X_0$ is a continuous mapping from X_0 into Y holds $(g \cdot f) \restriction X_0$ is a continuous mapping from X_0 into Z.

Let X, Y be topological spaces, and let X_0, X_1 be subspaces of X, and let g be a mapping from X_0 into Y. Let us assume that X_1 is a subspace of X_0. The functor $g \restriction X_1$ yielding a mapping from X_1 into Y is defined as follows:

(Def.4) $g \restriction X_1 = g \restriction$ the carrier of X_1.

For simplicity we follow a convention: X, Y denote topological spaces, X_0, X_1 denote subspaces of X, f denotes a mapping from X into Y, and g denotes a mapping from X_0 into Y. The following propositions are true:

(72) If X_1 is a subspace of X_0, then for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = (g \restriction X_1)(x_0)$.

(73) If X_1 is a subspace of X_0, then for every mapping g_1 from X_1 into Y such that for every point x_0 of X_0 such that $x_0 \in$ the carrier of X_1 holds $g(x_0) = g_1(x_0)$ holds $g \restriction X_1 = g_1$.

(74) $g = g \restriction X_0$.

(75) If X_1 is a subspace of X_0, then for every subset A of X_0 such that $A \subseteq$ the carrier of X_1 holds $g\,^\circ\,A = (g \restriction X_1)\,^\circ\,A$.

(76) If X_1 is a subspace of X_0, then for every subset B of Y such that $g^{-1}\,B \subseteq$ the carrier of X_1 holds $g^{-1}\,B = (g \restriction X_1)^{-1}\,B$.

(77) For every mapping g from X_0 into Y such that $g = f \restriction X_0$ holds if X_1 is a subspace of X_0, then $g \restriction X_1 = f \restriction X_1$.

(78) If X_1 is a subspace of X_0, then $f \restriction X_0 \restriction X_1 = f \restriction X_1$.

(79) For all subspaces X_0, X_1, X_2 of X such that X_1 is a subspace of X_0 and X_2 is a subspace of X_1 and for every mapping g from X_0 into Y holds $g \restriction X_1 \restriction X_2 = g \restriction X_2$.

(80) For every mapping f from X into Y and for every mapping f_0 from X_1 into Y and for every mapping g from X_0 into Y such that $X_0 = X$ and $f = g$ holds $g \restriction X_1 = f_0$ if and only if $f \restriction X_1 = f_0$.

We follow the rules: X_0, X_1, X_2 are subspaces of X, f is a mapping from X into Y, and g is a mapping from X_0 into Y. One can prove the following propositions:

(81) For every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if X_1 is a subspace of X_0 and g is continuous at x_0, then $g \restriction X_1$ is continuous at x_1.

(82) If X_1 is a subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if $f \restriction X_0$ is continuous at x_0, then $f \restriction X_1$ is continuous at x_1.

(83) If X_1 is a subspace of X_0, then for every subset A of X_0 and for every point x_0 of X_0 and for every point x_1 of X_1 such that $A \subseteq$ the carrier of

X_1 and A is a neighbourhood of x_0 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \restriction X_1$ is continuous at x_1.

(84) If X_1 is a subspace of X_0, then for every subset A of X_0 and for every point x_0 of X_0 and for every point x_1 of X_1 such that A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \restriction X_1$ is continuous at x_1.

(85) If X_1 is a subspace of X_0, then for every subset A of X and for every point x_0 of X_0 and for every point x_1 of X_1 such that A is open and $x_0 \in A$ and $A \subseteq$ the carrier of X_1 and $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \restriction X_1$ is continuous at x_1.

(86) If X_1 is an open subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \restriction X_1$ is continuous at x_1.

(87) If X_1 is an open subspace of X and X_1 is a subspace of X_0, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds g is continuous at x_0 if and only if $g \restriction X_1$ is continuous at x_1.

(88) If the topological structure of $X_1 = X_0$, then for every point x_0 of X_0 and for every point x_1 of X_1 such that $x_0 = x_1$ holds if $g \restriction X_1$ is continuous at x_1, then g is continuous at x_0.

(89) For every continuous mapping g from X_0 into Y such that X_1 is a subspace of X_0 holds $g \restriction X_1$ is a continuous mapping from X_1 into Y.

(90) If X_1 is a subspace of X_0 and X_2 is a subspace of X_1, then for every mapping g from X_0 into Y such that $g \restriction X_1$ is a continuous mapping from X_1 into Y holds $g \restriction X_2$ is a continuous mapping from X_2 into Y.

Let X be a topological space. The functor id_X yielding a mapping from X into X is defined as follows:

(Def.5) $\mathrm{id}_X = \mathrm{id}_{(\text{the carrier of } X)}$.

One can prove the following four propositions:

(91) For every point x of X holds $\mathrm{id}_X(x) = x$.

(92) For every mapping f from X into X such that for every point x of X holds $f(x) = x$ holds $f = \mathrm{id}_X$.

(93) For every mapping f from X into Y holds $f \cdot \mathrm{id}_X = f$ and $\mathrm{id}_Y \cdot f = f$.

(94) id_X is a continuous mapping from X into X.

We now define two new functors. Let X be a topological space, and let X_0 be a subspace of X. The functor $\overset{X_0}{\hookrightarrow}$ yielding a mapping from X_0 into X is defined by:

(Def.6) $\overset{X_0}{\hookrightarrow} = \mathrm{id}_X \restriction X_0$.

We introduce the functor $X_0 \hookrightarrow X$ as a synonym of $\overset{X_0}{\hookrightarrow}$.

Next we state four propositions:

(95) For every subspace X_0 of X and for every point x of X such that $x \in$ the carrier of X_0 holds $(\overset{X_0}{\hookrightarrow})(x) = x$.

(96) For every subspace X_0 of X and for every mapping f_0 from X_0 into X such that for every point x of X such that $x \in$ the carrier of X_0 holds $x = f_0(x)$ holds $\overset{X_0}{\hookrightarrow} = f_0$.

(97) For every subspace X_0 of X and for every mapping f from X into Y holds $f \restriction X_0 = f \cdot (\overset{X_0}{\hookrightarrow})$.

(98) For every subspace X_0 of X holds $\overset{X_0}{\hookrightarrow}$ is a continuous mapping from X_0 into X.

4. A Modification of the Topology of Topological Spaces

In the sequel X will denote a topological space and H, G will denote subsets of X. Let us consider X, and let A be a subset of X. The A-extension of the topology of X yielding a family of subsets of X is defined as follows:

(Def.7) the A-extension of the topology of $X = \{H \cup G \cap A : H \in$ the topology of $X \wedge G \in$ the topology of $X\}$.

We now state several propositions:

(99) For every subset A of X holds the topology of $X \subseteq$ the A-extension of the topology of X.

(100) For every subset A of X holds $\{G \cap A : G \in$ the topology of $X\} \subseteq$ the A-extension of the topology of X, where G ranges over subsets of X.

(101) For every subset A of X and for all subsets C, D of X such that $C \in$ the topology of X and $D \in \{G \cap A : G \in$ the topology of $X\}$, where G ranges over subsets of X holds $C \cup D \in$ the A-extension of the topology of X and $C \cap D \in$ the A-extension of the topology of X.

(102) For every subset A of X holds $A \in$ the A-extension of the topology of X.

(103) For every subset A of X holds $A \in$ the topology of X if and only if the topology of $X =$ the A-extension of the topology of X.

Let X be a topological space, and let A be a subset of X. The X modified w.r.t. A yields a strict topological space and is defined by:

(Def.8) the X modified w.r.t. $A = \langle$the carrier of X, the A-extension of the topology of $X\rangle$.

In the sequel A will be a subset of X. The following three propositions are true:

(104) The carrier of the X modified w.r.t. $A =$ the carrier of X and the topology of the X modified w.r.t. $A =$ the A-extension of the topology of X.

(105) For every subset B of the X modified w.r.t. A such that $B = A$ holds B is open.

(106) A is open if and only if the topological structure of $X =$ the X modified w.r.t. A.

Let X be a topological space, and let A be a subset of X. The functor $\mathrm{modid}_{X,A}$ yields a mapping from X into the X modified w.r.t. A and is defined as follows:

(Def.9) $\mathrm{modid}_{X,A} = \mathrm{id}_{(\text{the carrier of } X)}$.

We now state several propositions:

(107) If A is open, then $\mathrm{modid}_{X,A} = \mathrm{id}_X$.

(108) For every point x of X such that $x \notin A$ holds $\mathrm{modid}_{X,A}$ is continuous at x.

(109) For every subspace X_0 of X such that (the carrier of $X_0) \cap A = \emptyset$ and for every point x_0 of X_0 holds $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(110) For every subspace X_0 of X such that the carrier of $X_0 = A$ and for every point x_0 of X_0 holds $\mathrm{modid}_{X,A} \restriction X_0$ is continuous at x_0.

(111) For every subspace X_0 of X such that (the carrier of $X_0) \cap A = \emptyset$ holds $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. A.

(112) For every subspace X_0 of X such that the carrier of $X_0 = A$ holds $\mathrm{modid}_{X,A} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. A.

(113) For every subset A of X holds A is open if and only if $\mathrm{modid}_{X,A}$ is a continuous mapping from X into the X modified w.r.t. A.

Let X be a topological space, and let X_0 be a subspace of X. The X modified w.r.t. X_0 yielding a strict topological space is defined as follows:

(Def.10) for every subset A of X such that $A =$ the carrier of X_0 holds the X modified w.r.t. $X_0 =$ the X modified w.r.t. A.

In the sequel X_0 will denote a subspace of X. The following three propositions are true:

(114) The carrier of the X modified w.r.t. $X_0 =$ the carrier of X and for every subset A of X such that $A =$ the carrier of X_0 holds the topology of the X modified w.r.t. $X_0 =$ the A-extension of the topology of X.

(115) For every subspace Y_0 of the X modified w.r.t. X_0 such that the carrier of $Y_0 =$ the carrier of X_0 holds Y_0 is an open subspace of the X modified w.r.t. X_0.

(116) X_0 is an open subspace of X if and only if the topological structure of $X =$ the X modified w.r.t. X_0.

Let X be a topological space, and let X_0 be a subspace of X. The functor modid_{X,X_0} yielding a mapping from X into the X modified w.r.t. X_0 is defined as follows:

(Def.11) for every subset A of X such that $A =$ the carrier of X_0 holds $\mathrm{modid}_{X,X_0} = \mathrm{modid}_{X,A}$.

We now state several propositions:

(117) If X_0 is an open subspace of X, then $\mathrm{modid}_{X,X_0} = \mathrm{id}_X$.

(118) For all subspaces X_0, X_1 of X such that X_0 misses X_1 and for every point x_1 of X_1 holds $\mathrm{modid}_{X,X_0} \restriction X_1$ is continuous at x_1.

(119) For every subspace X_0 of X and for every point x_0 of X_0 holds $\mathrm{modid}_{X,X_0} \restriction X_0$
is continuous at x_0.

(120) For all subspaces X_0, X_1 of X such that X_0 misses X_1 holds $\mathrm{modid}_{X,X_0} \restriction X_1$ is a continuous mapping from X_1 into the X modified w.r.t. X_0.

(121) For every subspace X_0 of X holds $\mathrm{modid}_{X,X_0} \restriction X_0$ is a continuous mapping from X_0 into the X modified w.r.t. X_0.

(122) For every subspace X_0 of X holds X_0 is an open subspace of X if and only if modid_{X,X_0} is a continuous mapping from X into the X modified w.r.t. X_0.

5. Continuity of Mappings over the Union of Subspaces

In the sequel X, Y denote topological spaces. We now state three propositions:

(123) For all subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y and for every point x_1 of X_1 and for every point x_2 of X_2 and for every point x of $X_1 \cup X_2$ such that $x = x_1$ and $x = x_2$ holds g is continuous at x if and only if $g \restriction X_1$ is continuous at x_1 and $g \restriction X_2$ is continuous at x_2.

(124) Let f be a mapping from X into Y. Then for all subspaces X_1, X_2 of X and for every point x of $X_1 \cup X_2$ and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x = x_1$ and $x = x_2$ holds $f \restriction (X_1 \cup X_2)$ is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

(125) Let f be a mapping from X into Y. Then for all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every point x of X and for every point x_1 of X_1 and for every point x_2 of X_2 such that $x = x_1$ and $x = x_2$ holds f is continuous at x if and only if $f \restriction X_1$ is continuous at x_1 and $f \restriction X_2$ is continuous at x_2.

In the sequel X_1, X_2 will denote subspaces of X. One can prove the following propositions:

(126) If X_1 and X_2 are weakly separated, then for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(127) For all closed subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(128) For all open subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y.

(129) If X_1 and X_2 are weakly separated, then for every mapping f from X into Y holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(130) For every mapping f from X into Y and for all closed subspaces X_1, X_2 of X holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(131) For every mapping f from X into Y and for all open subspaces X_1, X_2 of X holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(132) For every mapping f from X into Y and for all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and X_1 and X_2 are weakly separated holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(133) For every mapping f from X into Y and for all closed subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(134) For every mapping f from X into Y and for all open subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds f is a continuous mapping from X into Y if and only if $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y.

(135) X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping g from $X_1 \cup X_2$ into Y such that $g \restriction X_1$ is a continuous mapping from X_1 into Y and $g \restriction X_2$ is a continuous mapping from X_2 into Y holds g is a continuous mapping from $X_1 \cup X_2$ into Y.

(136) X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping f from X into Y such that $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping from X_2 into Y holds $f \restriction (X_1 \cup X_2)$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(137) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ holds X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every mapping f from X into Y such that $f \restriction X_1$ is a continuous mapping from X_1 into Y and $f \restriction X_2$ is a continuous mapping

from X_2 into Y holds f is a continuous mapping from X into Y.

6. The Union of Continuous Mappings

Let X, Y be topological spaces, and let X_1, X_2 be subspaces of X, and let f_1 be a mapping from X_1 into Y, and let f_2 be a mapping from X_2 into Y. Let us assume that X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. The functor $f_1 \cup f_2$ yielding a mapping from $X_1 \cup X_2$ into Y is defined as follows:

(Def.12) $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$.

In the sequel X, Y will denote topological spaces. We now state a number of propositions:

(138) For all subspaces X_1, X_2 of X and for every mapping g from $X_1 \cup X_2$ into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(139) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every mapping g from X into Y holds $g = g \restriction X_1 \cup g \restriction X_2$.

(140) For all subspaces X_1, X_2 of X such that X_1 meets X_2 and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y holds $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ if and only if $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$.

(141) For all subspaces X_1, X_2 of X and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds X_1 is a subspace of X_2 if and only if $f_1 \cup f_2 = f_2$ but X_2 is a subspace of X_1 if and only if $f_1 \cup f_2 = f_1$.

(142) For all subspaces X_1, X_2 of X and for every mapping f_1 from X_1 into Y and for every mapping f_2 from X_2 into Y such that X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds $f_1 \cup f_2 = f_2 \cup f_1$.

(143) Let X_1, X_2, X_3 be subspaces of X. Let f_1 be a mapping from X_1 into Y. Let f_2 be a mapping from X_2 into Y. Let f_3 be a mapping from X_3 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ but X_1 misses X_3 or $f_1 \restriction (X_1 \cap X_3) = f_3 \restriction (X_1 \cap X_3)$ but X_2 misses X_3 or $f_2 \restriction (X_2 \cap X_3) = f_3 \restriction (X_2 \cap X_3)$. Then $(f_1 \cup f_2) \cup f_3 = f_1 \cup (f_2 \cup f_3)$.

(144) For all subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds if X_1 and X_2 are weakly separated, then $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(145) For all subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that X_1 and X_2 are weakly separated holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(146) For all closed subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous

mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(147) For all open subspaces X_1, X_2 of X such that X_1 meets X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$ holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(148) For all closed subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(149) For all open subspaces X_1, X_2 of X such that X_1 misses X_2 and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(150) For all subspaces X_1, X_2 of X holds X_1 and X_2 are separated if and only if X_1 misses X_2 and for every topological space Y and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y holds $f_1 \cup f_2$ is a continuous mapping from $X_1 \cup X_2$ into Y.

(151) For all subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$ and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ holds if X_1 and X_2 are weakly separated, then $f_1 \cup f_2$ is a continuous mapping from X into Y.

(152) For all closed subspaces X_1, X_2 of X and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ holds $f_1 \cup f_2$ is a continuous mapping from X into Y.

(153) For all open subspaces X_1, X_2 of X and for every continuous mapping f_1 from X_1 into Y and for every continuous mapping f_2 from X_2 into Y such that $X = X_1 \cup X_2$ and $(f_1 \cup f_2) \restriction X_1 = f_1$ and $(f_1 \cup f_2) \restriction X_2 = f_2$ holds $f_1 \cup f_2$ is a continuous mapping from X into Y.

Acknowledgments

I would like to express my thanks to Professors A. Trybulec and Cz. Byliński for their active interest in the publication of this article and for elucidating to me new advanced Mizar constructions.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[5] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.

[6] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[7] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[9] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.

[10] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[12] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[15] Eduard Čech. *Topological Spaces.* Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1966.

[16] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received May 22, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Functional Sequence from a Domain to a Domain

Beata Perkowska
Warsaw University
Białystok

Summary. Definitions of functional sequences and basic operations on functional sequences from a domain to a domain, point and uniform convergent, limit of functional sequence from a domain to the set of real numbers and facts about properties of the limit of functional sequences are proved.

MML Identifier: `SEQFUNC`.

The articles [11], [1], [2], [3], [13], [5], [6], [9], [8], [4], [12], [7], and [10] provide the notation and terminology for this paper. For simplicity we adopt the following rules: D, D_1, D_2 denote non-empty sets, n, k denote natural numbers, p, r denote real numbers, and f denotes a function. Let us consider D_1, D_2. A function is called a sequence of partial functions from D_1 into D_2 if:

(Def.1) $\operatorname{dom} \mathrm{it} = \mathbb{N}$ and $\operatorname{rng} \mathrm{it} \subseteq D_1 \dot{\rightarrow} D_2$.

In the sequel F, F_1, F_2 are sequences of partial functions from D_1 into D_2. Let us consider D_1, D_2, F, n. Then $F(n)$ is a partial function from D_1 to D_2.

In the sequel G, H, H_1, H_2, J are sequences of partial functions from D into $\mathbb{R}$. One can prove the following two propositions:

(1) f is a sequence of partial functions from D_1 into D_2 if and only if $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is a partial function from D_1 to D_2.

(2) For all F_1, F_2 such that for every n holds $F_1(n) = F_2(n)$ holds $F_1 = F_2$.

The scheme *ExFuncSeq* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a partial function from $\mathcal{A}$ to $\mathcal{B}$ and states that:

there exists a sequence G of partial functions from $\mathcal{A}$ into $\mathcal{B}$ such that for every n holds $G(n) = \mathcal{F}(n)$

for all values of the parameters.

ISSN 0777–4028

We now define several new functors. Let us consider D, H, r. The functor $r\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.2) for every n holds $(r\,H)(n) = r\,H(n)$.

Let us consider D, H. The functor H^{-1} yielding a sequence of partial functions from D into $\mathbb{R}$ is defined by:

(Def.3) for every n holds $H^{-1}(n) = \frac{1}{H(n)}$.

The functor $-H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def.4) for every n holds $(-H)(n) = -H(n)$.

The functor $|H|$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.5) for every n holds $|H|(n) = |H(n)|$.

Let us consider D, G, H. The functor $G + H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined by:

(Def.6) for every n holds $(G + H)(n) = G(n) + H(n)$.

The functor $G - H$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def.7) $G - H = G + -H$.

The functor $G\,H$ yields a sequence of partial functions from D into $\mathbb{R}$ and is defined as follows:

(Def.8) for every n holds $(G\,H)(n) = G(n)\,H(n)$.

Let us consider D, H, G. The functor $\frac{G}{H}$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined as follows:

(Def.9) $\frac{G}{H} = G\,H^{-1}$.

Next we state a number of propositions:

(3) $H_1 = \frac{G}{H}$ if and only if for every n holds $H_1(n) = \frac{G(n)}{H(n)}$.

(4) $H_1 = G - H$ if and only if for every n holds $H_1(n) = G(n) - H(n)$.

(5) $G + H = H + G$ and $(G + H) + J = G + (H + J)$.

(6) $G\,H = H\,G$ and $(G\,H)\,J = G\,(H\,J)$.

(7) $(G + H)\,J = G\,J + H\,J$ and $J\,(G + H) = J\,G + J\,H$.

(8) $-H = (-1)\,H$.

(9) $(G - H)\,J = G\,J - H\,J$ and $J\,G - J\,H = J\,(G - H)$.

(10) $r\,(G + H) = r\,G + r\,H$ and $r\,(G - H) = r\,G - r\,H$.

(11) $(r \cdot p)\,H = r\,(p\,H)$.

(12) $1\,H = H$.

(13) $--H = H$.

(14) $G^{-1}\,H^{-1} = (G\,H)^{-1}$.

(15) If $r \neq 0$, then $(r\,H)^{-1} = r^{-1}\,H^{-1}$.

(16) $|H|^{-1} = |H^{-1}|$.

(17) $|G\,H| = |G|\,|H|$.

(18) $|\frac{G}{H}| = \frac{|G|}{|H|}$.

(19) $|r\,H| = |r|\,|H|$.

In the sequel x is an element of D, X, Y are sets, and f is a partial function from D to $\mathbb{R}$. We now define three new constructions. Let us consider D_1, D_2, F, X. We say that X is common for elements of F if and only if:

(Def.10) $X \neq \emptyset$ and for every n holds $X \subseteq \operatorname{dom} F(n)$.

Let us consider D, H, x. The functor $H\#x$ yielding a sequence of real numbers is defined as follows:

(Def.11) for every n holds $(H\#x)(n) = H(n)(x)$.

Let us consider D, H, X. We say that H is point-convergent on X if and only if:

(Def.12) X is common for elements of H and there exists f such that $X = \operatorname{dom} f$ and for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

Next we state two propositions:

(20) H is point-convergent on X if and only if X is common for elements of H and there exists f such that $X = \operatorname{dom} f$ and for every x such that $x \in X$ holds $H\#x$ is convergent and $\lim(H\#x) = f(x)$.

(21) H is point-convergent on X if and only if X is common for elements of H and for every x such that $x \in X$ holds $H\#x$ is convergent.

We now define two new constructions. Let us consider D, H, X. We say that H is uniform-convergent on X if and only if:

(Def.13) X is common for elements of H and there exists f such that $X = \operatorname{dom} f$ and for every p such that $p > 0$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - f(x)| < p$.

Let us assume that H is point-convergent on X. The functor $\lim_X H$ yielding a partial function from D to $\mathbb{R}$ is defined as follows:

(Def.14) $\operatorname{dom} \lim_X H = X$ and for every x such that $x \in \operatorname{dom} \lim_X H$ holds $(\lim_X H)(x) = \lim(H\#x)$.

We now state a number of propositions:

(22) If H is point-convergent on X, then $f = \lim_X H$ if and only if $\operatorname{dom} f = X$ and for every x such that $x \in X$ and for every p such that $p > 0$ there exists k such that for every n such that $n \geq k$ holds $|H(n)(x) - f(x)| < p$.

(23) If H is uniform-convergent on X, then H is point-convergent on X.

(24) If $Y \subseteq X$ and $Y \neq \emptyset$ and X is common for elements of H, then Y is common for elements of H.

(25) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is point-convergent on X, then H is point-convergent on Y and $\lim_X H \restriction Y = \lim_Y H$.

(26) If $Y \subseteq X$ and $Y \neq \emptyset$ and H is uniform-convergent on X, then H is uniform-convergent on Y.

(27) If X is common for elements of H, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(28) If H is point-convergent on X, then for every x such that $x \in X$ holds $\{x\}$ is common for elements of H.

(29) Suppose $\{x\}$ is common for elements of H_1 and $\{x\}$ is common for elements of H_2. Then $H_1\#x+H_2\#x = (H_1+H_2)\#x$ and $H_1\#x-H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(30) If $\{x\}$ is common for elements of H, then $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(31) If $\{x\}$ is common for elements of H, then $(r\,H)\#x = r\,(H\#x)$.

(32) Suppose X is common for elements of H_1 and X is common for elements of H_2. Then for every x such that $x \in X$ holds $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(33) If X is common for elements of H, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(34) If X is common for elements of H, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(35) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then for every x such that $x \in X$ holds $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$ and $(H_1\#x)\,(H_2\#x) = (H_1\,H_2)\#x$.

(36) If H is point-convergent on X, then for every x such that $x \in X$ holds $|H|\#x = |H\#x|$ and $(-H)\#x = -H\#x$.

(37) If H is point-convergent on X, then for every x such that $x \in X$ holds $(r\,H)\#x = r\,(H\#x)$.

(38) If X is common for elements of H_1 and X is common for elements of H_2, then X is common for elements of $H_1 + H_2$ and X is common for elements of $H_1 - H_2$ and X is common for elements of $H_1\,H_2$.

(39) If X is common for elements of H, then X is common for elements of $|H|$ and X is common for elements of $-H$.

(40) If X is common for elements of H, then X is common for elements of $r\,H$.

(41) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then
- (i) $H_1 + H_2$ is point-convergent on X,
- (ii) $\lim_X(H_1 + H_2) = \lim_X H_1 + \lim_X H_2$,
- (iii) $H_1 - H_2$ is point-convergent on X,
- (iv) $\lim_X(H_1 - H_2) = \lim_X H_1 - \lim_X H_2$,
- (v) $H_1\,H_2$ is point-convergent on X,
- (vi) $\lim_X(H_1\,H_2) = \lim_X H_1 \lim_X H_2$.

(42) If H is point-convergent on X, then $|H|$ is point-convergent on X and $\lim_X|H| = |\lim_X H|$ and $-H$ is point-convergent on X and $\lim_X(-H) =$

$-\lim_X H$.

(43) If H is point-convergent on X, then $r\,H$ is point-convergent on X and $\lim_X(r\,H) = r\lim_X H$.

(44) H is uniform-convergent on X if and only if X is common for elements of H and H is point-convergent on X and for every r such that $0 < r$ there exists k such that for all n, x such that $n \geq k$ and $x \in X$ holds $|H(n)(x) - (\lim_X H)(x)| < r$.

In the sequel H will be a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$. Let us consider n, k. Then $\max(n,k)$ is a natural number.

We now state the proposition

(45) If H is uniform-convergent on X and for every n holds $H(n)$ is continuous on X, then $\lim_X H$ is continuous on X.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[5] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.

[6] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.

[7] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[9] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[10] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[12] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received May 22, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Reper Algebras

Michał Muzalewski
Warsaw University
Białystok

Summary. We shall describe n-dimensional spaces with the reper operation [10, pages 72–79]. An inspiration to such approach comes from the monograph [12] and so-called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes. The Euclidean geometry in Szmielew's approach [12] is a theory of structures $\langle S; \parallel, \oplus, O\rangle$, where $\langle S; \parallel, \oplus, O\rangle$ is Desarguean midpoint plane and $O \subseteq S \times S \times S$ is the relation of equi-orthogonal basis. Points o, p, q are in relation O if they form an isosceles triangle with the right angle in vertex a. If we fix vertices a, p, then there exist exactly two points q, q' such that $O(apq)$, $O(apq')$. Moreover $q \oplus q' = a$. In accordance with the Leibniz program we replace the relation of equi-orthogonal basis by a binary operation $* : S \times S \to S$, called the reper operation. A standard model for the Euclidean geometry in the above sense is the oriented plane over the field of real numbers with the reper operations $*$ defined by the condition: $a * b = q$ iff the point q is the result of rotating of p about right angle around the center a.

MML Identifier: `MIDSP_3`.

The terminology and notation used here are introduced in the following articles: [13], [5], [6], [3], [7], [2], [4], [1], [8], [11], and [9].

1. Substitutions in tuples

For simplicity we adopt the following rules: n, i, j, k, l are natural numbers, D is a non-empty set, c, d are elements of D, and p, q, r are finite sequences of elements of D. The following propositions are true:

(1) If $\operatorname{len} p = j + 1 + k$, then there exist q, r, c such that $\operatorname{len} q = j$ and $\operatorname{len} r = k$ and $p = q \frown \langle c\rangle \frown r$.

(2) If $i \in \operatorname{Seg} n$, then there exist j, k such that $n = j + 1 + k$ and $i = j + 1$.

ISSN 0777–4028

(3) Suppose $p = q \frown \langle c \rangle \frown r$ and $i = \operatorname{len} q + 1$. Then for every l such that $1 \leq l$ and $l \leq \operatorname{len} q$ holds $p(l) = q(l)$ and $p(i) = c$ and for every l such that $i + 1 \leq l$ and $l \leq \operatorname{len} p$ holds $p(l) = r(l - i)$.

(4) $l \leq j$ or $l = j + 1$ or $j + 2 \leq l$.

(5) If $l \in \operatorname{Seg} n \setminus \{i\}$ and $i = j + 1$, then $1 \leq l$ and $l \leq j$ or $i + 1 \leq l$ and $l \leq n$.

Let us consider n, i, D, d, and let p be an element of D^{n+1}. Let us assume that $i \in \operatorname{Seg}(n + 1)$. The functor $p(i/d)$ yielding an element of D^{n+1} is defined as follows:

(Def.1) $p(i/d)(i) = d$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} p \setminus \{i\}$ holds $p(i/d)(l) = p(l)$.

2. Reper Algebra Structure and its Properties

Let us consider n. We consider structures of reper algebra over n which are extension of a midpoint algebra structure and are systems

⟨a carrier, a midpoint operation, a reper⟩,

where the carrier is a non-empty set, the midpoint operation is a binary operation on the carrier, and the reper is a function from (the carrier)^n into the carrier. Let us observe that there exists a structure of reper algebra over $n + 2$ which is midpoint algebra-like.

We adopt the following rules: R_1 will denote a midpoint algebra-like structure of reper algebra over $n + 2$ and a, b, d, p_1, p'_1 will denote points of R_1. We now define two new modes. Let us consider i, D. A tuple of i and D is an element of D^i.

Let us consider n, R_1, i. A tuple of i and R_1 is a tuple of i and the carrier of R_1.

In the sequel p, q will denote tuples of $n + 1$ and R_1. Let us consider n, R_1, a. Then $\langle a \rangle$ is a tuple of 1 and R_1. Let us consider n, R_1, i, j, and let p be a tuple of i and R_1, and let q be a tuple of j and R_1. Then $p \frown q$ is a tuple of $i + j$ and R_1.

We now state the proposition

(6) $\langle a \rangle \frown p$ is a tuple of $n + 2$ and R_1.

We now define two new functors. Let us consider n, R_1, a, p. The functor $*(a, p)$ yielding a point of R_1 is defined as follows:

(Def.2) $*(a, p) = (\text{the reper of } R_1)(\langle a \rangle \frown p)$.

Let us consider n, i, R_1, d, p. The functor $p_{\restriction i \dot{\to} d}$ yields a tuple of $n + 1$ and R_1 and is defined as follows:

(Def.3) for every D and for every element p' of D^{n+1} and for every element d' of D such that $D =$ the carrier of R_1 and $p' = p$ and $d' = d$ holds $p_{\restriction i \dot{\to} d} = p'(i/d')$.

We now state the proposition

(7) If $i \in \operatorname{Seg}(n+1)$, then $p_{\restriction i \dot{\rightarrow} d}(i) = d$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} p \setminus \{i\}$ holds $p_{\restriction i \dot{\rightarrow} d}(l) = p(l)$.

Let us consider n. A natural number is said to be a natural number of n if:

(Def.4) $1 \leq$ it and it $\leq n+1$.

In the sequel m is a natural number of n. We now state several propositions:

(8) i is a natural number of n if and only if $i \in \operatorname{Seg}(n+1)$.

(9) $1 \leq i+1$.

(10) If $i \leq n$, then $i+1$ is a natural number of n.

(11) If for every m holds $p(m) = q(m)$, then $p = q$.

(12) For every natural number l of n such that $l = i$ holds $p_{\restriction i \dot{\rightarrow} d}(l) = d$ and for all natural numbers l, i of n such that $l \neq i$ holds $p_{\restriction i \dot{\rightarrow} d}(l) = p(l)$.

We now define three new predicates. Let us consider n, D, and let p be an element of D^{n+1}, and let us consider m. Then $p(m)$ is an element of D. Let us consider n, R_1. We say that R_1 is invariance if and only if:

(Def.5) for all a, b, p, q such that for every m holds $a \oplus q(m) = b \oplus p(m)$ holds $a \oplus *(b, q) = b \oplus *(a, p)$.

Let us consider n, i, R_1. We say that R_1 has property of zero in i if and only if:

(Def.6) for all a, p holds $*(a, p_{\restriction i \dot{\rightarrow} a}) = a$.

We say that R_1 is semi additive in i if and only if:

(Def.7) for all a, p_1, p such that $p(i) = p_1$ holds $*(a, p_{\restriction i \dot{\rightarrow} a \oplus p_1}) = a \oplus *(a, p)$.

The following proposition is true

(13) If R_1 is semi additive in m, then for all a, d, p, q such that $q = p_{\restriction m \dot{\rightarrow} d}$ holds $*(a, p_{\restriction m \dot{\rightarrow} a \oplus d}) = a \oplus *(a, q)$.

We now define two new predicates. Let us consider n, i, R_1. We say that R_1 is additive in i if and only if:

(Def.8) for all a, p_1, p'_1, p such that $p(i) = p_1$ holds $*(a, p_{\restriction i \dot{\rightarrow} p_1 \oplus p'_1}) = *(a, p) \oplus *(a, p_{\restriction i \dot{\rightarrow} p'_1})$.

We say that R_1 is alternative in i if and only if:

(Def.9) for all a, p, p_1 such that $p(i) = p_1$ holds $*(a, p_{\restriction i+1 \dot{\rightarrow} p_1}) = a$.

In the sequel W is an atlas of R_1 and v is a vector of W. Let us consider n, R_1, W, i. A tuple of i and W is a tuple of i and the carrier of the algebra of W.

In the sequel x, y are tuples of $n+1$ and W. Let us consider n, R_1, W, x, i, v. The functor $x_{\restriction i \dot{\rightarrow} v}$ yields a tuple of $n+1$ and W and is defined by:

(Def.10) for every D and for every element x' of D^{n+1} and for every element v' of D such that $D =$ the carrier of the algebra of W and $x' = x$ and $v' = v$ holds $x_{\restriction i \dot{\rightarrow} v} = x'(i/v')$.

Next we state three propositions:

(14) If $i \in \operatorname{Seg}(n+1)$, then $x_{\restriction i \dot{\rightarrow} v}(i) = v$ and for every l such that $l \in \operatorname{Seg} \operatorname{len} x \setminus \{i\}$ holds $x_{\restriction i \dot{\rightarrow} v}(l) = x(l)$.

(15) For every natural number l of n such that $l = i$ holds $x_{\restriction i \dot{\rightarrow} v}(l) = v$ and for all natural numbers l, i of n such that $l \neq i$ holds $x_{\restriction i \dot{\rightarrow} v}(l) = x(l)$.

(16) If for every m holds $x(m) = y(m)$, then $x = y$.

The scheme *SeqLambdaD'* concerns a natural number $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a finite sequence z of elements of $\mathcal{B}$ such that $\operatorname{len} z = \mathcal{A} + 1$ and for every natural number j of $\mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

We now define two new functors. Let us consider n, R_1, W, a, x. The functor $(a, x).W$ yielding a tuple of $n + 1$ and R_1 is defined as follows:

(Def.11) $((a, x).W)(m) = (a, x(m)).W$.

Let us consider n, R_1, W, a, p. The functor $W(a,\, p)$ yielding a tuple of $n + 1$ and W is defined by:

(Def.12) $W(a,\, p)(m) = W(a,\, p(m))$.

The following three propositions are true:

(17) $W(a,\, p) = x$ if and only if $(a, x).W = p$.

(18) $W(a,\, (a, x).W) = x$.

(19) $(a, W(a,\, p)).W = p$.

Let us consider n, R_1, W, a, x. The functor $\Phi(a, x)$ yields a vector of W and is defined by:

(Def.13) $\Phi(a, x) = W(a,\, *(a, (a, x).W))$.

One can prove the following propositions:

(20) If $W(a,\, p) = x$ and $W(a,\, b) = v$, then $*(a, p) = b$ if and only if $\Phi(a, x) = v$.

(21) R_1 is invariance if and only if for all a, b, x holds $\Phi(a, x) = \Phi(b, x)$.

(22) $1 \in \operatorname{Seg}(n + 1)$.

(23) 1 is an element of $\operatorname{Seg}(n + 1)$.

(24) 1 is a natural number of n.

3. Reper Algebra and its Atlas

Let us consider n. A midpoint algebra-like structure of reper algebra over $n + 2$ is called a reper algebra of n if:

(Def.14) it is invariance.

For simplicity we adopt the following convention: R_1 will be a reper algebra of n, a, b will be points of R_1, p will be a tuple of $n + 1$ and R_1, W will be an atlas of R_1, v will be a vector of W, and x will be a tuple of $n + 1$ and W. Next we state the proposition

(25) $\Phi(a, x) = \Phi(b, x)$.

Let us consider n, R_1, W, x. The functor $\Phi(x)$ yields a vector of W and is defined by:

(Def.15) for every a holds $\Phi(x) = \Phi(a, x)$.

We now state a number of propositions:

(26) If $W(a, p) = x$ and $W(a, b) = v$ and $\Phi(x) = v$, then $*(a, p) = b$.

(27) If $(a, x).W = p$ and $(a, v).W = b$ and $*(a, p) = b$, then $\Phi(x) = v$.

(28) If $W(a, p) = x$ and $W(a, b) = v$, then $W(a, p_{\restriction m \dot{\to} b}) = x_{\restriction m \dot{\to} v}$.

(29) If $(a, x).W = p$ and $(a, v).W = b$, then $(a, x_{\restriction m \dot{\to} v}).W = p_{\restriction m \dot{\to} b}$.

(30) R_1 has property of zero in m if and only if for every x holds $\Phi((x_{\restriction m \dot{\to} 0_W})) = 0_W$.

(31) R_1 is semi additive in m if and only if for every x holds $\Phi((x_{\restriction m \dot{\to} 2x(m)})) = 2\,\Phi(x)$.

(32) If R_1 has property of zero in m and R_1 is additive in m, then R_1 is semi additive in m.

(33) If R_1 has property of zero in m, then R_1 is additive in m if and only if for all x, v holds $\Phi((x_{\restriction m \dot{\to} x(m)+v})) = \Phi(x) + \Phi((x_{\restriction m \dot{\to} v}))$.

(34) If $W(a, p) = x$ and $m \leq n$, then $W(a, p_{\restriction m+1 \dot{\to} p(m)}) = x_{\restriction m+1 \dot{\to} x(m)}$.

(35) If $(a, x).W = p$ and $m \leq n$, then $(a, x_{\restriction m+1 \dot{\to} x(m)}).W = p_{\restriction m+1 \dot{\to} p(m)}$.

(36) If $m \leq n$, then R_1 is alternative in m if and only if for every x holds $\Phi((x_{\restriction m+1 \dot{\to} x(m)})) = 0_W$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[9] Michał Muzalewski. Atlas of Midpoint Algebra. *Formalized Mathematics*, 2(**4**):487–491, 1991.

[10] Michał Muzalewski. *Foundations of Metric-Affine Geometry*. Dział Wydawnictw Filii UW w Białymstoku, Filia UW w Białymstoku, 1990.

[11] Michał Muzalewski. Midpoint algebras. *Formalized Mathematics*, 1(**3**):483–488, 1990.

[12] Wanda Szmielew. *From Affine to Euclidean Geometry*. Volume 27, PWN – D.Reidel Publ. Co., Warszawa – Dordrecht, 1983.

[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received May 28, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Isomorphisms of Cyclic Groups. Some Properties of Cyclic Groups

Dariusz Surowik
Warsaw University
Białystok

Summary. Some theorems and properties of cyclic groups have been proved with special regard to isomorphisms of these groups. Among other things it has been proved that an arbitrary cyclic group is isomorphic with groups of integers with addition or group of integers with addition modulo m. Moreover, it has been proved that two arbitrary cyclic groups of the same order are isomorphic and that the class of cyclic groups is closed in consideration of homomorphism images. Some other properties of groups of this type have been proved too.

MML Identifier: `GR_CY_2`.

The terminology and notation used in this paper have been introduced in the following articles: [19], [6], [11], [7], [12], [2], [18], [1], [10], [4], [14], [17], [21], [13], [31], [25], [29], [23], [3], [27], [26], [24], [30], [15], [16], [5], [28], [22], [20], [9], and [8]. For simplicity we adopt the following rules: F, G will be groups, G_1 will be a subgroup of G, G_2 will be a cyclic group, H will be a subgroup of G_2, f will be a homomorphism from G to G_2, a, b will be elements of G, g will be an element of G_2, a_1 will be an element of G_1, k, m, n, p, s will be natural numbers, and i, i_1, i_2 will be integers. The following propositions are true:

(1) For all n, m such that $0 < m$ holds $n \bmod m = n - m \cdot (n \div m)$.

(2) If $i_2 > 0$, then $i_1 \bmod i_2 \geq 0$.

(3) If $i_2 > 0$, then $i_1 \bmod i_2 < i_2$.

(4) $i_1 = (i_1 \div i_2) \cdot i_2 + (i_1 \bmod i_2)$.

(5) For all m, n such that $m > 0$ or $n > 0$ there exist i, i_1 such that $i \cdot m + i_1 \cdot n = \gcd(m, n)$.

(6) If $\operatorname{ord}(a) > 1$ and $a = b^k$, then $k \neq 0$.

(7) If G is finite, then $\operatorname{ord}(G) > 0$.

(8) $a \in \operatorname{gr}(\{a\})$.

ISSN 0777–4028

(9) If $a = a_1$, then $\mathrm{gr}(\{a\}) = \mathrm{gr}(\{a_1\})$.

(10) $\mathrm{gr}(\{a\})$ is a cyclic group.

(11) For every strict group G and for every element b of G holds for every element a of G there exists i such that $a = b^i$ if and only if $G = \mathrm{gr}(\{b\})$.

(12) For every strict group G and for every element b of G such that G is finite holds for every element a of G there exists p such that $a = b^p$ if and only if $G = \mathrm{gr}(\{b\})$.

(13) For every strict group G and for every element a of G such that G is finite and $G = \mathrm{gr}(\{a\})$ and for every strict subgroup G_1 of G there exists p such that $G_1 = \mathrm{gr}(\{a^p\})$.

(14) If G is finite and $G = \mathrm{gr}(\{a\})$ and $\mathrm{ord}(G) = n$ and $n = p \cdot s$, then $\mathrm{ord}(a^p) = s$.

(15) If $s \mid k$, then $a^k \in \mathrm{gr}(\{a^s\})$.

(16) If G is finite and $\mathrm{ord}(\mathrm{gr}(\{a^s\})) = \mathrm{ord}(\mathrm{gr}(\{a^k\}))$ and $a^k \in \mathrm{gr}(\{a^s\})$, then $\mathrm{gr}(\{a^s\}) = \mathrm{gr}(\{a^k\})$.

(17) If G is finite and $\mathrm{ord}(G) = n$ and $G = \mathrm{gr}(\{a\})$ and $\mathrm{ord}(G_1) = p$ and $G_1 = \mathrm{gr}(\{a^k\})$, then $n \mid k \cdot p$.

(18) For every strict group G and for every element a of G such that G is finite and $G = \mathrm{gr}(\{a\})$ and $\mathrm{ord}(G) = n$ holds $G = \mathrm{gr}(\{a^k\})$ if and only if $\gcd(k, n) = 1$.

(19) If $G_2 = \mathrm{gr}(\{g\})$ and $g \in H$, then the half group structure of $G_2 =$ the half group structure of H.

(20) If $G_2 = \mathrm{gr}(\{g\})$, then G_2 is finite if and only if there exist i, i_1 such that $i \neq i_1$ and $g^i = g^{i_1}$.

Let us consider n satisfying the condition: $n > 0$. Let h be an element of $\mathbb{Z}_n^+$. The functor ${}^{@}h$ yielding a natural number is defined as follows:

(Def.1) ${}^{@}h = h$.

The following propositions are true:

(21) For every strict cyclic group G_2 such that G_2 is finite and $\mathrm{ord}(G_2) = n$ holds $\mathbb{Z}_n^+$ and G_2 are isomorphic.

(22) For every strict cyclic group G_2 such that G_2 is infinite holds $\mathbb{Z}^+$ and G_2 are isomorphic.

(23) For all strict cyclic groups G_2, H_1 such that H_1 is finite and G_2 is finite and $\mathrm{ord}(H_1) = \mathrm{ord}(G_2)$ holds H_1 and G_2 are isomorphic.

(24) For all strict groups F, G such that F is finite and G is finite and $\mathrm{ord}(F) = p$ and $\mathrm{ord}(G) = p$ and p is prime holds F and G are isomorphic.

(25) For all strict groups F, G such that F is finite and G is finite and $\mathrm{ord}(F) = 2$ and $\mathrm{ord}(G) = 2$ holds F and G are isomorphic.

(26) For every strict group G such that G is finite and $\mathrm{ord}(G) = 2$ and for every strict subgroup H of G holds $H = \{\mathbf{1}\}_G$ or $H = G$.

(27) For every strict group G such that G is finite and $\mathrm{ord}(G) = 2$ holds G is a cyclic group.

(28) For every strict group G such that G is finite and G is a cyclic group and $\mathrm{ord}(G) = n$ and for every p such that $p \mid n$ there exists a strict subgroup G_1 of G such that $\mathrm{ord}(G_1) = p$ and for every strict subgroup G_3 of G such that $\mathrm{ord}(G_3) = p$ holds $G_3 = G_1$.

Let us note that every group which is cyclic is also Abelian.

We now state two propositions:

(29) If $G_2 = \mathrm{gr}(\{g\})$, then for all G, f such that $g \in \mathrm{Im}\, f$ holds f is an epimorphism.

(30) For every strict cyclic group G_2 such that G_2 is finite and $\mathrm{ord}(G_2) = n$ and there exists k such that $n = 2 \cdot k$ there exists an element g_1 of G_2 such that $\mathrm{ord}(g_1) = 2$ and for every element g_2 of G_2 such that $\mathrm{ord}(g_2) = 2$ holds $g_1 = g_2$.

Let us consider G. Then $\mathrm{Z}(G)$ is a strict normal subgroup of G.

One can prove the following propositions:

(31) For every strict cyclic group G_2 such that G_2 is finite and $\mathrm{ord}(G_2) = n$ and there exists k such that $n = 2 \cdot k$ there exists a subgroup H of G_2 such that $\mathrm{ord}(H) = 2$ and H is a cyclic group.

(32) For every strict group G and for every homomorphism g from G to F such that G is a cyclic group holds $\mathrm{Im}\, g$ is a cyclic group.

(33) For all strict groups G, F such that G and F are isomorphic but G is a cyclic group or F is a cyclic group holds G is a cyclic group and F is a cyclic group.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.

[9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[13] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(**5**):829–832, 1990.
[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[15] Dariusz Surowik. Cyclic groups and some of their properties - part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[18] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[21] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[22] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1(**5**):979–981, 1990.
[23] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.
[24] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(**4**):461–466, 1991.
[25] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[26] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.
[27] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.
[28] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[29] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.
[30] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[31] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received June 5, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Some Isomorphisms Between Functor Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. We define some well known isomorphisms between functor categories: between $A^{\dot{\bigcirc}(o,m)}$ and A, between $C^{[:A,B:]}$ and $(C^B)^A$, and between $[:B,C:]^A$ and $[:B^A,C^A:]$. Compare [12] and [11]. Unfortunately in this paper "functor" is used in two different meanings, as a lingual function and as a functor between categories.

MML Identifier: `ISOCAT_2`.

The notation and terminology used in this paper are introduced in the following papers: [17], [18], [4], [5], [3], [7], [1], [2], [10], [13], [8], [14], [6], [9], [16], and [15].

1. Preliminaries

The scheme *ChoiceD* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function h from $\mathcal{A}$ into $\mathcal{B}$ such that for every element a of $\mathcal{A}$ holds $\mathcal{P}[a, h(a)]$

provided the parameters meet the following requirement:

- for every element a of $\mathcal{A}$ there exists an element b of $\mathcal{B}$ such that $\mathcal{P}[a, b]$.

Let A, B, C be non-empty sets, and let f be a function from A into C^B. Then uncurry f is a function from $[: A, B :]$ into C.

We now state several propositions:

(1) For all non-empty sets A, B, C and for every function f from A into C^B holds curry uncurry $f = f$.

ISSN 0777–4028

(2) For all non-empty sets A, B, C and for every function f from A into C^B and for every element a of A and for every element b of B holds $(\mathrm{uncurry}\, f)(\langle a, b \rangle) = f(a)(b)$.

(3) For an arbitrary x and for every non-empty set A and for all functions f, g from $\{x\}$ into A such that $f(x) = g(x)$ holds $f = g$.

(4) For all non-empty sets A, B and for every element x of A and for every function f from A into B holds $f(x) \in \mathrm{rng}\, f$.

(5) For all non-empty sets A, B, C and for all functions f, g from A into $[\!:B, C:\!]$ such that $\pi_1(B \times C) \cdot f = \pi_1(B \times C) \cdot g$ and $\pi_2(B \times C) \cdot f = \pi_2(B \times C) \cdot g$ holds $f = g$.

We adopt the following rules: A, B, C will be categories and F, F_1, F_2 will be functors from A to B. The following two propositions are true:

(6) For every morphism f of A holds $\mathrm{id}_{\mathrm{cod}\, f} \cdot f = f$.

(7) For every morphism f of A holds $f \cdot \mathrm{id}_{\mathrm{dom}\, f} = f$.

In the sequel o, m will be arbitrary. The following two propositions are true:

(8) o is an object of B^A if and only if o is a functor from A to B.

(9) For every morphism f of B^A there exist functors F_1, F_2 from A to B and there exists a natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 and $\mathrm{dom}\, f = F_1$ and $\mathrm{cod}\, f = F_2$ and $f = \langle \langle F_1, F_2 \rangle, t \rangle$.

2. The isomorphism between $A^{\dot{\circlearrowleft}(o,m)}$ and A

Let us consider A, B, and let a be an object of A. The functor $a \mapsto B$ yields a functor from B^A to B and is defined by:

(Def.1) for all functors F_1, F_2 from A to B and for every natural transformation t from F_1 to F_2 such that F_1 is naturally transformable to F_2 holds $(a \mapsto B)(\langle \langle F_1, F_2 \rangle, t \rangle) = t(a)$.

One can prove the following two propositions:

(10) The objects of $\dot{\circlearrowleft}(o, m) = \{o\}$ and the morphisms of $\dot{\circlearrowleft}(o, m) = \{m\}$.

(11) $A^{\dot{\circlearrowleft}(o,m)} \cong A$.

3. The isomorphism between $C^{[\!:A,B:\!]}$ and $(C^B)^A$

Next we state four propositions:

(12) For every functor F from $[\!:A, B:\!]$ to C and for every object a of A and for every object b of B holds $F(a, -)(b) = F(\langle a, b \rangle)$.

(13) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B holds $\hom(a_1, a_2) \neq \emptyset$ and $\hom(b_1, b_2) \neq \emptyset$ if and only if $\hom(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) \neq \emptyset$.

(14) Let a_1, a_2 be objects of A. Then for all objects b_1, b_2 of B such that $\hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) \neq \emptyset$ and for every morphism f of A and for every morphism g of B holds $\langle f, g\rangle$ is a morphism from $\langle a_1, b_1\rangle$ to $\langle a_2, b_2\rangle$ if and only if f is a morphism from a_1 to a_2 and g is a morphism from b_1 to b_2.

(15) For all functors F_1, F_2 from $[\!: A, B :\!]$ to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 and for every object a of A holds $F_1(a, -)$ is naturally transformable to $F_2(a, -)$ and $(\mathrm{curry}\, t)(a)$ is a natural transformation from $F_1(a, -)$ to $F_2(a, -)$.

Let us consider A, B, C, and let F be a functor from $[\!: A, B :\!]$ to C, and let f be a morphism of A. The functor $\mathrm{curry}(F, f)$ yields a function from the morphisms of B into the morphisms of C and is defined by:

(Def.2) $\mathrm{curry}(F, f) = (\mathrm{curry}\, F)(f)$.

The following two propositions are true:

(16) For all objects a_1, a_2 of A and for all objects b_1, b_2 of B and for every morphism f of A and for every morphism g of B such that $f \in \hom(a_1, a_2)$ and $g \in \hom(b_1, b_2)$ holds $\langle f, g\rangle \in \hom(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle)$.

(17) For every functor F from $[\!: A, B :\!]$ to C and for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ and for every morphism f from a to b holds $F(a, -)$ is naturally transformable to $F(b, -)$ and $\mathrm{curry}(F, f) \cdot$ the id-map of B is a natural transformation from $F(a, -)$ to $F(b, -)$.

Let us consider A, B, C, and let F be a functor from $[\!: A, B :\!]$ to C, and let f be a morphism of A. The functor $F(f, -)$ yielding a natural transformation from $F(\mathrm{dom}\, f, -)$ to $F(\mathrm{cod}\, f, -)$ is defined by:

(Def.3) $F(f, -) = \mathrm{curry}(F, f) \cdot$ the id-map of B.

We now state four propositions:

(18) For every functor F from $[\!: A, B :\!]$ to C and for every morphism g of A holds $F(\mathrm{dom}\, g, -)$ is naturally transformable to $F(\mathrm{cod}\, g, -)$.

(19) For every functor F from $[\!: A, B :\!]$ to C and for every morphism f of A and for every object b of B holds $F(f, -)(b) = F(\langle f, \mathrm{id}_b\rangle)$.

(20) For every functor F from $[\!: A, B :\!]$ to C and for every object a of A holds $\mathrm{id}_{F(a,-)} = F(\mathrm{id}_a, -)$.

(21) For every functor F from $[\!: A, B :\!]$ to C and for all morphisms g, f of A such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ and for every natural transformation t from $F(\mathrm{dom}\, f, -)$ to $F(\mathrm{dom}\, g, -)$ such that $t = F(f, -)$ holds $F(g \cdot f, -) = F(g, -) \circ t$.

Let us consider A, B, C, and let F be a functor from $[\!: A, B :\!]$ to C. The functor $\mathrm{export}(F)$ yielding a functor from A to C^B is defined as follows:

(Def.4) for every morphism f of A holds $(\mathrm{export}(F))(f) = \langle\langle F(\mathrm{dom}\, f, -), F(\mathrm{cod}\, f, -)\rangle, F(f, -)\rangle$.

We now state several propositions:

(22) For every functor F from $[\![A, B]\!]$ to C and for every morphism f of A holds $(\mathrm{export}(F))(f) = \langle\langle F(\mathrm{dom}\, f, -), F(\mathrm{cod}\, f, -)\rangle, F(f, -)\rangle$.

(23) For all functors F_1, F_2 from A to B such that F_1 is transformable to F_2 and for every transformation t from F_1 to F_2 and for every object a of A holds $t(a) \in \mathrm{hom}(F_1(a), F_2(a))$.

(24) For every functor F from $[\![A, B]\!]$ to C and for every object a of A holds $(\mathrm{export}(F))(a) = F(a, -)$.

(25) For every functor F from $[\![A, B]\!]$ to C and for every object a of A holds $(\mathrm{export}(F))(a)$ is a functor from B to C.

(26) For all functors F_1, F_2 from $[\![A, B]\!]$ to C such that $\mathrm{export}(F_1) = \mathrm{export}(F_2)$ holds $F_1 = F_2$.

(27) Let F_1, F_2 be functors from $[\![A, B]\!]$ to C. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. Then $\mathrm{export}(F_1)$ is naturally transformable to $\mathrm{export}(F_2)$ and there exists a natural transformation G from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ such that for every function s from $[\![$ the objects of A, the objects of $B]\!]$ into the morphisms of C such that $t = s$ and for every object a of A holds $G(a) = \langle\langle(\mathrm{export}(F_1))(a), (\mathrm{export}(F_2))(a)\rangle, (\mathrm{curry}\, s)(a)\rangle$.

Let us consider A, B, C, and let F_1, F_2 be functors from $[\![A, B]\!]$ to C satisfying the condition: F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2. The functor $\mathrm{export}(t)$ yielding a natural transformation from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ is defined as follows:

(Def.5) for every function s from $[\![$ the objects of A, the objects of $B]\!]$ into the morphisms of C such that $t = s$ and for every object a of A holds $(\mathrm{export}(t))(a) = \langle\langle(\mathrm{export}(F_1))(a), (\mathrm{export}(F_2))(a)\rangle, (\mathrm{curry}\, s)(a)\rangle$.

We now state several propositions:

(28) For every functor F from $[\![A, B]\!]$ to C holds $\mathrm{id}_{\mathrm{export}(F)} = \mathrm{export}(\mathrm{id}_F)$.

(29) For all functors F_1, F_2, F_3 from $[\![A, B]\!]$ to C such that F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3 and for every natural transformation t_1 from F_1 to F_2 and for every natural transformation t_2 from F_2 to F_3 holds $\mathrm{export}(t_2 \circ t_1) = \mathrm{export}(t_2) \circ \mathrm{export}(t_1)$.

(30) For all functors F_1, F_2 from $[\![A, B]\!]$ to C such that F_1 is naturally transformable to F_2 and for all natural transformations t_1, t_2 from F_1 to F_2 such that $\mathrm{export}(t_1) = \mathrm{export}(t_2)$ holds $t_1 = t_2$.

(31) For every functor G from A to C^B there exists a functor F from $[\![A, B]\!]$ to C such that $G = \mathrm{export}(F)$.

(32) For all functors F_1, F_2 from $[\![A, B]\!]$ to C such that $\mathrm{export}(F_1)$ is naturally transformable to $\mathrm{export}(F_2)$ and for every natural transformation t from $\mathrm{export}(F_1)$ to $\mathrm{export}(F_2)$ holds F_1 is naturally transformable to F_2 and there exists a natural transformation u from F_1 to F_2 such that $t = \mathrm{export}(u)$.

Let us consider A, B, C. The functor $\mathbf{export}_{A,B,C}$ yields a functor from $C^{[: A, B :]}$ to $(C^B)^A$ and is defined by:

(Def.6) for all functors F_1, F_2 from $[: A, B :]$ to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $\mathbf{export}_{A,B,C}(\langle\langle F_1, F_2\rangle, t\rangle) = \langle\langle \mathrm{export}(F_1), \mathrm{export}(F_2)\rangle, \mathrm{export}(t)\rangle$.

Next we state two propositions:

(33) $\mathbf{export}_{A,B,C}$ is an isomorphism.

(34) $C^{[: A, B :]} \cong (C^B)^A$.

4. The isomorphism between $[:B, C:]^A$ and $[:B^A, C^A:]$

We now state the proposition

(35) For all functors F_1, F_2 from A to B and for every functor G from B to C such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $G \cdot t = G \cdot t$ **qua** a function .

We now define two new functors. Let us consider A, B. Then $\pi_1(A \times B)$ is a functor from $[: A, B :]$ to A. Then $\pi_2(A \times B)$ is a functor from $[: A, B :]$ to B. Let us consider A, B, C, and let F be a functor from A to B, and let G be a functor from A to C. Then $\langle F, G\rangle$ is a functor from A to $[: B, C :]$. Let F be a functor from A to $[: B, C :]$. The functor $\pi_1 \cdot F$ yielding a functor from A to B is defined as follows:

(Def.7) $\pi_1 \cdot F = \pi_1(B \times C) \cdot F$.

The functor $\pi_2 \cdot F$ yielding a functor from A to C is defined by:

(Def.8) $\pi_2 \cdot F = \pi_2(B \times C) \cdot F$.

The following two propositions are true:

(36) For every functor F from A to B and for every functor G from A to C holds $\pi_1 \cdot \langle F, G\rangle = F$ and $\pi_2 \cdot \langle F, G\rangle = G$.

(37) For all functors F, G from A to $[: B, C :]$ such that $\pi_1 \cdot F = \pi_1 \cdot G$ and $\pi_2 \cdot F = \pi_2 \cdot G$ holds $F = G$.

We now define two new functors. Let us consider A, B, C, and let F_1, F_2 be functors from A to $[: B, C :]$, and let t be a natural transformation from F_1 to F_2. The functor $\pi_1 \cdot t$ yielding a natural transformation from $\pi_1 \cdot F_1$ to $\pi_1 \cdot F_2$ is defined as follows:

(Def.9) $\pi_1 \cdot t = \pi_1(B \times C) \cdot t$.

The functor $\pi_2 \cdot t$ yielding a natural transformation from $\pi_2 \cdot F_1$ to $\pi_2 \cdot F_2$ is defined as follows:

(Def.10) $\pi_2 \cdot t = \pi_2(B \times C) \cdot t$.

We now state several propositions:

(38) For all functors F, G from A to $[: B, C :]$ such that F is naturally transformable to G holds $\pi_1 \cdot F$ is naturally transformable to $\pi_1 \cdot G$ and $\pi_2 \cdot F$ is naturally transformable to $\pi_2 \cdot G$.

(39) For all functors F_1, F_2, G_1, G_2 from A to $[: B, C :]$ such that F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 and for every natural transformation s from F_1 to F_2 and for every natural transformation t from G_1 to G_2 such that $\pi_1 \cdot s = \pi_1 \cdot t$ and $\pi_2 \cdot s = \pi_2 \cdot t$ holds $s = t$.

(40) For every functor F from A to $[: B, C :]$ holds $\mathrm{id}_{\pi_1 \cdot F} = \pi_1 \cdot (\mathrm{id}_F)$ and $\mathrm{id}_{\pi_2 \cdot F} = \pi_2 \cdot (\mathrm{id}_F)$.

(41) For all functors F, G, H from A to $[: B, C :]$ such that F is naturally transformable to G and G is naturally transformable to H and for every natural transformation s from F to G and for every natural transformation t from G to H holds $\pi_1 \cdot (t \circ s) = \pi_1 \cdot t \circ \pi_1 \cdot s$ and $\pi_2 \cdot (t \circ s) = \pi_2 \cdot t \circ \pi_2 \cdot s$.

(42) For every functor F from A to B and for every functor G from A to C and for all objects a, b of A such that $\hom(a, b) \neq \emptyset$ and for every morphism f from a to b holds $\langle F, G \rangle (f) = \langle\!\langle F(f), G(f) \rangle\!\rangle$.

(43) For every functor F from A to B and for every functor G from A to C and for every object a of A holds $\langle F, G \rangle (a) = \langle\!\langle F(a), G(a) \rangle\!\rangle$.

(44) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is transformable to G_1 and F_2 is transformable to G_2 holds $\langle F_1, F_2 \rangle$ is transformable to $\langle G_1, G_2 \rangle$.

Let us consider A, B, C, and let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C satisfying the condition: F_1 is transformable to G_1 and F_2 is transformable to G_2. Let t_1 be a transformation from F_1 to G_1, and let t_2 be a transformation from F_2 to G_2. The functor $\langle t_1, t_2 \rangle$ yielding a transformation from $\langle F_1, F_2 \rangle$ to $\langle G_1, G_2 \rangle$ is defined as follows:

(Def.11) $\langle t_1, t_2 \rangle = \langle t_1, t_2 \rangle$.

One can prove the following propositions:

(45) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is transformable to G_1 and F_2 is transformable to G_2 and for every transformation t_1 from F_1 to G_1 and for every transformation t_2 from F_2 to G_2 and for every object a of A holds $\langle t_1, t_2 \rangle (a) = \langle\!\langle t_1(a), t_2(a) \rangle\!\rangle$.

(46) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2 holds $\langle F_1, F_2 \rangle$ is naturally transformable to $\langle G_1, G_2 \rangle$.

Let us consider A, B, C, and let F_1, G_1 be functors from A to B, and let F_2, G_2 be functors from A to C satisfying the conditions: F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2. Let t_1 be a natural transformation from F_1 to G_1, and let t_2 be a natural transformation from F_2 to G_2. The functor $\langle t_1, t_2 \rangle$ yielding a natural transformation from $\langle F_1, F_2 \rangle$ to $\langle G_1, G_2 \rangle$ is defined as follows:

(Def.12) $\langle t_1, t_2 \rangle = \langle t_1, t_2 \rangle$.

Next we state the proposition

(47) For all functors F_1, G_1 from A to B and for all functors F_2, G_2 from A to C such that F_1 is naturally transformable to G_1 and F_2 is naturally transformable to G_2 and for every natural transformation t_1 from F_1 to G_1 and for every natural transformation t_2 from F_2 to G_2 holds $\pi_1 \cdot \langle t_1, t_2 \rangle = t_1$ and $\pi_2 \cdot \langle t_1, t_2 \rangle = t_2$.

Let us consider A, B, C. The functor $\mathbf{distribute}_{A,B,C}$ yielding a functor from $[\![B, C]\!]^A$ to $[\![B^A, C^A]\!]$ is defined by:

(Def.13) for all functors F_1, F_2 from A to $[\![B, C]\!]$ such that F_1 is naturally transformable to F_2 and for every natural transformation t from F_1 to F_2 holds $\mathbf{distribute}_{A,B,C}(\langle\langle F_1, F_2\rangle, t\rangle) = \langle\langle\langle \pi_1 \cdot F_1, \pi_1 \cdot F_2\rangle, \pi_1 \cdot t\rangle, \langle\langle \pi_2 \cdot F_1, \pi_2 \cdot F_2\rangle, \pi_2 \cdot t\rangle\rangle$.

One can prove the following two propositions:

(48) $\mathbf{distribute}_{A,B,C}$ is an isomorphism.

(49) $[\![B, C]\!]^A \cong [\![B^A, C^A]\!]$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[3] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.
[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Saunders Mac Lane. *Categories for the Working Mathematician.* Volume 5 of *Graduate Texts in Mathematics*, Springer Verlag, New York, Heidelberg, Berlin, 1971.
[12] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów.* Volume 45 of *Biblioteka Matematyczna*, PWN, Warszawa, 1978.
[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[15] Andrzej Trybulec. Isomorphisms of categories. *Formalized Mathematics*, 2(**5**):629–634, 1991.
[16] Andrzej Trybulec. Natural transformations. Discrete categories. *Formalized Mathematics*, 2(**4**):467–474, 1991.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[18] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received June 5, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

The Lattice of Domains of a Topological Space [1]

Toshihiko Watanabe
Shinshu University
Nagano

Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *closed domain* of T if $A = \overline{\mathrm{Int}\, A}$ and A is said to be an *open domain* of T if $A = \mathrm{Int}\, \overline{A}$ (see e.g. [8], [15]). Some simple generalization of these notions is the following one. A is said to be a *domain* of T provided $\mathrm{Int}\, \overline{A} \subseteq A \subseteq \overline{\mathrm{Int}\, A}$ (see [15] and compare [7]). In this paper certain connections between these concepts are introduced and studied.

Our main results are concerned with the following well–known theorems (see e.g. [9], [2]). For a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too. It is proved that *all domains of a given topological space form a complemented lattice.* Moreover, it is shown that both *the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains.* In the beginning some useful theorems about subsets of topological spaces are proved and certain properties of domains, closed domains and open domains are discussed.

MML Identifier: `TDLAT_1`.

The terminology and notation used in this paper are introduced in the following articles: [14], [11], [4], [5], [16], [3], [13], [10], [15], [1], [12], and [6].

1. Preliminary Theorems on Subset of Topological Spaces

In the sequel T is a topological space. We now state a number of propositions:

(1) For all subsets A, B of T holds $A \cup B = \Omega_T$ if and only if $A^{\mathrm{c}} \subseteq B$.

(2) For all subsets A, B of T holds $A \cap B = \emptyset_T$ if and only if $B \subseteq A^{\mathrm{c}}$.

[1]This paper was done under the supervision of Z. Karno while the author was visiting the Institute of Mathematics of Warsaw University in Białystok.

ISSN 0777–4028

(3) For every subset A of T holds $\overline{\mathrm{Int}\,\overline{A}} \subseteq \overline{A}$.

(4) For every subset A of T holds $\mathrm{Int}\,A \subseteq \mathrm{Int}\,\overline{\mathrm{Int}\,A}$.

(5) For every subset A of T holds $\mathrm{Int}\,\overline{A} = \mathrm{Int}\,\overline{\mathrm{Int}\,\overline{A}}$.

(6) For all subsets A, B of T such that A is closed or B is closed holds $\overline{\mathrm{Int}\,A} \cup \overline{\mathrm{Int}\,B} = \overline{\mathrm{Int}(A \cup B)}$.

(7) For all subsets A, B of T such that A is open or B is open holds $\mathrm{Int}\,\overline{A} \cap \mathrm{Int}\,\overline{B} = \mathrm{Int}\,\overline{A \cap B}$.

(8) For every subset A of T holds $\mathrm{Int}(A \cap \overline{A^{\mathrm{c}}}) = \emptyset_T$.

(9) For every subset A of T holds $\overline{A \cup \mathrm{Int}(A^{\mathrm{c}})} = \Omega_T$.

(10) For all subsets A, B of T holds $\mathrm{Int}\,\overline{A \cup (\mathrm{Int}\,\overline{B} \cup B)} \cup (A \cup (\mathrm{Int}\,\overline{B} \cup B)) = \mathrm{Int}\,\overline{A \cup B} \cup (A \cup B)$.

(11) For all subsets A, C of T holds $\mathrm{Int}\,\overline{\mathrm{Int}\,\overline{A} \cup A \cup C} \cup (\mathrm{Int}\,\overline{A} \cup A \cup C) = \mathrm{Int}\,\overline{A \cup C} \cup (A \cup C)$.

(12) For all subsets A, B of T holds $\overline{\mathrm{Int}(A \cap (\overline{\mathrm{Int}\,B} \cap B))} \cap (A \cap (\overline{\mathrm{Int}\,B} \cap B)) = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

(13) For all subsets A, C of T holds $\overline{\mathrm{Int}(\overline{\mathrm{Int}\,A} \cap A \cap C)} \cap (\overline{\mathrm{Int}\,A} \cap A \cap C) = \overline{\mathrm{Int}(A \cap C)} \cap (A \cap C)$.

2. Properties of Domains of Topological Spaces

In the sequel T will be a topological space. Next we state a number of propositions:

(14) $\emptyset_T$ is a domain.

(15) Ω_T is a domain.

(16) For every subset A of T such that A is a domain holds A^{c} is a domain.

(17) For all subsets A, B of T such that A is a domain and B is a domain holds $\mathrm{Int}\,\overline{A \cup B} \cup (A \cup B)$ is a domain and $\overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$ is a domain.

(18) $\emptyset_T$ is a closed domain.

(19) Ω_T is a closed domain.

(20) $\emptyset_T$ is an open domain.

(21) Ω_T is an open domain.

(22) For every subset A of T holds $\overline{\mathrm{Int}\,A}$ is a closed domain.

(23) For every subset A of T holds $\mathrm{Int}\,\overline{A}$ is an open domain.

(24) For every subset A of T such that A is a domain holds $\overline{A}$ is a closed domain.

(25) For every subset A of T such that A is a domain holds $\mathrm{Int}\,A$ is an open domain.

(26) For every subset A of T such that A is a domain holds $\overline{A^{\mathrm{c}}}$ is a closed domain.

(27) For every subset A of T such that A is a domain holds $\mathrm{Int}(A^{\mathrm{c}})$ is an open domain.

(28) For all subsets A, B, C of T such that A is a closed domain and B is a closed domain and C is a closed domain holds $\overline{\mathrm{Int}(A \cap \overline{\mathrm{Int}(B \cap C)})} = \overline{\mathrm{Int}(\overline{\mathrm{Int}(A \cap B)} \cap C)}$.

(29) For all subsets A, B, C of T such that A is an open domain and B is an open domain and C is an open domain holds $\mathrm{Int}\,\overline{A \cup \mathrm{Int}\,\overline{B \cup C}} = \mathrm{Int}\,\overline{\mathrm{Int}\,\overline{A \cup B} \cup C}$.

3. The Lattice of Domains

We now define five new functors. Let T be a topological space. The domains of T yields a non-empty family of subsets of the carrier of T and is defined as follows:

(Def.1) the domains of $T = \{A : A$ is a domain$\}$, where A ranges over subsets of T.

The domains union of T yielding a binary operation on the domains of T is defined by:

(Def.2) for all elements A, B of the domains of T holds (the domains union of $T)(A,\, B) = \mathrm{Int}\,\overline{A \cup B} \cup (A \cup B)$.

We introduce the functor D-Union(T) as a synonym of the domains union of T. The domains meet of T yields a binary operation on the domains of T and is defined as follows:

(Def.3) for all elements A, B of the domains of T holds (the domains meet of $T)(A,\, B) = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

We introduce the functor D-Meet(T) as a synonym of the domains meet of T.

One can prove the following proposition

(30) For every topological space T holds $\langle$the domains of T, D-Union(T), D-Meet(T)$\rangle$ is a complemented lattice.

Let T be a topological space. The lattice of domains of T yields a complemented lattice and is defined by:

(Def.4) the lattice of domains of $T = \langle$the domains of T, the domains union of T, the domains meet of $T\rangle$.

4. The Lattice of Closed Domains

Let T be a topological space. The closed domains of T yielding a non-empty family of subsets of the carrier of T is defined as follows:

(Def.5) the closed domains of $T = \{A : A$ is a closed domain$\}$, where A ranges over subsets of T.

Next we state the proposition

(31) For every topological space T holds the closed domains of $T \subseteq$ the domains of T.

We now define two new functors. Let T be a topological space. The closed domains union of T yielding a binary operation on the closed domains of T is defined by:

(Def.6) for all elements A, B of the closed domains of T holds (the closed domains union of $T)(A, B) = A \cup B$.

We introduce the functor CLD-Union(T) as a synonym of the closed domains union of T.

Next we state the proposition

(32) For all elements A, B of the closed domains of T holds (CLD-Union(T))$(A, B) =$ (D-Union(T))(A, B).

We now define two new functors. Let T be a topological space. The closed domains meet of T yielding a binary operation on the closed domains of T is defined as follows:

(Def.7) for all elements A, B of the closed domains of T holds (the closed domains meet of $T)(A, B) = \overline{\mathrm{Int}(A \cap B)}$.

We introduce the functor CLD-Meet(T) as a synonym of the closed domains meet of T.

One can prove the following two propositions:

(33) For all elements A, B of the closed domains of T holds (CLD-Meet(T))$(A, B) =$ (D-Meet(T))(A, B).

(34) For every topological space T holds $\langle$the closed domains of $T, \mathrm{CLD\text{-}Union}(T), \mathrm{CLD\text{-}Meet}(T)\rangle$ is a Boolean lattice.

Let T be a topological space. The lattice of closed domains of T yielding a Boolean lattice is defined as follows:

(Def.8) the lattice of closed domains of $T = \langle$the closed domains of $T,$ the closed domains union of $T,$ the closed domains meet of $T\rangle$.

5. The Lattice of Open Domains

Let T be a topological space. The open domains of T yields a non-empty family of subsets of the carrier of T and is defined by:

(Def.9) the open domains of $T = \{A : A$ is an open domain$\}$, where A ranges over subsets of T.

Next we state the proposition

(35) For every topological space T holds the open domains of $T \subseteq$ the domains of T.

We now define two new functors. Let T be a topological space. The open domains union of T yielding a binary operation on the open domains of T is defined by:

(Def.10) for all elements A, B of the open domains of T holds (the open domains union of T)$(A, B) = \mathrm{Int}\,\overline{A \cup B}$.

We introduce the functor OPD-Union(T) as a synonym of the open domains union of T.

One can prove the following proposition

(36) For all elements A, B of the open domains of T holds (OPD-Union(T))$(A, B) =$ (D-Union(T))(A, B).

We now define two new functors. Let T be a topological space. The open domains meet of T yielding a binary operation on the open domains of T is defined by:

(Def.11) for all elements A, B of the open domains of T holds (the open domains meet of T)$(A, B) = A \cap B$.

We introduce the functor OPD-Meet(T) as a synonym of the open domains meet of T.

We now state two propositions:

(37) For all elements A, B of the open domains of T holds (OPD-Meet(T))$(A, B) =$ (D-Meet(T))(A, B).

(38) For every topological space T holds $\langle$the open domains of T, OPD-Union(T), OPD-Meet(T)$\rangle$ is a Boolean lattice.

Let T be a topological space. The lattice of open domains of T yielding a Boolean lattice is defined by:

(Def.12) the lattice of open domains of $T = \langle$the open domains of T, the open domains union of T, the open domains meet of $T\rangle$.

6. Connections between Lattices of Domains

In the sequel T will be a topological space. The following propositions are true:

(39) CLD-Union(T) $=$ D-Union(T) $\restriction$ [: the closed domains of T, the closed domains of T :].

(40) CLD-Meet(T) $=$ D-Meet(T) $\restriction$ [: the closed domains of T, the closed domains of T :].

(41) The lattice of closed domains of T is a sublattice of the lattice of domains of T.

(42) OPD-Union(T) $=$ D-Union(T) $\restriction$ [: the open domains of T, the open domains of T :].

(43) OPD-Meet(T) = D-Meet$(T) \restriction [\!:$the open domains of T, the open domains of $T :\!]$.

(44) The lattice of open domains of T is a sublattice of the lattice of domains of T.

Acknowledgments

The author wishes to express his thanks to Professors A. Trybulec and Z. Karno for their active interest in the publication of this article and for useful suggestions and many valuable comments.

References

[1] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.

[2] Garrett Birkhoff. *Lattice Theory.* Providence, Rhode Island, New York, 1967.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Formalized Mathematics*, 2(**4**):453–459, 1991.

[7] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.

[8] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[9] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory (with an introduction to descriptive set theory).* Volume 86 of *Studies in Logic and The Foundations of Mathematics*, PWN - Polish Scientific Publishers and North-Holland Publishing Company, Warsaw-Amsterdam, 1976.

[10] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[15] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

[16] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received June 12, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Submodules

Michał Muzalewski
Warsaw University
Białystok

Summary. This article contains the notions of trivial and non-trivial leftmodules and rings, cyclic submodules and inclusion of submodules. A few basic theorems related to these notions are proved.

MML Identifier: `LMOD_6`.

The notation and terminology used here are introduced in the following papers: [15], [16], [3], [4], [2], [1], [5], [6], [7], [14], [9], [13], [12], [10], [11], and [8].

1. Preliminaries

For simplicity we adopt the following rules: x is arbitrary, K denotes an associative ring, r denotes a scalar of K, V, M, M_1, M_2, N denote left modules over K, a denotes a vector of V, m, m_1, m_2 denote vectors of M, n, n_1, n_2 denote vectors of N, A denotes a subset of V, l denotes a linear combination of A, and W, W_1, W_2, W_3 denote submodules of V. Next we state four propositions:

(1) If $M_1 =$ the left module structure of M_2, then $x \in M_1$ if and only if $x \in M_2$.

(2) For every vector v of the left module structure of V such that $a = v$ holds $r \cdot a = r \cdot v$.

(3) The left module structure of V is a strict submodule of V.

(4) V is a submodule of Ω_V.

ISSN 0777–4028

2. Trivial and non-trivial modules and rings

We now define two new predicates. Let us consider K, V. We say that V is non-trivial if and only if:

(Def.1) there exists a vector a of V such that $a \neq \Theta_V$.

Let us consider K. We say that K is non-trivial if and only if:

(Def.2) $0_K \neq 1_K$.

We now state three propositions:

(5) If K is trivial, then for every r holds $r = 0_K$ and for every a holds $a = \Theta_V$.

(6) If K is trivial, then V is trivial.

(7) V is trivial if and only if the left module structure of $V = \mathbf{0}_V$.

3. Submodules and subsets

We now define two new functors. Let us consider K, V, and let W be a strict submodule of V. The functor $\ddot{\mathrm{e}}(W)$ yields an element of $\mathrm{Sub}(V)$ and is defined by:

(Def.3) $\ddot{\mathrm{e}}(W) = W$.

The functor ç(V) yields a non-empty subset of V and is defined as follows:

(Def.4) ç$(V) =$ the carrier of V.

The following two propositions are true:

(8) For all sets X, Y, A such that $X \subseteq Y$ and A is a subset of X holds A is a subset of Y.

(9) Every subset of W is a subset of V.

Let us consider K, V, W, and let A be a subset of W. The functor ï(A) yields a subset of V and is defined by:

(Def.5) ï$(A) = A$.

Let A be a non-empty subset of W. Then ï(A) is a non-empty subset of V.

The following propositions are true:

(10) $x \in$ ç(V) if and only if $x \in V$.

(11) $x \in$ ï(ç(W)) if and only if $x \in W$.

(12) $A \subseteq$ ç$(\mathrm{Lin}(A))$.

(13) If $A \neq \emptyset$ and A is linearly closed, then $\sum l \in A$.

(14) If $\Theta_V \in A$ and A is linearly closed, then $\sum l \in A$.

(15) If $\Theta_V \in A$ and A is linearly closed, then $A =$ ç$(\mathrm{Lin}(A))$.

4. Cyclic submodules

Let us consider K, V, a. Then $\{a\}$ is a non-empty subset of V. The functor $\prod^* a$ yielding a strict submodule of V is defined by:

(Def.6) $\prod^* a = \mathrm{Lin}(\{a\})$.

5. Inclusion of left R-modules

Let us consider K, M, N. The predicate $M \subseteq N$ is defined as follows:

(Def.7) M is a submodule of N.

We now state a number of propositions:

(16) If $M \subseteq N$, then if $x \in M$, then $x \in N$ but if x is a vector of M, then x is a vector of N.

(17) Suppose $M \subseteq N$. Then

(i) $\Theta_M = \Theta_N$,

(ii) if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 + m_2 = n_1 + n_2$,

(iii) if $m = n$, then $r \cdot m = r \cdot n$,

(iv) if $m = n$, then $-n = -m$,

(v) if $m_1 = n_1$ and $m_2 = n_2$, then $m_1 - m_2 = n_1 - n_2$,

(vi) $\Theta_N \in M$,

(vii) $\Theta_M \in N$,

(viii) if $n_1 \in M$ and $n_2 \in M$, then $n_1 + n_2 \in M$,

(ix) if $n \in M$, then $r \cdot n \in M$,

(x) if $n \in M$, then $-n \in M$,

(xi) if $n_1 \in M$ and $n_2 \in M$, then $n_1 - n_2 \in M$.

(18) Suppose $M_1 \subseteq N$ and $M_2 \subseteq N$. Then

(i) $\Theta_{M_1} = \Theta_{M_2}$,

(ii) $\Theta_{M_1} \in M_2$,

(iii) if the carrier of $M_1 \subseteq$ the carrier of M_2, then $M_1 \subseteq M_2$,

(iv) if for every n such that $n \in M_1$ holds $n \in M_2$, then $M_1 \subseteq M_2$,

(v) if the carrier of $M_1 =$ the carrier of M_2 and M_1 is strict and M_2 is strict, then $M_1 = M_2$,

(vi) $\mathbf{0}_{M_1} \subseteq M_2$.

(19) $W_1 + W_2 \subseteq V$ and $W_1 \cap W_2 \subseteq V$.

(20) $N \subseteq N$.

(21) For all strict left modules V, M over K such that $V \subseteq M$ and $M \subseteq V$ holds $V = M$.

(22) If $V \subseteq M$ and $M \subseteq N$, then $V \subseteq N$.

(23) If $M \subseteq N$, then $\mathbf{0}_M \subseteq N$.

(24) If $M \subseteq N$, then $\mathbf{0}_N \subseteq M$.

(25) If $M \subseteq N$, then $M \subseteq \Omega_N$.

(26) $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

(27) $W_1 \cap W_2 \subseteq W_1$ and $W_1 \cap W_2 \subseteq W_2$.

(28) If $W_1 \subseteq W_2$, then $W_1 \cap W_3 \subseteq W_2 \cap W_3$.

(29) If $W_1 \subseteq W_3$, then $W_1 \cap W_2 \subseteq W_3$.

(30) If $W_1 \subseteq W_2$ and $W_1 \subseteq W_3$, then $W_1 \subseteq W_2 \cap W_3$.

(31) $W_1 \cap W_2 \subseteq W_1 + W_2$.

(32) $W_1 \cap W_2 + W_2 \cap W_3 \subseteq W_2 \cap (W_1 + W_3)$.

(33) If $W_1 \subseteq W_2$, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3 \subseteq (W_1 + W_2) \cap (W_2 + W_3)$.

(35) If $W_1 \subseteq W_2$, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If $W_1 \subseteq W_2$, then $W_1 \subseteq W_2 + W_3$.

(37) If $W_1 \subseteq W_3$ and $W_2 \subseteq W_3$, then $W_1 + W_2 \subseteq W_3$.

(38) For all subsets A, B of V such that $A \subseteq B$ holds $\mathrm{Lin}(A) \subseteq \mathrm{Lin}(B)$.

(39) For all subsets A, B of V holds $\mathrm{Lin}(A \cap B) \subseteq \mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(40) If $M_1 \subseteq M_2$, then $\text{ç}(M_1) \subseteq \text{ç}(M_2)$.

(41) $W_1 \subseteq W_2$ if and only if for every a such that $a \in W_1$ holds $a \in W_2$.

(42) $W_1 \subseteq W_2$ if and only if $\text{ç}(W_1) \subseteq \text{ç}(W_2)$.

(43) $W_1 \subseteq W_2$ if and only if $\text{ï}(\text{ç}(W_1)) \subseteq \text{ï}(\text{ç}(W_2))$.

(44) $\mathbf{0}_W \subseteq V$ and $\mathbf{0}_V \subseteq W$ and $\mathbf{0}_{W_1} \subseteq W_2$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[7] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.

[8] Michał Muzalewski. Free modules. *Formalized Mathematics*, 2(**4**):587–589, 1991.

[9] Michał Muzalewski and Wojciech Skaba. Finite sums of vectors in left module over associative ring. *Formalized Mathematics*, 2(**2**):279–282, 1991.

[10] Michał Muzalewski and Wojciech Skaba. Linear combinations in left module over associative ring. *Formalized Mathematics*, 2(**2**):295–300, 1991.

[11] Michał Muzalewski and Wojciech Skaba. Linear independence in left module over domain. *Formalized Mathematics*, 2(**2**):301–303, 1991.

[12] Michał Muzalewski and Wojciech Skaba. Operations on submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):289–293, 1991.

[13] Michał Muzalewski and Wojciech Skaba. Submodules and cosets of submodules in left module over associative ring. *Formalized Mathematics*, 2(**2**):283–287, 1991.

[14] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received June 19, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Oriented Metric-Affine Plane - Part II

Jarosław Zajkowski
Warsaw University
Białystok

Summary. Axiomatic description of properties of the oriented orthogonality relation. Next we construct (with the help of the oriented orthogonality relation) vector space and give the definitions of left-, right-, and semi-transitives.

MML Identifier: DIRORT.

The articles [1], [6], [7], [5], [3], [2], [4], and [8] provide the notation and terminology for this paper. In the sequel V will be a real linear space, A_1 will be an affine structure, and x, y will be vectors of V. One can prove the following propositions:

(1) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of $\mathrm{CESpace}(V, x, y)$ holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, w$, then $u, v \top^{>} u_1, w$ but if $u, u_1 \top^{>} v, v_1$, then $v, v_1 \top^{>} u, u_1$ or $v, v_1 \top^{>} u_1, u$,

(ii) for every elements u, v, w of the carrier of $\mathrm{CESpace}(V, x, y)$ there exists an element u_1 of the carrier of $\mathrm{CESpace}(V, x, y)$ such that $w \neq u_1$ and $w, u_1 \top^{>} u, v$,

(iii) for every elements u, v, w of the carrier of $\mathrm{CESpace}(V, x, y)$ there exists an element u_1 of the carrier of $\mathrm{CESpace}(V, x, y)$ such that $w \neq u_1$ and $u, v \top^{>} w, u_1$.

(2) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of $\mathrm{CMSpace}(V, x, y)$ holds $u, u \top^{>} v, w$ and $u, v \top^{>} w, w$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^{>} u_1, v_1$ and $u, v \top^{>} u_1, w$, then $u, v \top^{>} v_1, w$ or $u, v \top^{>} w, v_1$ but if $u, v \top^{>} u_1, v_1$, then $v, u \top^{>} v_1, u_1$ but if $u, v \top^{>} u_1, v_1$

ISSN 0777–4028

and $u, v \top^> v_1, w$, then $u, v \top^> u_1, w$ but if $u, u_1 \top^> v, v_1$, then $v, v_1 \top^> u, u_1$ or $v, v_1 \top^> u_1, u$,

(ii) for every elements u, v, w of the carrier of CMSpace(V, x, y) there exists an element u_1 of the carrier of CMSpace(V, x, y) such that $w \neq u_1$ and $w, u_1 \top^> u, v$,

(iii) for every elements u, v, w of the carrier of CMSpace(V, x, y) there exists an element u_1 of the carrier of CMSpace(V, x, y) such that $w \neq u_1$ and $u, v \top^> w, u_1$.

We now define two new constructions. An affine structure is oriented orthogonality if it satisfies the conditions (Def.1).

(Def.1) (i) For all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of it holds $u, u \top^> v, w$ and $u, v \top^> w, w$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> u_1, w$, then $u, v \top^> v_1, w$ or $u, v \top^> w, v_1$ but if $u, v \top^> u_1, v_1$, then $v, u \top^> v_1, u_1$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> v_1, w$, then $u, v \top^> u_1, w$ but if $u, u_1 \top^> v, v_1$, then $v, v_1 \top^> u, u_1$ or $v, v_1 \top^> u_1, u$,

(ii) for every elements u, v, w of the carrier of it there exists an element u_1 of the carrier of it such that $w \neq u_1$ and $w, u_1 \top^> u, v$,

(iii) for every elements u, v, w of the carrier of it there exists an element u_1 of the carrier of it such that $w \neq u_1$ and $u, v \top^> w, u_1$.

An oriented orthogonality space is an oriented orthogonality affine structure.

Next we state three propositions:

(3) The following conditions are equivalent:

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of the carrier of A_1 holds $u, u \top^> v, w$ and $u, v \top^> w, w$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> v_1, u_1$, then $u = v$ or $u_1 = v_1$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> u_1, w$, then $u, v \top^> v_1, w$ or $u, v \top^> w, v_1$ but if $u, v \top^> u_1, v_1$, then $v, u \top^> v_1, u_1$ but if $u, v \top^> u_1, v_1$ and $u, v \top^> v_1, w$, then $u, v \top^> u_1, w$ but if $u, u_1 \top^> v, v_1$, then $v, v_1 \top^> u, u_1$ or $v, v_1 \top^> u_1, u$ and for every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $w \neq u_1$ and $w, u_1 \top^> u, v$ and for every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $w \neq u_1$ and $u, v \top^> w, u_1$,

(ii) A_1 is an oriented orthogonality space.

(4) If x, y span the space, then CMSpace(V, x, y) is an oriented orthogonality space.

(5) If x, y span the space, then CESpace(V, x, y) is an oriented orthogonality space.

We follow a convention: A_1 will denote an oriented orthogonality space and u, u_1, u_2, v, v_1, v_2, w, w_1 will denote elements of the carrier of A_1. We now state three propositions:

(6) For every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $u_1, w \top^> u, v$ and $u_1 \neq w$.

(7) For all elements u, v, w of the carrier of A_1 holds $u, v \top^> w, w$.

(8) For every elements u, v, w of the carrier of A_1 there exists an element u_1 of the carrier of A_1 such that $u \neq u_1$ but $v, w \top^{>} u, u_1$ or $v, w \top^{>} u_1, u$.

We now define several new constructions. Let A_1 be an oriented orthogonality space, and let a, b, c, d be elements of the carrier of A_1. The predicate $a, b \perp c, d$ is defined by:

(Def.2) $a, b \top^{>} c, d$ or $a, b \top^{>} d, c$.

Let a, b, c, d be elements of the carrier of A_1. The predicate $a, b \upharpoonleft\!\!\upharpoonright c, d$ is defined as follows:

(Def.3) there exist elements e, f of the carrier of A_1 such that $e \neq f$ and $e, f \top^{>} a, b$ and $e, f \top^{>} c, d$.

An oriented orthogonality space is semi transitive if:

(Def.4) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $w, w_1 \top^{>} v, v_1$ and $w, w_1 \top^{>} u_2, v_2$ holds $w = w_1$ or $v = v_1$ or $u, u_1 \top^{>} u_2, v_2$.

An oriented orthogonality space is right transitive if:

(Def.5) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \top^{>} w, w_1$ and $u_2, v_2 \top^{>} w, w_1$ holds $w = w_1$ or $v = v_1$ or $u, u_1 \top^{>} u_2, v_2$.

An oriented orthogonality space is left transitive if:

(Def.6) for all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \top^{>} w, w_1$ and $u, u_1 \top^{>} u_2, v_2$ holds $u = u_1$ or $v = v_1$ or $u_2, v_2 \top^{>} w, w_1$.

An oriented orthogonality space is Euclidean like if:

(Def.7) for all elements u, u_1, v, v_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ holds $v, v_1 \top^{>} u_1, u$.

An oriented orthogonality space is Minkowskian like if:

(Def.8) for all elements u, u_1, v, v_1 of the carrier of it such that $u, u_1 \top^{>} v, v_1$ holds $v, v_1 \top^{>} u, u_1$.

One can prove the following propositions:

(9) $u, u_1 \upharpoonleft\!\!\upharpoonright w, w$ and $w, w \upharpoonleft\!\!\upharpoonright u, u_1$.

(10) If $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$, then $v, v_1 \upharpoonleft\!\!\upharpoonright u, u_1$.

(11) If $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$, then $u_1, u \upharpoonleft\!\!\upharpoonright v_1, v$.

(12) A_1 is left transitive if and only if for all v, v_1, w, w_1, u_2, v_2 such that $v, v_1 \upharpoonleft\!\!\upharpoonright u_2, v_2$ and $v, v_1 \top^{>} w, w_1$ and $v \neq v_1$ holds $u_2, v_2 \top^{>} w, w_1$.

(13) A_1 is semi transitive if and only if for all u, u_1, u_2, v, v_1, v_2 such that $u, u_1 \top^{>} v, v_1$ and $v, v_1 \upharpoonleft\!\!\upharpoonright u_2, v_2$ and $v \neq v_1$ holds $u, u_1 \top^{>} u_2, v_2$.

(14) If A_1 is semi transitive, then for all u, u_1, v, v_1, w, w_1 such that $u, u_1 \upharpoonleft\!\!\upharpoonright v, v_1$ and $v, v_1 \upharpoonleft\!\!\upharpoonright w, w_1$ and $v \neq v_1$ holds $u, u_1 \upharpoonleft\!\!\upharpoonright w, w_1$.

(15) If x, y span the space and $A_1 = \mathrm{CESpace}(V, x, y)$, then A_1 is Euclidean like, left transitive, right transitive and semi transitive.

One can readily verify that there exists an oriented orthogonality space which is Euclidean like, left transitive, right transitive and semi transitive.

We now state the proposition

(16) If x, y span the space and $A_1 = \mathrm{CMSpace}(V, x, y)$, then A_1 is Minkowskian like, left transitive, right transitive and semi transitive.

Let us note that there exists an oriented orthogonality space which is Minkowskian like, left transitive, right transitive and semi transitive.

Next we state four propositions:

(17) If A_1 is left transitive, then A_1 is right transitive.

(18) If A_1 is left transitive, then A_1 is semi transitive.

(19) If A_1 is semi transitive, then A_1 is right transitive if and only if for all u, u_1, v, v_1, u_2, v_2 such that $u, u_1 \top^{>} u_2, v_2$ and $v, v_1 \top^{>} u_2, v_2$ and $u_2 \neq v_2$ holds $u, u_1 \upuparrows v, v_1$.

(20) If A_1 is right transitive but A_1 is Euclidean like or A_1 is Minkowskian like, then A_1 is left transitive.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical metric affine spaces and planes. *Formalized Mathematics*, 1(**5**):891–899, 1990.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. A construction of analytical ordered trapezium spaces. *Formalized Mathematics*, 2(**3**):315–322, 1991.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[6] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[7] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[8] Jarosław Zajkowski. Oriented metric-affine plane - part I. *Formalized Mathematics*, 2(**4**):591–597, 1991.

Received June 19, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Opposite Rings, Modules and their Morphisms

Michał Muzalewski
Warsaw University
Białystok

Summary. Let $\mathbb{K} = \langle S; K, 0, 1, +, \cdot\rangle$ be a ring. The structure ${}^{\rm op}\mathbb{K} = \langle S; K, 0, 1, +, \bullet\rangle$ is called anti-ring, if $\alpha \bullet \beta = \beta \cdot \alpha$ for elements α, β of K [12, pages 5–7]. It is easily seen that ${}^{\rm op}\mathbb{K}$ is also a ring. If V is a left module over $\mathbb{K}$, then V is a right module over ${}^{\rm op}\mathbb{K}$. If W is a right module over $\mathbb{K}$, then W is a left module over ${}^{\rm op}\mathbb{K}$. Let K, L be rings. A morphism $J : K \longrightarrow L$ is called anti-homomorphism, if $J(\alpha \cdot \beta) = J(\beta) \cdot J(\alpha)$ for elements α, β of K. If $J : K \longrightarrow L$ is a homomorphism, then $J : K \longrightarrow {}^{\rm op}L$ is an anti-homomorphism. Let K, L be rings, V, W left modules over K, L respectively and $J : K \longrightarrow L$ an anti-monomorphism. A map $f : V \longrightarrow W$ is called J - semilinear, if $f(x + y) = f(x) + f(y)$ and $f(\alpha \cdot x) = J(\alpha) \cdot f(x)$ for vectors x, y of V and a scalar α of K.

MML Identifier: MOD_4.

The papers [19], [18], [21], [3], [4], [1], [20], [17], [2], [7], [8], [11], [14], [15], [16], [5], [6], [9], [13], and [10] provide the notation and terminology for this paper.

1. Opposite functions

In the sequel A, B, C are non-empty sets and f is a function from $[\!:A, B:\!]$ into C. Let us consider A, B, C, f. Then $\curvearrowleft f$ is a function from $[\!:B, A:\!]$ into C.

We now state the proposition

(1) For every element x of A and for every element y of B holds $f(x, y) = (\curvearrowleft f)(y, x)$.

ISSN 0777–4028

2. Opposite rings

In the sequel K, L will be field structures. Let us consider K. The functor ${}^{\rm op}K$ yielding a strict field structure is defined by:

(Def.1) ${}^{\rm op}K = \langle$the carrier of $K, \curvearrowleft($the multiplication of $K)$, the addition of K, the reverse-map of K, the unity of K, the zero of $K\rangle$.

We now state four propositions:

(2) The group structure of ${}^{\rm op}K$ = the group structure of K and for an arbitrary x holds x is a scalar of ${}^{\rm op}K$ if and only if x is a scalar of K.

(3) ${}^{\rm op}({}^{\rm op}K)$ = the field structure of K.

(4) (i) $0_K = 0_{{}^{\rm op}K}$,

(ii) $1_K = 1_{{}^{\rm op}K}$,

(iii) for all scalars x, y, z, u of K and for all scalars a, b, c, d of ${}^{\rm op}K$ such that $x = a$ and $y = b$ and $z = c$ and $u = d$ holds $x+y = a+b$ and $x \cdot y = b \cdot a$ and $-x = -a$ and $x+y+z = a+b+c$ and $x+(y+z) = a+(b+c)$ and $(x \cdot y) \cdot z = c \cdot (b \cdot a)$ and $x \cdot (y \cdot z) = (c \cdot b) \cdot a$ and $x \cdot (y+z) = (b+c) \cdot a$ and $(y+z) \cdot x = a \cdot (b+c)$ and $x \cdot y + z \cdot u = b \cdot a + d \cdot c$.

(5) For every ring K holds ${}^{\rm op}K$ is a strict ring.

Let K be a ring. Then ${}^{\rm op}K$ is a strict ring.

One can prove the following proposition

(6) For every associative ring K holds ${}^{\rm op}K$ is an associative ring.

Let K be an associative ring. Then ${}^{\rm op}K$ is a strict associative ring.

Next we state the proposition

(7) For every skew field K holds ${}^{\rm op}K$ is a skew field.

Let K be a skew field. Then ${}^{\rm op}K$ is a strict skew field.

One can prove the following proposition

(8) For every field K holds ${}^{\rm op}K$ is a strict field.

Let K be a field. Then ${}^{\rm op}K$ is a strict field.

3. Opposite modules

In the sequel V denotes a left module structure over K. Let us consider K, V. The functor ${}^{\rm op}V$ yields a strict right module structure over ${}^{\rm op}K$ and is defined as follows:

(Def.2) for every function o from $[\!:$the carrier of V, the carrier of ${}^{\rm op}K:\!]$ into the carrier of V such that $o = \curvearrowleft($the left multiplication of $V)$ holds ${}^{\rm op}V = \langle$the carrier of V, the addition of V, the reverse-map of V, the zero of $V, o\rangle$.

The following proposition is true

(9) The group structure of ${}^{\rm op}V$ = the group structure of V and for an arbitrary x holds x is a vector of V if and only if x is a vector of ${}^{\rm op}V$.

Let us consider K, V, and let o be a function from $[\!:$the carrier of K, the carrier of $V:\!]$ into the carrier of V. The functor ${}^{\rm op}o$ yields a function from $[\!:$the carrier of ${}^{\rm op}V$, the carrier of ${}^{\rm op}K:\!]$ into the carrier of ${}^{\rm op}V$ and is defined by:

(Def.3) ${}^{\rm op}o = \curvearrowleft o$.

One can prove the following two propositions:

(10) The right multiplication of ${}^{\rm op}V = {}^{\rm op}$(the left multiplication of V).

(11) ${}^{\rm op}V = \langle$the carrier of ${}^{\rm op}V$, the addition of ${}^{\rm op}V$, the reverse-map of ${}^{\rm op}V$, the zero of ${}^{\rm op}V$, ${}^{\rm op}$(the left multiplication of V)$\rangle$.

In the sequel W denotes a right module structure over K. Let us consider K, W. The functor ${}^{\rm op}W$ yields a strict left module structure over ${}^{\rm op}K$ and is defined by:

(Def.4) for every function o from $[\!:$the carrier of ${}^{\rm op}K$, the carrier of $W:\!]$ into the carrier of W such that $o = \curvearrowleft$(the right multiplication of W) holds ${}^{\rm op}W = \langle$the carrier of W, the addition of W, the reverse-map of W, the zero of $W, o\rangle$.

We now state the proposition

(12) The group structure of ${}^{\rm op}W =$ the group structure of W and for an arbitrary x holds x is a vector of W if and only if x is a vector of ${}^{\rm op}W$.

Let us consider K, W, and let o be a function from $[\!:$the carrier of W, the carrier of $K:\!]$ into the carrier of W. The functor ${}^{\rm op}o$ yielding a function from $[\!:$the carrier of ${}^{\rm op}K$, the carrier of ${}^{\rm op}W:\!]$ into the carrier of ${}^{\rm op}W$ is defined as follows:

(Def.5) ${}^{\rm op}o = \curvearrowleft o$.

The following propositions are true:

(13) The left multiplication of ${}^{\rm op}W = {}^{\rm op}$(the right multiplication of W).

(14) ${}^{\rm op}W = \langle$the carrier of ${}^{\rm op}W$, the addition of ${}^{\rm op}W$, the reverse-map of ${}^{\rm op}W$, the zero of ${}^{\rm op}W$, ${}^{\rm op}$(the right multiplication of W)$\rangle$.

(15) For every function o from $[\!:$the carrier of K, the carrier of $V:\!]$ into the carrier of V holds ${}^{\rm op}({}^{\rm op}o) = o$.

(16) For every function o from $[\!:$the carrier of K, the carrier of $V:\!]$ into the carrier of V and for every scalar x of K and for every scalar y of ${}^{\rm op}K$ and for every vector v of V and for every vector w of ${}^{\rm op}V$ such that $x = y$ and $v = w$ holds $({}^{\rm op}o)(w, y) = o(x, v)$.

(17) Let K, L be rings. Then for every V being a left module structure over K and for every W being a right module structure over L and for every scalar x of K and for every scalar y of L and for every vector v of V and for every vector w of W such that $L = {}^{\rm op}K$ and $W = {}^{\rm op}V$ and $x = y$ and $v = w$ holds $w \cdot y = x \cdot v$.

(18) For all rings K, L and for every V being a left module structure over K and for every W being a right module structure over L and for all vectors v_1, v_2 of V and for all vectors w_1, w_2 of W such that $L = {}^{\rm op}K$ and $W = {}^{\rm op}V$ and $v_1 = w_1$ and $v_2 = w_2$ holds $w_1 + w_2 = v_1 + v_2$.

(19) For every function o from $[\!:$ the carrier of W, the carrier of $K :\!]$ into the carrier of W holds ${}^{\rm op}({}^{\rm op}o) = o$.

(20) For every function o from $[\!:$ the carrier of W, the carrier of $K :\!]$ into the carrier of W and for every scalar x of K and for every scalar y of ${}^{\rm op}K$ and for every vector v of W and for every vector w of ${}^{\rm op}W$ such that $x = y$ and $v = w$ holds $({}^{\rm op}o)(y, w) = o(v, x)$.

(21) Let K, L be rings. Then for every V being a left module structure over K and for every W being a right module structure over L and for every scalar x of K and for every scalar y of L and for every vector v of V and for every vector w of W such that $K = {}^{\rm op}L$ and $V = {}^{\rm op}W$ and $x = y$ and $v = w$ holds $w \cdot y = x \cdot v$.

(22) For all rings K, L and for every V being a left module structure over K and for every W being a right module structure over L and for all vectors v_1, v_2 of V and for all vectors w_1, w_2 of W such that $K = {}^{\rm op}L$ and $V = {}^{\rm op}W$ and $v_1 = w_1$ and $v_2 = w_2$ holds $w_1 + w_2 = v_1 + v_2$.

(23) For every K being a strict field structure and for every V being a left module structure over K holds ${}^{\rm op}({}^{\rm op}V) =$ the left module structure of V.

(24) For every K being a strict field structure and for every W being a right module structure over K holds ${}^{\rm op}({}^{\rm op}W) =$ the right module structure of W.

(25) For every associative ring K and for every left module V over K holds ${}^{\rm op}V$ is a strict right module over ${}^{\rm op}K$.

Let K be an associative ring, and let V be a left module over K. Then ${}^{\rm op}V$ is a strict right module over ${}^{\rm op}K$.

One can prove the following proposition

(26) For every associative ring K and for every right module W over K holds ${}^{\rm op}W$ is a strict left module over ${}^{\rm op}K$.

Let K be an associative ring, and let W be a right module over K. Then ${}^{\rm op}W$ is a strict left module over ${}^{\rm op}K$.

4. Morphisms of rings

We now define several new attributes. Let us consider K, L. A map from K into L is antilinear if:

(Def.6) for all scalars x, y of K holds ${\rm it}(x+y) = {\rm it}(x) + {\rm it}(y)$ and for all scalars x, y of K holds ${\rm it}(x \cdot y) = {\rm it}(y) \cdot {\rm it}(x)$ and ${\rm it}(1_K) = 1_L$.

A map from K into L is monomorphism if:

(Def.7) it is linear and it is one-to-one.

A map from K into L is antimonomorphism if:

(Def.8) it is antilinear and it is one-to-one.

A map from K into L is epimorphism if:

(Def.9) it is linear and rng it = the carrier of L.

A map from K into L is antiepimorphism if:

(Def.10) it is antilinear and rng it = the carrier of L.

A map from K into L is isomorphism if:

(Def.11) it is monomorphism and rng it = the carrier of L.

A map from K into L is antiisomorphism if:

(Def.12) it is antimonomorphism and rng it = the carrier of L.

In the sequel J denotes a map from K into K. We now define four new attributes. Let us consider K. A map from K into K is endomorphism if:

(Def.13) it is linear.

A map from K into K is antiendomorphism if:

(Def.14) it is antilinear.

A map from K into K is automorphism if:

(Def.15) it is isomorphism.

A map from K into K is antiautomorphism if:

(Def.16) it is antiisomorphism.

One can prove the following propositions:

(27) J is automorphism if and only if the following conditions are satisfied:
  (i) for all scalars x, y of K holds $J(x+y) = J(x) + J(y)$,
  (ii) for all scalars x, y of K holds $J(x \cdot y) = J(x) \cdot J(y)$,
  (iii) $J(1_K) = 1_K$,
  (iv) J is one-to-one,
  (v) $\operatorname{rng} J$ = the carrier of K.

(28) J is antiautomorphism if and only if the following conditions are satisfied:
  (i) for all scalars x, y of K holds $J(x+y) = J(x) + J(y)$,
  (ii) for all scalars x, y of K holds $J(x \cdot y) = J(y) \cdot J(x)$,
  (iii) $J(1_K) = 1_K$,
  (iv) J is one-to-one,
  (v) $\operatorname{rng} J$ = the carrier of K.

(29) id_K is automorphism.

We follow the rules: K, L will denote rings, J will denote a map from K into L, and x, y will denote scalars of K. Next we state three propositions:

(30) If J is linear, then $J(0_K) = 0_L$ and $J(-x) = -J(x)$ and $J(x-y) = J(x) - J(y)$.

(31) If J is antilinear, then $J(0_K) = 0_L$ and $J(-x) = -J(x)$ and $J(x-y) = J(x) - J(y)$.

(32) For every associative ring K holds id_K is antiautomorphism if and only if K is a commutative ring.

One can prove the following proposition

(33) For every skew field K holds id_K is antiautomorphism if and only if K is a field.

5. Opposite morphisms to morphisms of rings

In the sequel K, L will be field structures and J will be a map from K into L. Let us consider K, L, J. The functor ${}^{\mathrm{op}}J$ yielding a map from K into ${}^{\mathrm{op}}L$ is defined by:

(Def.17) ${}^{\mathrm{op}}J = J$.

Next we state several propositions:

(34) ${}^{\mathrm{op}}({}^{\mathrm{op}}J) = J$.

(35) J is linear if and only if ${}^{\mathrm{op}}J$ is antilinear.

(36) J is antilinear if and only if ${}^{\mathrm{op}}J$ is linear.

(37) J is monomorphism if and only if ${}^{\mathrm{op}}J$ is antimonomorphism.

(38) J is antimonomorphism if and only if ${}^{\mathrm{op}}J$ is monomorphism.

(39) J is epimorphism if and only if ${}^{\mathrm{op}}J$ is antiepimorphism.

(40) J is antiepimorphism if and only if ${}^{\mathrm{op}}J$ is epimorphism.

(41) J is isomorphism if and only if ${}^{\mathrm{op}}J$ is antiisomorphism.

(42) J is antiisomorphism if and only if ${}^{\mathrm{op}}J$ is isomorphism.

In the sequel J will be a map from K into K. We now state four propositions:

(43) J is endomorphism if and only if ${}^{\mathrm{op}}J$ is antilinear.

(44) J is antiendomorphism if and only if ${}^{\mathrm{op}}J$ is linear.

(45) J is automorphism if and only if ${}^{\mathrm{op}}J$ is antiisomorphism.

(46) J is antiautomorphism if and only if ${}^{\mathrm{op}}J$ is isomorphism.

6. Morphisms of groups

In the sequel G, H will denote groups. Let us consider G, H. A map from G into H is said to be a homomorphism from G to H if:

(Def.18) for all elements x, y of G holds $\mathrm{it}(x + y) = \mathrm{it}(x) + \mathrm{it}(y)$.

Then $\mathrm{zero}(G, H)$ is a homomorphism from G to H.

In the sequel f is a homomorphism from G to H. We now define four new constructions. Let us consider G, H. A homomorphism from G to H is monomorphism if:

(Def.19) it is one-to-one.

A homomorphism from G to H is epimorphism if:

(Def.20) $\mathrm{rng}\,\mathrm{it} =$ the carrier of H.

A homomorphism from G to H is isomorphism if:

(Def.21) it is one-to-one and rng it = the carrier of H.

Let us consider G. An endomorphism of G is a homomorphism from G to G.

We now state the proposition

(47) For every element x of G holds $\mathrm{id}_G(x) = x$.

We now define two new constructions. Let us consider G. An endomorphism of G is automorphism-like if:

(Def.22) it is isomorphism.

An automorphism of G is an automorphism-like endomorphism of G.

Then id_G is an automorphism of G.

In the sequel x, y will be elements of G. We now state the proposition

(48) $f(0_G) = 0_H$ and $f(-x) = -f(x)$ and $f(x -' y) = f(x) -' f(y)$.

We adopt the following convention: G, H denote Abelian groups, f denotes a homomorphism from G to H, and x, y denote elements of G. The following proposition is true

(49) $f(x - y) = f(x) - f(y)$.

7. Semilinear morphisms

For simplicity we adopt the following rules: K, L are associative rings, J is a map from K into L, V is a left module over K, and W is a left module over L. Let us consider K, L, J, V, W. A map from V into W is said to be a homomorphism from V to W by J if:

(Def.23) for all vectors x, y of V holds $\mathrm{it}(x + y) = \mathrm{it}(x) + \mathrm{it}(y)$ and for every scalar a of K and for every vector x of V holds $\mathrm{it}(a \cdot x) = J(a) \cdot \mathrm{it}(x)$.

The following proposition is true

(50) $\mathrm{zero}(V, W)$ is a homomorphism from V to W by J.

In the sequel f denotes a homomorphism from V to W by J. We now define three new predicates. Let us consider K, L, J, V, W, f. We say that f is a monomorphism wrp J if and only if:

(Def.24) f is one-to-one.

We say that f is a epimorphism wrp J if and only if:

(Def.25) $\mathrm{rng}\, f =$ the carrier of W.

We say that f is a isomorphism wrp J if and only if:

(Def.26) f is one-to-one and $\mathrm{rng}\, f =$ the carrier of W.

In the sequel J will denote a map from K into K and f will denote a homomorphism from V to V by J. We now define two new constructions. Let us consider K, J, V. An endomorphism of J and V is a homomorphism from V to V by J.

Let us consider K, J, V, f. We say that f is a automorphism wrp J if and only if:

(Def.27) f is one-to-one and $\operatorname{rng} f =$ the carrier of V.

In the sequel W is a left module over K. Let us consider K, V, W. A homomorphism from V to W is a homomorphism from V to W by id_K.

Next we state the proposition

(51) For every map f from V into W holds f is a homomorphism from V to W if and only if for all vectors x, y of V holds $f(x+y) = f(x)+f(y)$ and for every scalar a of K and for every vector x of V holds $f(a \cdot x) = a \cdot f(x)$.

We now define five new constructions. Let us consider K, V, W. A homomorphism from V to W is monomorphism if:

(Def.28) it is one-to-one.

A homomorphism from V to W is epimorphism if:

(Def.29) rng it = the carrier of W.

A homomorphism from V to W is isomorphism if:

(Def.30) it is one-to-one and rng it = the carrier of W.

Let us consider K, V. An endomorphism of V is a homomorphism from V to V.

An endomorphism of V is automorphism if:

(Def.31) it is one-to-one and rng it = the carrier of V.

8. Annex

Next we state three propositions:

(52) For every skew field K holds K is a field if and only if for all scalars x, y of K holds $x \cdot y = y \cdot x$.

(53) For every K being a field structure holds K is a field if and only if K is a skew field and for all scalars x, y of K holds $x \cdot y = y \cdot x$.

(54) For every group G and for all elements x, y, z of G such that $x+y = x+z$ holds $y = z$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.
[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[9] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[10] Michał Muzalewski. Category of rings. *Formalized Mathematics*, 2(**5**):643–648, 1991.
[11] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[12] Michał Muzalewski. *Foundations of Metric-Affine Geometry.* Dział Wydawnictw Filii UW w Białymstoku, Filia UW w Białymstoku, 1990.
[13] Michał Muzalewski. Rings and modules - part II. *Formalized Mathematics*, 2(**4**):579–585, 1991.
[14] Michał Muzalewski and Wojciech Skaba. Groups, rings, left- and right-modules. *Formalized Mathematics*, 2(**2**):275–278, 1991.
[15] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):97–104, 1991.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[18] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received June 22, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Properties of Caratheodor's Measure

Józef Białas
University of Łódź

Summary. The paper contains definitions and basic properties of Caratheodor's measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [14]. The article includes the text being a continuation of the paper [3]. Caratheodor's theorem and some theorems concerning basic properties of Caratheodor's measure are proved. The work is the sixth part of the series of articles concerning the Lebesgue measure theory.

MML Identifier: `MEASURE4`.

The terminology and notation used in this paper have been introduced in the following papers: [16], [15], [10], [11], [8], [9], [1], [13], [2], [12], [4], [5], [7], [6], [3], and [17]. One can prove the following propositions:

(1) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$ holds $(x+y)+z = x+(y+z)$.

(2) For all *Real numbers* x, y, z such that $x \neq -\infty$ and $x \neq +\infty$ holds $y+x \leq z$ if and only if $y \leq z-x$.

(3) For all *Real numbers* x, y such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ holds $x+y = y+x$.

(4) For every set X and for every σ-field S of subsets of X and for every function F from $\mathbb{N}$ into S and for every element A of S and for every function G from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $G(n) = A \cap F(n)$ holds $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(5) Let X be a set. Let S be a σ-field of subsets of X. Let F be a function from $\mathbb{N}$ into S. Let G be a function from $\mathbb{N}$ into S. Suppose $G(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(n+1) \cup G(n)$. Then for every function H from $\mathbb{N}$ into S such that $H(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $H(n+1) = F(n+1) \setminus G(n)$ holds $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} H$.

(6) For every set X holds 2^X is a σ-field of subsets of X.

ISSN 0777–4028

Let X be a set, and let F be a function from $\mathbb{N}$ into 2^X. Then $\operatorname{rng} F$ is a non-empty family of subsets of X. Let A be a non-empty family of subsets of X. Then $\bigcup A$ is an element of 2^X. Let F be a function from 2^X into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

(Def.1) for every element A of 2^X holds $0_{\overline{\mathbb{R}}} \leq F(A)$.

Let F be a function from $\mathbb{N}$ into 2^X, and let M be a function from 2^X into $\overline{\mathbb{R}}$. Then $M \cdot F$ is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

One can prove the following propositions:

(7) For every set X and for every *Real numbers* a, b there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds if $A = \emptyset$, then $M(A) = a$ but if $A \neq \emptyset$, then $M(A) = b$.

(8) For every set X there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(9) For every set X and for every function F from $\mathbb{N}$ into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative holds $M \cdot F$ is non-negative.

(10) For every set X and for every function F from $\mathbb{N}$ into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ and for every natural number n holds $(M \cdot F)(n) = M(F(n))$.

(11) Let X be a set. Then there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $M(A) \leq M(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$.

We now define two new constructions. Let X be a set. A function from 2^X into $\overline{\mathbb{R}}$ is said to be a Caratheodor's measure on X if:

(Def.2) it is non-negative and $\mathrm{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $\mathrm{it}(A) \leq \mathrm{it}(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $\mathrm{it}(\bigcup \operatorname{rng} F) \leq \sum(\mathrm{it} \cdot F)$.

Let C be a Caratheodor's measure on X. The functor σ-Field(C) yielding a non-empty family of subsets of X is defined by:

(Def.3) for every element A of 2^X holds $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) \leq C(W \cup Z)$.

The following propositions are true:

(12) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(W \cup Z) \leq C(W) + C(Z)$.

(13) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(Z) + C(W) = C(W) + C(Z)$.

(14) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X holds $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) = C(W \cup Z)$.

(15) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X such that $W \in \sigma\text{-Field}(C)$ and $Z \in \sigma\text{-Field}(C)$ and $Z \cap W = \emptyset$ holds $C(W \cup Z) = C(W) + C(Z)$.

(16) For every set X and for every Caratheodor's measure C on X and for every set A such that $A \in \sigma\text{-Field}(C)$ holds $X \setminus A \in \sigma\text{-Field}(C)$.

(17) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cup B \in \sigma\text{-Field}(C)$.

(18) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cap B \in \sigma\text{-Field}(C)$.

(19) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \setminus B \in \sigma\text{-Field}(C)$.

(20) For every set X and for every σ-field S of subsets of X and for every function N from $\mathbb{N}$ into S and for every element A of S there exists a function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) = A \cap N(n)$.

(21) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Field}(C)$ is a σ-field of subsets of X.

Let X be a set, and let C be a Caratheodor's measure on X. Then $\sigma\text{-Field}(C)$ is a σ-field of subsets of X. Let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S. The functor $\sigma\text{-Meas}(C)$ yields a function from $\sigma\text{-Field}(C)$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.4) for every element A of 2^X such that $A \in \sigma\text{-Field}(C)$ holds $(\sigma\text{-Meas}(C))(A) = C(A)$.

One can prove the following proposition

(22) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X, and let A be an element of $\sigma\text{-Field}(C)$. Then $C(A)$ is a *Real number*.

One can prove the following proposition

(23) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a σ-measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X. Then $\sigma\text{-Meas}(C)$ is a σ-measure on $\sigma\text{-Field}(C)$.

The following propositions are true:

(24) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X such that $C(A) = 0_{\overline{\mathbb{R}}}$ holds $A \in \sigma\text{-Field}(C)$.

(25) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is complete on $\sigma\text{-Field}(C)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Józef Biał as. Completeness of the σ-additive measure. measure theory. *Formalized Mathematics*, 2(**5**):689–693, 1991.
[4] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[6] Józef Białas. Several properties of the σ-additive measure. *Formalized Mathematics*, 2(**4**):493–497, 1991.
[7] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[8] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[9] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[10] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[11] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[14] R. Sikorski. *Rachunek różniczkowy i całkowy - funkcje wielu zmiennych. Biblioteka Matematyczna*, PWN - Warszawa, 1968.
[15] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received June 25, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Completeness of the Lattices of Domains of a Topological Space [1]

Zbigniew Karno
Warsaw University
Białystok

Toshihiko Watanabe
Shinshu University
Nagano

Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *domain* in T provided $\operatorname{Int}\overline{A} \subseteq A \subseteq \overline{\operatorname{Int} A}$ (see [24] and comp. [14]). This notion is a simple generalization of the notions of open and closed domains in T (see [24]). Our main result is concerned with an extension of the following well–known theorem (see e.g. [5], [17], [13]). For a given topological space the Boolean lattices of all its closed domains and all its open domains are complete. It is proved here, using Mizar System, that *the complemented lattice of all domains of a given topological space is complete*, too (comp. [23]).

It is known that both the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains [23]. However, the following two problems remain open.

> **Problem 1.** Let L be a sublattice of the lattice of all domains. Suppose L is complete, is smallest with respect to inclusion, and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all domains ?

A domain in T is said to be a *Borel domain* provided it is a Borel set. Of course every open (closed) domain is a Borel domain. It can be proved that all Borel domains form a sublattice of the lattice of domains.

> **Problem 2.** Let L be a sublattice of the lattice of all domains. Suppose L is smallest with respect to inclusion and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all Borel domains ?

Note that in the beginning the closure and the interior operations for families of subsets of topological spaces are introduced and their important properties are presented (comp. [16], [15], [17]). Using these notions, certain properties of domains, closed domains and open domains are studied (comp. [15], [13]).

MML Identifier: `TDLAT_2`.

[1]This paper was done while the second author was visiting the Institute of Mathematics of Warsaw University in Białystok.

ISSN 0777–4028

The papers [20], [22], [21], [18], [8], [9], [12], [4], [3], [19], [24], [11], [6], [7], [25], [10], [2], [1], and [23] provide the notation and terminology for this paper.

1. Preliminary Theorems about Subsets of Topological Spaces

In the sequel T will denote a topological space. One can prove the following propositions:

(1) For every subset A of T holds $\operatorname{Int}\overline{\operatorname{Int}A} \subseteq \operatorname{Int}\overline{A}$ and $\operatorname{Int}\overline{\operatorname{Int}A} \subseteq \overline{\operatorname{Int}A}$.

(2) For every subset A of T holds $\overline{\operatorname{Int}A} \subseteq \overline{\operatorname{Int}\overline{A}}$ and $\operatorname{Int}\overline{A} \subseteq \overline{\operatorname{Int}\overline{A}}$.

(3) For all subsets A, B of T such that B is closed holds if $\overline{\operatorname{Int}(A \cap B)} = A$, then $A \subseteq B$.

(4) For all subsets A, B of T such that A is open holds if $\operatorname{Int}\overline{A \cup B} = B$, then $A \subseteq B$.

(5) For every subset A of T such that $A \subseteq \overline{\operatorname{Int}A}$ holds $A \cup \operatorname{Int}\overline{A} \subseteq \overline{\operatorname{Int}(A \cup \operatorname{Int}\overline{A})}$.

(6) For every subset A of T such that $\operatorname{Int}\overline{A} \subseteq A$ holds $\operatorname{Int}\overline{A \cap \overline{\operatorname{Int}A}} \subseteq A \cap \overline{\operatorname{Int}A}$.

2. The Closure and the Interior Operations for Families of Subsets of a Topological Space

In the sequel T will be a topological space. Let us consider T, and let F be a family of subsets of T. We introduce the functor $\overline{F}$ as a synonym of clf F.

One can prove the following propositions:

(7) For every family F of subsets of T holds $\overline{F} = \{A : \bigvee_{B}[A = \overline{B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(8) For every family F of subsets of T holds $\overline{F} = \overline{\overline{F}}$.

(9) For every family F of subsets of T holds $F = \emptyset$ if and only if $\overline{F} = \emptyset$.

(10) For all families F, G of subsets of T holds $\overline{F \cap G} \subseteq \overline{F} \cap \overline{G}$.

(11) For all families F, G of subsets of T holds $\overline{F} \setminus \overline{G} \subseteq \overline{F \setminus G}$.

(12) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\bigcap \overline{F} \subseteq \overline{A}$ and $\overline{A} \subseteq \bigcup \overline{F}$.

(13) For every family F of subsets of T holds $\bigcap F \subseteq \bigcap \overline{F}$.

(14) For every family F of subsets of T holds $\overline{\bigcap F} \subseteq \bigcap \overline{F}$.

(15) For every family F of subsets of T holds $\bigcup \overline{F} \subseteq \overline{\bigcup F}$.

Let us consider T, and let F be a family of subsets of T. The functor $\operatorname{Int} F$ yielding a family of subsets of T is defined as follows:

(Def.1) for every subset A of T holds $A \in \operatorname{Int} F$ if and only if there exists a subset B of T such that $A = \operatorname{Int} B$ and $B \in F$.

The following propositions are true:

(16) For every family F of subsets of T holds $\mathrm{Int}\,F = \{A : \bigvee_B[A = \mathrm{Int}\,B \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(17) For every family F of subsets of T holds $\mathrm{Int}\,F = \mathrm{Int}\,\mathrm{Int}\,F$.

(18) For every family F of subsets of T holds $\mathrm{Int}\,F$ is open.

(19) For every family F of subsets of T holds $F = \emptyset$ if and only if $\mathrm{Int}\,F = \emptyset$.

(20) For every subset A of T and for every family F of subsets of T such that $F = \{A\}$ holds $\mathrm{Int}\,F = \{\mathrm{Int}\,A\}$.

(21) For all families F, G of subsets of T such that $F \subseteq G$ holds $\mathrm{Int}\,F \subseteq \mathrm{Int}\,G$.

(22) For all families F, G of subsets of T holds $\mathrm{Int}(F \cup G) = \mathrm{Int}\,F \cup \mathrm{Int}\,G$.

(23) For all families F, G of subsets of T holds $\mathrm{Int}(F \cap G) \subseteq \mathrm{Int}\,F \cap \mathrm{Int}\,G$.

(24) For all families F, G of subsets of T holds $\mathrm{Int}\,F \setminus \mathrm{Int}\,G \subseteq \mathrm{Int}(F \setminus G)$.

(25) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\mathrm{Int}\,A \subseteq \bigcup \mathrm{Int}\,F$ and $\bigcap \mathrm{Int}\,F \subseteq \mathrm{Int}\,A$.

(26) For every family F of subsets of T holds $\bigcup \mathrm{Int}\,F \subseteq \bigcup F$.

(27) For every family F of subsets of T holds $\bigcap \mathrm{Int}\,F \subseteq \bigcap F$.

(28) For every family F of subsets of T holds $\bigcup \mathrm{Int}\,F \subseteq \mathrm{Int} \bigcup F$.

(29) For every family F of subsets of T holds $\mathrm{Int} \bigcap F \subseteq \bigcap \mathrm{Int}\,F$.

(30) For every family F of subsets of T such that F is finite holds $\mathrm{Int} \bigcap F = \bigcap \mathrm{Int}\,F$.

In the sequel F denotes a family of subsets of T. The following propositions are true:

(31) $\overline{\mathrm{Int}\,F} = \{A : \bigvee_B[A = \overline{\mathrm{Int}\,B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(32) $\mathrm{Int}\,\overline{F} = \{A : \bigvee_B[A = \mathrm{Int}\,\overline{B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(33) $\overline{\mathrm{Int}\,\overline{F}} = \{A : \bigvee_B[A = \overline{\mathrm{Int}\,\overline{B}} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(34) $\mathrm{Int}\,\overline{\mathrm{Int}\,F} = \{A : \bigvee_B[A = \mathrm{Int}\,\overline{\mathrm{Int}\,B} \wedge B \in F]\}$, where A ranges over subsets of T, and B ranges over subsets of T.

(35) $\overline{\mathrm{Int}\,\overline{\mathrm{Int}\,F}} = \overline{\mathrm{Int}\,F}$.

(36) $\mathrm{Int}\,\overline{\mathrm{Int}\,\overline{F}} = \mathrm{Int}\,\overline{F}$.

(37) $\bigcup \mathrm{Int}\,\overline{F} \subseteq \bigcup \overline{\mathrm{Int}\,\overline{F}}$.

(38) $\bigcap \mathrm{Int}\,\overline{F} \subseteq \bigcap \overline{\mathrm{Int}\,\overline{F}}$.

(39) $\bigcup \overline{\mathrm{Int}\,F} \subseteq \bigcup \overline{\mathrm{Int}\,\overline{F}}$.

(40) $\bigcap \overline{\mathrm{Int}\,F} \subseteq \bigcap \overline{\mathrm{Int}\,\overline{F}}$.

(41) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcup \mathrm{Int}\,\overline{F}$.

(42) $\bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcap \mathrm{Int}\,\overline{F}$.

(43) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcup \overline{\mathrm{Int}\,F}$.

(44) $\bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcap \overline{\mathrm{Int}\,F}$.

(45) $\bigcup \overline{\mathrm{Int}\,\overline{F}} \subseteq \bigcup \overline{F}$.

(46) $\bigcap \overline{\mathrm{Int}\,\overline{F}} \subseteq \bigcap \overline{F}$.

(47) $\bigcup \mathrm{Int}\,F \subseteq \bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F}$.

(48) $\bigcap \mathrm{Int}\,F \subseteq \bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F}$.

(49) $\bigcup \overline{\mathrm{Int}\,F} \subseteq \overline{\mathrm{Int} \bigcup F}$.

(50) $\overline{\mathrm{Int} \bigcap F} \subseteq \bigcap \overline{\mathrm{Int}\,F}$.

(51) $\bigcup \mathrm{Int}\,\overline{F} \subseteq \mathrm{Int}\,\overline{\bigcup F}$.

(52) $\mathrm{Int}\,\overline{\bigcap F} \subseteq \bigcap \mathrm{Int}\,\overline{F}$.

(53) $\bigcup \overline{\mathrm{Int}\,\overline{F}} \subseteq \overline{\mathrm{Int}\,\overline{\bigcup F}}$.

(54) $\overline{\mathrm{Int}\,\overline{\bigcap F}} \subseteq \bigcap \overline{\mathrm{Int}\,\overline{F}}$.

(55) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \mathrm{Int}\,\overline{\mathrm{Int} \bigcup F}$.

(56) $\mathrm{Int}\,\overline{\mathrm{Int} \bigcap F} \subseteq \bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F}$.

(57) For every family F of subsets of T such that for every subset A of T such that $A \in F$ holds $A \subseteq \overline{\mathrm{Int}\,A}$ holds $\bigcup F \subseteq \overline{\mathrm{Int} \bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int}\,\overline{\bigcup F}}$.

(58) For every family F of subsets of T such that for every subset A of T such that $A \in F$ holds $\mathrm{Int}\,\overline{A} \subseteq A$ holds $\mathrm{Int}\,\overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int}\,\overline{\mathrm{Int} \bigcap F} = \mathrm{Int} \bigcap F$.

3. Selected Properties of Domains of a Topological Space

In the sequel T is a topological space. We now state several propositions:

(59) For all subsets A, B of T such that B is a domain holds $\mathrm{Int}\,\overline{A \cup B} \cup (A \cup B) = B$ if and only if $A \subseteq B$.

(60) For all subsets A, B of T such that A is a domain holds $\overline{\mathrm{Int}(A \cap B)} \cap (A \cap B) = A$ if and only if $A \subseteq B$.

(61) For all subsets A, B of T such that A is a closed domain and B is a closed domain holds $\mathrm{Int}\,A \subseteq \mathrm{Int}\,B$ if and only if $A \subseteq B$.

(62) For all subsets A, B of T such that A is an open domain and B is an open domain holds $\overline{A} \subseteq \overline{B}$ if and only if $A \subseteq B$.

(63) For all subsets A, B of T such that A is a closed domain holds if $A \subseteq B$, then $\overline{\mathrm{Int}(A \cap B)} = A$.

(64) For all subsets A, B of T such that B is an open domain holds if $A \subseteq B$, then $\mathrm{Int}\,\overline{A \cup B} = B$.

Let us consider T. A family of subsets of T is domains-family if:

(Def.2) for every subset A of T such that $A \in$ it holds A is a domain.

The following propositions are true:

(65) For every family F of subsets of T holds $F \subseteq$ the domains of T if and only if F is domains-family.

(66) For every family F of subsets of T such that F is domains-family holds $\bigcup F \subseteq \overline{\mathrm{Int} \bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int} \overline{\bigcup F}}$.

(67) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int} \overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int} \overline{\mathrm{Int} \bigcap F} = \mathrm{Int} \bigcap F$.

(68) For every family F of subsets of T such that F is domains-family holds $\bigcup F \cup \mathrm{Int} \overline{\bigcup F}$ is a domain.

(69) Let F be a family of subsets of T. Then for every subset B of T such that $B \in F$ holds $B \subseteq \bigcup F \cup \mathrm{Int} \overline{\bigcup F}$ and for every subset A of T such that A is a domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\bigcup F \cup \mathrm{Int} \overline{\bigcup F} \subseteq A$.

(70) For every family F of subsets of T such that F is domains-family holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F}$ is a domain.

(71) Let F be a family of subsets of T. Then
  (i) for every subset B of T such that $B \in F$ holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F} \subseteq B$,
  (ii) $F = \emptyset$ or for every subset A of T such that A is a domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$.

Let us consider T. A family of subsets of T is closed-domains-family if:

(Def.3) for every subset A of T such that $A \in$ it holds A is a closed domain.

We now state several propositions:

(72) For every family F of subsets of T holds $F \subseteq$ the closed domains of T if and only if F is closed-domains-family.

(73) For every family F of subsets of T such that F is closed-domains-family holds F is domains-family.

(74) For every family F of subsets of T such that F is closed-domains-family holds F is closed.

(75) For every family F of subsets of T such that F is domains-family holds $\overline{F}$ is closed-domains-family.

(76) For every family F of subsets of T such that F is closed-domains-family holds $\overline{\bigcup F}$ is a closed domain and $\overline{\mathrm{Int} \bigcap F}$ is a closed domain.

(77) For every family F of subsets of T holds for every subset B of T such that $B \in F$ holds $B \subseteq \overline{\bigcup F}$ and for every subset A of T such that A is a closed domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\overline{\bigcup F} \subseteq A$.

(78) Let F be a family of subsets of T. Then if F is closed, then for every subset B of T such that $B \in F$ holds $\overline{\mathrm{Int} \bigcap F} \subseteq B$ but $F = \emptyset$ or for every subset A of T such that A is a closed domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \overline{\mathrm{Int} \bigcap F}$.

Let us consider T. A family of subsets of T is open-domains-family if:

(Def.4) for every subset A of T such that $A \in$ it holds A is an open domain.

We now state several propositions:

(79) For every family F of subsets of T holds $F \subseteq$ the open domains of T if and only if F is open-domains-family.

(80) For every family F of subsets of T such that F is open-domains-family holds F is domains-family.

(81) For every family F of subsets of T such that F is open-domains-family holds F is open.

(82) For every family F of subsets of T such that F is domains-family holds $\operatorname{Int} F$ is open-domains-family.

(83) For every family F of subsets of T such that F is open-domains-family holds $\operatorname{Int} \bigcap F$ is an open domain and $\operatorname{Int} \overline{\bigcup F}$ is an open domain.

(84) For every family F of subsets of T holds if F is open, then for every subset B of T such that $B \in F$ holds $B \subseteq \operatorname{Int} \overline{\bigcup F}$ but for every subset A of T such that A is an open domain holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\operatorname{Int} \overline{\bigcup F} \subseteq A$.

(85) For every family F of subsets of T holds for every subset B of T such that $B \in F$ holds $\operatorname{Int} \bigcap F \subseteq B$ but $F = \emptyset$ or for every subset A of T such that A is an open domain holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \operatorname{Int} \bigcap F$.

4. Completeness of the Lattice of Domains

In the sequel T denotes a topological space. Next we state several propositions:

(86) The carrier of the lattice of domains of $T =$ the domains of T.

(87) For all elements a, b of the lattice of domains of T and for all elements A, B of the domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = \operatorname{Int} \overline{A \cup B} \cup (A \cup B)$ and $a \sqcap b = \overline{\operatorname{Int}(A \cap B)} \cap (A \cap B)$.

(88) $\perp_{\text{the lattice of domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of domains of } T} = \Omega_T$.

(89) For all elements a, b of the lattice of domains of T and for all elements A, B of the domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(90) For every subset X of the lattice of domains of T there exists an element a of the lattice of domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(91) The lattice of domains of T is complete.

(92) For every family F of subsets of T such that F is domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of domains of } T)} X = \bigcup F \cup \operatorname{Int} \overline{\bigcup F}$.

(93) For every family F of subsets of T such that F is domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds

if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \bigcap F \cap \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

5. Completeness of the Lattices of Closed Domains and Open Domains

In the sequel T will be a topological space. The following propositions are true:

(94) The carrier of the lattice of closed domains of T = the closed domains of T.

(95) For all elements a, b of the lattice of closed domains of T and for all elements A, B of the closed domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = A \cup B$ and $a \sqcap b = \overline{\operatorname{Int}(A \cap B)}$.

(96) $\bot_{\text{the lattice of closed domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of closed domains of } T} = \Omega_T$.

(97) For all elements a, b of the lattice of closed domains of T and for all elements A, B of the closed domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(98) For every subset X of the lattice of closed domains of T there exists an element a of the lattice of closed domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of closed domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(99) The lattice of closed domains of T is complete.

(100) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of closed domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of closed domains of } T)} X = \overline{\bigcup F}$.

(101) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of closed domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \Omega_T$.

(102) For every family F of subsets of T such that F is closed-domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \overline{\operatorname{Int} \bigcap F}$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

(103) The carrier of the lattice of open domains of T = the open domains of T.

(104) For all elements a, b of the lattice of open domains of T and for all elements A, B of the open domains of T such that $a = A$ and $b = B$ holds $a \sqcup b = \operatorname{Int} \overline{A \cup B}$ and $a \sqcap b = A \cap B$.

(105) $\bot_{\text{the lattice of open domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of open domains of } T} = \Omega_T$.

(106) For all elements a, b of the lattice of open domains of T and for all elements A, B of the open domains of T such that $a = A$ and $b = B$ holds $a \sqsubseteq b$ if and only if $A \subseteq B$.

(107) For every subset X of the lattice of open domains of T there exists an element a of the lattice of open domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of open domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(108) The lattice of open domains of T is complete.

(109) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of open domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of open domains of } T)} X = \operatorname{Int}\overline{\bigcup F}$.

(110) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of open domains of T such that $X = F$ holds if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \operatorname{Int}\bigcap F$ but if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \Omega_T$.

(111) For every family F of subsets of T such that F is open-domains-family and for every subset X of the lattice of domains of T such that $X = F$ holds $\bigsqcup_{(\text{the lattice of domains of } T)} X = \operatorname{Int}\overline{\bigcup F}$.

Acknowledgments

The authors would like to thank to Professors A. Trybulec and Cz. Byliński for many helpful conversations during the preparation of this paper. The authors are also very grateful to G. Bancerek for acquainting them with the Miz TEX interface system for automated typesetting and translation.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Garrett Birkhoff. *Lattice Theory*. Providence, Rhode Island, New York, 1967.
[6] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Formalized Mathematics*, 2(**4**):453–459, 1991.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Ryszard Engelking. *General Topology*. Volume 60 of *Monografie Matematyczne*, PWN - Polish Scientific Publishers, Warsaw, 1977.
[14] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.

[15] Kazimierz Kuratowski. Sur l'opération $\overline{A}$ de l'analysis situs. *Fundamenta Mathematicae*, 3:182–199, 1922.

[16] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[17] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory (with an introduction to descriptive set theory).* Volume 86 of *Studies in Logic and The Foundations of Mathematics*, PWN - Polish Scientific Publishers and North-Holland Publishing Company, Warsaw-Amsterdam, 1976.

[18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[22] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[23] Toshihiko Watanabe. The lattice of domains of a topological space. *Formalized Mathematics*, 3(**1**):41–46, 1992.

[24] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

[25] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received July 16, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

On Paracompactness of Metrizable Spaces

Leszek Borys
Warsaw University
Białystok

Summary. The aim is to prove, using Mizar System, one of the most important result in general topology, namely the Stone Theorem on paracompactness of metrizable spaces [19]. Our proof is based on [18] (and also [16]). We prove first auxiliary fact that every open cover of any metrizable space has a locally finite open refinement. We show next the main theorem that every metrizable space is paracompact. The remaining material is devoted to concepts and certain properties needed for the formulation and the proof of that theorem (see also [5]).

MML Identifier: PCOMPS_2.

The notation and terminology used here are introduced in the following articles: [21], [7], [8], [13], [26], [15], [10], [20], [11], [23], [1], [14], [9], [5], [12], [17], [24], [2], [3], [4], [25], [6], and [22].

1. Selected Properties of Real Numbers

We adopt the following rules: r, u, v, w, y are real numbers and k is a natural number. One can prove the following propositions:

(1) $r^0_{\mathbb{N}} = 1$.

(2) $r^1_{\mathbb{N}} = r$.

(3) If $r > 0$ and $u > 0$, then there exists a natural number k such that $\frac{u}{2^k_{\mathbb{N}}} \leq r$.

(4) If $k \geq n$ and $r \geq 1$, then $r^k_{\mathbb{N}} \geq r^n_{\mathbb{N}}$.

ISSN 0777–4028

2. Certain Functions Defined on Families of Sets

We adopt the following convention: R will be a binary relation, A, B, C will be sets, and t will be arbitrary. The following proposition is true

(5) If R well orders A, then $R\,|^2\,A$ well orders A and $A = \mathrm{field}(R\,|^2\,A)$.

The scheme *MinSet* concerns a set $\mathcal{A}$, a binary relation $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists arbitrary X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$ and for an arbitrary Y such that $Y \in \mathcal{A}$ and $\mathcal{P}[Y]$ holds $\langle X, Y\rangle \in \mathcal{B}$

provided the parameters meet the following conditions:

- $\mathcal{B}$ well orders $\mathcal{A}$,
- there exists arbitrary X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$.

We now define three new functors. Let F_1 be a family of sets, and let R be a binary relation, and let B be an element of F_1. The functor $\bigcup_{\beta <_R B} \beta$ yields a family of sets and is defined as follows:

(Def.1) $\bigcup_{\beta <_R B} \beta = \bigcup(R\text{–}\mathrm{Seg}(B))$.

Let F_1 be a family of sets, and let R be a binary relation. The disjoint family of F_1, R yielding a family of sets is defined by:

(Def.2) $A \in$ the disjoint family of F_1, R if and only if there exists an element B of F_1 such that $B \in F_1$ and $A = B \setminus \bigcup_{\beta <_R B} \beta$.

Let X be a set, and let n be a natural number, and let f be a function from $\mathbb{N}$ into 2^X. The functor $\bigcup_{\kappa < n} f(\kappa)$ yields a set and is defined as follows:

(Def.3) $\bigcup_{\kappa < n} f(\kappa) = \bigcup(f\,^\circ\,(\mathrm{Seg}\,n \setminus \{n\}))$.

3. Paracompactness of Metrizable Spaces

We adopt the following convention: P_1 will denote a topological space, F_1, G_1 will denote families of subsets of P_1, and W, X will denote subsets of P_1. We now state several propositions:

(6) If P_1 is a T_3 space, then for every F_1 such that F_1 is a cover of P_1 and F_1 is open there exists H_1 such that H_1 is open and H_1 is a cover of P_1 and for every V such that $V \in H_1$ there exists W such that $W \in F_1$ and $\overline{V} \subseteq W$.

(7) For all P_1, F_1 such that P_1 is a T_2 space and P_1 is paracompact and F_1 is a cover of P_1 and F_1 is open there exists G_1 such that G_1 is open and G_1 is a cover of P_1 and $\mathrm{clf}\,G_1$ is finer than F_1 and G_1 is locally finite.

(8) For every function f from $[\!:$ the carrier of P_1, the carrier of $P_1 :\!]$ into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the carrier of $P_1), f)$, then the carrier of $P_2 =$ the carrier of P_1.

(9) For every function f from $[\!:$ the carrier of P_1, the carrier of $P_1 :\!]$ into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the

carrier of $P_1), f)$, then x is a point of P_1 if and only if x is an element of the carrier of P_2.

(10) For every function f from [: the carrier of P_1, the carrier of P_1 :] into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the carrier of $P_1), f)$, then X is a subset of P_1 if and only if X is a subset of the carrier of P_2.

(11) For every function f from [: the carrier of P_1, the carrier of P_1 :] into $\mathbb{R}$ such that f is a metric of the carrier of P_1 holds if $P_2 = \mathrm{MetrSp}(($the carrier of $P_1), f)$, then F_1 is a family of subsets of P_1 if and only if F_1 is a family of subsets of the carrier of P_2.

In the sequel k is a natural number. Let P_2 be a non-empty set, and let g be a function from $\mathbb{N}$ into $(2^{2^{P_2}})^*$, and let us consider n. Then $g(n)$ is a finite sequence of elements of $2^{2^{P_2}}$.

The following propositions are true:

(12) If P_1 is metrizable, then for every family F_1 of subsets of P_1 such that F_1 is a cover of P_1 and F_1 is open there exists a family G_1 of subsets of P_1 such that G_1 is open and G_1 is a cover of P_1 and G_1 is finer than F_1 and G_1 is locally finite.

(13) If P_1 is metrizable, then P_1 is paracompact.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.
[3] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[10] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Hanna Patkowska. *Wstęp do Topologii*. PWN, Warszawa, 1974.

[17] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[18] M. E. Rudin. A new proof that metric spaces are paracompact. *Proc. Amer. Math. Soc.*, 20:603, 1969.

[19] A. H. Stone. Paracompactness and product spaces. *Bull. Amer. Math. Soc.*, 54:977–982, 1948.

[20] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[22] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[25] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

[26] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received July 23, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

The Brouwer Fixed Point Theorem for Intervals [1]

Toshihiko Watanabe
Shinshu University
Nagano

Summary. The aim is to prove, using Mizar System, the following simplest version of the Brouwer Fixed Point Theorem [2]. *For every continuous mapping $f : \mathbb{I} \to \mathbb{I}$ of the topological unit interval $\mathbb{I}$ there exists a point x such that $f(x) = x$* (see e.g. [9], [3]).

MML Identifier: `TREAL_1`.

The terminology and notation used here are introduced in the following papers: [23], [22], [25], [16], [5], [6], [20], [4], [18], [10], [24], [14], [19], [17], [7], [15], [11], [1], [21], [8], [13], and [12].

1. Properties of Topological Intervals

The following three propositions are true:

(1) For all real numbers a, b, c, d such that $a \leq c$ and $d \leq b$ and $c \leq d$ holds $[c, d] \subseteq [a, b]$.

(2) For all real numbers a, b, c, d such that $a \leq c$ and $b \leq d$ and $c \leq b$ holds $[a, b] \cup [c, d] = [a, d]$.

(3) For all real numbers a, b, c, d such that $a \leq c$ and $b \leq d$ and $c \leq b$ holds $[a, b] \cap [c, d] = [c, b]$.

In the sequel a, b, c, d are real numbers. We now state four propositions:

(4) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = [a, b]$ holds A is closed.

(5) If $a \leq b$, then $[a,\ b]_{\mathrm{T}}$ is a closed subspace of $\mathbb{R}^{\mathbf{1}}$.

(6) If $a \leq c$ and $d \leq b$ and $c \leq d$, then $[c,\ d]_{\mathrm{T}}$ is a closed subspace of $[a,\ b]_{\mathrm{T}}$.

[1] This paper was done under the supervision of Z. Karno while the author was visiting the Institute of Mathematics of Warsaw University in Białystok.

ISSN 0777–4028

(7) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a, d]_{\mathrm{T}} = [a, b]_{\mathrm{T}} \cup [c, d]_{\mathrm{T}}$ and $[c, b]_{\mathrm{T}} = [a, b]_{\mathrm{T}} \cap [c, d]_{\mathrm{T}}$.

We now define two new functors. Let a, b be real numbers. Let us assume that $a \leq b$. The functor $a_{[a,b]_{\mathrm{T}}}$ yields a point of $[a, b]_{\mathrm{T}}$ and is defined by:

(Def.1) $a_{[a,b]_{\mathrm{T}}} = a$.

The functor $b_{[a,b]_{\mathrm{T}}}$ yields a point of $[a, b]_{\mathrm{T}}$ and is defined by:

(Def.2) $b_{[a,b]_{\mathrm{T}}} = b$.

One can prove the following two propositions:

(8) $0_{\mathbb{I}} = 0_{[0,1]_{\mathrm{T}}}$ and $1_{\mathbb{I}} = 1_{[0,1]_{\mathrm{T}}}$.

(9) If $a \leq b$ and $b \leq c$, then $a_{[a,b]_{\mathrm{T}}} = a_{[a,c]_{\mathrm{T}}}$ and $c_{[b,c]_{\mathrm{T}}} = c_{[a,c]_{\mathrm{T}}}$.

2. Continuous Mappings Between Topological Intervals

Let a, b be real numbers satisfying the condition: $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\mathrm{T}}$. The functor $\mathrm{L}_{01}(t_1, t_2)$ yielding a mapping from $[0, 1]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$ is defined as follows:

(Def.3) for every point s of $[0, 1]_{\mathrm{T}}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{L}_{01}(t_1, t_2))(s) = (1 - r) \cdot r_1 + r \cdot r_2$.

We now state four propositions:

(10) Let a, b be real numbers. Then if $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\mathrm{T}}$ and for every point s of $[0, 1]_{\mathrm{T}}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{L}_{01}(t_1, t_2))(s) = (r_2 - r_1) \cdot r + r_1$.

(11) For all real numbers a, b such that $a \leq b$ and for all points t_1, t_2 of $[a, b]_{\mathrm{T}}$ holds $\mathrm{L}_{01}(t_1, t_2)$ is a continuous mapping from $[0, 1]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$.

(12) For all real numbers a, b such that $a \leq b$ and for all points t_1, t_2 of $[a, b]_{\mathrm{T}}$ holds $(\mathrm{L}_{01}(t_1, t_2))(0_{[0,1]_{\mathrm{T}}}) = t_1$ and $(\mathrm{L}_{01}(t_1, t_2))(1_{[0,1]_{\mathrm{T}}}) = t_2$.

(13) $\mathrm{L}_{01}(0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}) = \mathrm{id}_{([0,\,1]_{\mathrm{T}})}$.

Let a, b be real numbers satisfying the condition: $a < b$. Let t_1, t_2 be points of $[0, 1]_{\mathrm{T}}$. The functor $\mathrm{P}_{01}(a, b, t_1, t_2)$ yielding a mapping from $[a, b]_{\mathrm{T}}$ into $[0, 1]_{\mathrm{T}}$ is defined as follows:

(Def.4) for every point s of $[a, b]_{\mathrm{T}}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{P}_{01}(a, b, t_1, t_2))(s) = \frac{(b-r) \cdot r_1 + (r-a) \cdot r_2}{b-a}$.

The following propositions are true:

(14) Let a, b be real numbers. Suppose $a < b$. Let t_1, t_2 be points of $[0, 1]_{\mathrm{T}}$. Let s be a point of $[a, b]_{\mathrm{T}}$. Then for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{P}_{01}(a, b, t_1, t_2))(s) = \frac{r_2 - r_1}{b-a} \cdot r + \frac{b \cdot r_1 - a \cdot r_2}{b-a}$.

(15) For all real numbers a, b such that $a < b$ and for all points t_1, t_2 of $[0, 1]_{\mathrm{T}}$ holds $\mathrm{P}_{01}(a, b, t_1, t_2)$ is a continuous mapping from $[a, b]_{\mathrm{T}}$ into $[0, 1]_{\mathrm{T}}$.

(16) For all real numbers a, b such that $a < b$ and for all points t_1, t_2 of $[0, 1]_{\mathrm{T}}$ holds $(\mathrm{P}_{01}(a, b, t_1, t_2))(a_{[a,b]_{\mathrm{T}}}) = t_1$ and $(\mathrm{P}_{01}(a, b, t_1, t_2))(b_{[a,b]_{\mathrm{T}}}) = t_2$.

(17) $\mathrm{P}_{01}(0, 1, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}) = \mathrm{id}_{([0,1]_{\mathrm{T}})}$.

(18) Let a, b be real numbers. Then if $a < b$, then
$\mathrm{id}_{([a,b]_{\mathrm{T}})} = \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$
and $\mathrm{id}_{([0,1]_{\mathrm{T}})} = \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}) \cdot \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$.

(19) Let a, b be real numbers. Then if $a < b$, then
$\mathrm{id}_{([a,b]_{\mathrm{T}})} = \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}}) \cdot \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$
and $\mathrm{id}_{([0,1]_{\mathrm{T}})} = \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}) \cdot \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$.

(20) Let a, b be real numbers. Suppose $a < b$. Then
(i) $\mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$ is a homeomorphism,
(ii) $(\mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}}))^{-1} = \mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$,
(iii) $\mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}})$ is a homeomorphism,
(iv) $(\mathrm{P}_{01}(a, b, 0_{[0,1]_{\mathrm{T}}}, 1_{[0,1]_{\mathrm{T}}}))^{-1} = \mathrm{L}_{01}(a_{[a,b]_{\mathrm{T}}}, b_{[a,b]_{\mathrm{T}}})$.

(21) Let a, b be real numbers. Suppose $a < b$. Then
(i) $\mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$ is a homeomorphism,
(ii) $(\mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}}))^{-1} = \mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$,
(iii) $\mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}})$ is a homeomorphism,
(iv) $(\mathrm{P}_{01}(a, b, 1_{[0,1]_{\mathrm{T}}}, 0_{[0,1]_{\mathrm{T}}}))^{-1} = \mathrm{L}_{01}(b_{[a,b]_{\mathrm{T}}}, a_{[a,b]_{\mathrm{T}}})$.

3. Connectedness of Intervals and Brouwer Fixed Point Theorem for Intervals

We now state several propositions:

(22) $\mathbb{I}$ is connected.

(23) For all real numbers a, b such that $a \leq b$ holds $[a, b]_{\mathrm{T}}$ is connected.

(24) For every continuous mapping f from $\mathbb{I}$ into $\mathbb{I}$ there exists a point x of $\mathbb{I}$ such that $f(x) = x$.

(25) For all real numbers a, b such that $a \leq b$ and for every continuous mapping f from $[a, b]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$ there exists a point x of $[a, b]_{\mathrm{T}}$ such that $f(x) = x$.

(26) Let X, Y be subspaces of $\mathbb{R}^{\mathbf{1}}$. Then for every continuous mapping f from X into Y such that there exist real numbers a, b such that $a \leq b$ and $[a, b] \subseteq$ the carrier of X and $[a, b] \subseteq$ the carrier of Y and $f\,^\circ\,[a, b] \subseteq [a, b]$ there exists a point x of X such that $f(x) = x$.

(27) For all subspaces X, Y of $\mathbb{R}^{\mathbf{1}}$ and for every continuous mapping f from X into Y such that there exist real numbers a, b such that $a \leq b$ and $[a, b] \subseteq$ the carrier of X and $f\,^\circ\,[a, b] \subseteq [a, b]$ there exists a point x of X such that $f(x) = x$.

Acknowledgments

The author wishes to express his thanks to Professors A. Trybulec and Z. Karno for their useful suggestions and many valuable comments.

References

[1] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[2] L. Brouwer. Über Abbildungen von Mannigfaltigkeiten. *Mathematische Annalen*, 38(71):97–115, 1912.

[3] Robert H. Brown. *The Lefschetz Fixed Point Theorem*. Scott–Foresman, New York, 1971.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[8] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[9] James Dugundji and Andrzej Granas. *Fixed Point Theory*. Volume I, PWN - Polish Scientific Publishers, Warsaw, 1982.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[12] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.

[13] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.

[14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[15] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[16] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[18] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[20] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[21] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.

[22] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[24] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received August 17, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

On Powers of Cardinals

Grzegorz Bancerek
IM PAN, Warsaw
Warsaw University, Białystok

Summary. In the first section the results of [23, axiom (30)][1], i.e. the correspondence between natural and ordinal (cardinal) numbers are shown. The next section is concerned with the concepts of infinity and cofinality (see [3]), and introduces alephs as infinite cardinal numbers. The arithmetics of alephs, i.e. some facts about addition and multiplication, is present in the third section. The concepts of regular and irregular alephs are introduced in the fourth section, and the fact that $\aleph_0$ and every non-limit cardinal number are regular is proved there. Finally, for every alephs α and β

$$\alpha^\beta = \begin{cases} 2^\beta, & \text{if } \alpha \leq \beta, \\ \sum_{\gamma<\alpha} \gamma^\beta, & \text{if } \beta < \mathrm{cf}\alpha \text{ and } \alpha \text{ is limit cardinal,} \\ \left(\sum_{\gamma<\alpha} \gamma^\beta\right)^{\mathrm{cf}\alpha}, & \text{if } \mathrm{cf}\alpha \leq \beta \leq \alpha. \end{cases}$$

Some proofs are based on [20].

MML Identifier: `CARD_5`.

The papers [24], [6], [16], [14], [21], [19], [26], [10], [17], [12], [15], [13], [25], [22], [11], [2], [18], [5], [9], [1], [8], [7], [4], and [3] provide the notation and terminology for this paper.

1. Results of [23, axiom (30)]

One can readily check that every set which is cardinal is also ordinal-like.

For simplicity we adopt the following convention: n denotes a natural number, A, B denote ordinal numbers, X denotes a set, and x, y are arbitrary. We now state several propositions:

[1]Axiom (30) – $n = \{k \in \mathbb{N} : k < n\}$ for every natural number n.

ISSN 0777–4028

(1) $0 = \emptyset$ and $1 = \{0\}$ and $2 = \{0, 1\}$.

(2) $\operatorname{succ} n = n + 1$.

(3) For every n holds $\operatorname{ord}(n) = n$ and $\overline{\overline{n}} = n$.

(4) $\mathbf{0} = 0$ and $\mathbf{1} = 1$.

(5) $\overline{\mathbf{0}} = 0$ and $\overline{\mathbf{1}} = 1$ and $\overline{\mathbf{2}} = 2$.

(6) If X is finite, then $\operatorname{card} X = \overline{\overline{X}}$.

(7) $\mathbb{N} = \omega$ and $\mathbb{N} = \aleph_{\mathbf{0}}$.

(8) $\operatorname{Seg} n = (n + 1) \setminus \{0\}$.

2. Infinity, alephs and cofinality

We adopt the following rules: f is a function, K, M, N are cardinal numbers, and p_1, p_2 are sequences of ordinal numbers. The following propositions are true:

(9) $\overline{\overline{X}}^{+} = X^{+}$.

(10) $y \in \bigcup f$ if and only if there exists x such that $x \in \operatorname{dom} f$ and $y \in f(x)$.

(11) $\aleph_A$ is not finite.

(12) If M is not finite, then there exists A such that $M = \aleph_A$.

(13) There exists n such that $M = \overline{\overline{n}}$ or there exists A such that $M = \aleph_A$.

Let us consider p_1. Then $\bigcup p_1$ is an ordinal number.

Next we state a number of propositions:

(14) If $X \subseteq A$, then there exists p_1 such that $p_1 =$ the canonical isomorphism between $\subseteq_{\overline{\subseteq_X}}$ and $\subseteq_X$ and p_1 is increasing and $\operatorname{dom} p_1 = \overline{\subseteq_X}$ and $\operatorname{rng} p_1 = X$.

(15) If $X \subseteq A$, then $\sup X$ is cofinal with $\overline{\subseteq_X}$.

(16) If $X \subseteq A$, then $\overline{\overline{X}} = \overline{\overline{\overline{\subseteq_X}}}$.

(17) There exists B such that $B \subseteq \overline{\overline{A}}$ and A is cofinal with B.

(18) There exists M such that $M \leq \overline{\overline{A}}$ and A is cofinal with M and for every B such that A is cofinal with B holds $M \subseteq B$.

(19) If $\operatorname{rng} p_1 = \operatorname{rng} p_2$ and p_1 is increasing and p_2 is increasing, then $p_1 = p_2$.

(20) If p_1 is increasing, then p_1 is one-to-one.

(21) $(p_1 \frown p_2) \restriction \operatorname{dom} p_1 = p_1$.

(22) If $X \neq \emptyset$, then $\overline{\overline{\{Y : \overline{\overline{Y}} < M\}}} \leq M \cdot \overline{\overline{X}}^{M}$, where Y ranges over elements of 2^X.

(23) $M < \overline{\mathbf{2}}^{M}$.

We now define four new constructions. A set is infinite if:

(Def.1) it is not finite.

Let us observe that there exists a set which is infinite. One can readily check that there exists a cardinal number which is infinite. One can readily check that every set which is infinite is also non-empty.

An aleph is an infinite cardinal number.

Let us consider M. The functor $\mathrm{cf}\, M$ yielding a cardinal number is defined by:

(Def.2) M is cofinal with $\mathrm{cf}\, M$ and for every N such that M is cofinal with N holds $\mathrm{cf}\, M \leq N$.

Let us consider N. The functor $(\alpha \mapsto \alpha^N)_{\alpha \in M}$ yielding a function yielding cardinal numbers is defined as follows:

(Def.3) for every x holds $x \in \mathrm{dom}((\alpha \mapsto \alpha^N)_{\alpha \in M})$ if and only if $x \in M$ and x is a cardinal number and for every K such that $K \in M$ holds $(\alpha \mapsto \alpha^N)_{\alpha \in M}(K) = K^N$.

Let us consider A. Then $\aleph_A$ is an aleph.

3. Arithmetics of alephs

In the sequel a, b will be alephs. The following propositions are true:

(24) There exists A such that $a = \aleph_A$.

(25) $a \neq \overline{\mathbf{0}}$ and $a \neq \overline{\mathbf{1}}$ and $a \neq \overline{\mathbf{2}}$ and $a \neq \overline{\overline{n}}$ and $\overline{\overline{n}} < a$ and $\aleph_{\mathbf{0}} \leq a$.

(26) If $a \leq M$ or $a < M$, then M is an aleph.

(27) If $a \leq M$ or $a < M$, then $a + M = M$ and $M + a = M$ and $a \cdot M = M$ and $M \cdot a = M$.

(28) $a + a = a$ and $a \cdot a = a$.

(29) If $M \leq a$ or $M < a$, then $a + M = a$ and $M + a = a$.

(30) If $\overline{\mathbf{0}} < M$ but $M \leq a$ or $M < a$, then $a \cdot M = a$ and $M \cdot a = a$.

(31) $M \leq M^a$.

(32) $\bigcup a = a$.

Let us consider a, M. Then $a + M$ is an aleph. Let us consider M, a. Then $M + a$ is an aleph. Let us consider a, b. Then $a + b$ is an aleph. Then $a \cdot b$ is an aleph. Then a^b is an aleph.

4. Regular alephs

We now define two new attributes. An aleph is regular if:

(Def.4) $\mathrm{cf}\,\mathrm{it} = \mathrm{it}$.

An aleph is irregular if:

(Def.5) $\mathrm{cf}\,\mathrm{it} < \mathrm{it}$.

Let us consider a. Then a^+ is an aleph. We see that the element of a is an ordinal number.

One can prove the following propositions:

(33) $\mathrm{cf}\, M \leq M$.

(34) $\mathrm{cf}(\aleph_{\mathbf{0}}) = \aleph_{\mathbf{0}}$.

(35) $\mathrm{cf}(a^+) = a^+$.

(36) $\aleph_{\mathbf{0}} \leq \mathrm{cf}\, a$.

(37) $\mathrm{cf}\, \overline{\mathbf{0}} = \overline{\mathbf{0}}$ and $\mathrm{cf}\, \overline{\overline{n+1}} = \overline{\mathbf{1}}$.

(38) If $X \subseteq M$ and $\overline{\overline{X}} < \mathrm{cf}\, M$, then $\sup X \in M$ and $\bigcup X \in M$.

(39) If $\mathrm{dom}\, p_1 = M$ and $\mathrm{rng}\, p_1 \subseteq N$ and $M < \mathrm{cf}\, N$, then $\sup p_1 \in N$ and $\bigcup p_1 \in N$.

Let us consider a. Then $\mathrm{cf}\, a$ is an aleph.

One can prove the following propositions:

(40) If $\mathrm{cf}\, a < a$, then a is a limit cardinal number.

(41) If $\mathrm{cf}\, a < a$, then there exists a sequence x_1 of ordinal numbers such that $\mathrm{dom}\, x_1 = \mathrm{cf}\, a$ and $\mathrm{rng}\, x_1 \subseteq a$ and x_1 is increasing and $a = \sup x_1$ and x_1 is a function yielding cardinal numbers and $\overline{\mathbf{0}} \notin \mathrm{rng}\, x_1$.

(42) $\aleph_{\mathbf{0}}$ is regular and a^+ is regular.

5. Infinite powers

In the sequel a, b will denote alephs. The following propositions are true:

(43) If $a \leq b$, then $a^b = \overline{\mathbf{2}}^b$.

(44) $(a^+)^b = a^b \cdot (a^+)$.

(45) $\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}) \leq a^b$.

(46) If a is a limit cardinal number and $b < \mathrm{cf}\, a$, then $a^b = \sum((\alpha \mapsto \alpha^b)_{\alpha \in a})$.

(47) If $\mathrm{cf}\, a \leq b$ and $b < a$, then $a^b = (\sum((\alpha \mapsto \alpha^b)_{\alpha \in a}))^{\mathrm{cf}\, a}$.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. Consequences of the reflection theorem. *Formalized Mathematics*, 1(**5**):989–993, 1990.

[4] Grzegorz Bancerek. Countable sets and Hessenberg's theorem. *Formalized Mathematics*, 2(**1**):65–69, 1991.

[5] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[6] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[7] Grzegorz Bancerek. Increasing and continuous ordinal sequences. *Formalized Mathematics*, 1(**4**):711–714, 1990.

[8] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[9] Grzegorz Bancerek. Ordinal arithmetics. *Formalized Mathematics*, 1(**3**):515–519, 1990.
[10] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[11] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[12] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.
[13] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.
[14] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[15] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[16] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[17] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[18] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[19] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[20] Wojciech Guzicki and Paweł Zbierski. *Podstawy teorii mnogości.* PWN, Warszawa, 1978.
[21] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[22] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[23] Andrzej Trybulec. Built-in concepts. *Formalized Mathematics*, 1(**1**):13–15, 1990.
[24] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[25] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Basic Properties of Connecting Points with Line Segments in $\mathcal{E}^2_{\rm T}$

Yatsuka Nakamura
Shinshu University
Nagano

Jarosław Kotowicz[1]
Warsaw University
Białystok

Summary. Some properties of line segments in 2-dimensional Euclidean space and some relations between line segments and balls are proved.

MML Identifier: `TOPREAL3`.

The terminology and notation used in this paper have been introduced in the following papers: [17], [13], [1], [7], [2], [8], [4], [15], [16], [18], [6], [14], [5], [9], [10], [3], [11], and [12].

1. Real Numbers Preliminaries

For simplicity we follow the rules: p, p_1, p_2, p_3, q will denote points of $\mathcal{E}^2_{\rm T}$, f, h will denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r, r_1, r_2, s, s_1, s_2 will denote real numbers, u, u_1, u_2 will denote points of $\mathcal{E}^2$, n, m, i, j, k will denote natural numbers, and x, y, z will be arbitrary. One can prove the following propositions:

(1) $3-2=1$ and $3-1=2$ and $\frac{1}{2}=1-\frac{1}{2}$.

(2) $0 \leq \frac{1}{2}$ and $\frac{1}{2} \leq 1$.

(3) If $r < s$, then $r < \frac{r+s}{2}$ and $\frac{r+s}{2} < s$ and $r < \frac{s+r}{2}$ and $\frac{s+r}{2} < s$.

(4) If $r \neq s$, then $r \neq \frac{r+s}{2}$ and $\frac{r+s}{2} \neq s$.

(5) If $r_1 > s_1$ and $r_2 \geq s_2$ or $r_1 \geq s_1$ and $r_2 > s_2$, then $r_1+r_2 > s_1+s_2$.

[1]The article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

2. Properties of Line Segments

We now state a number of propositions:

(6) $1 \in \operatorname{Seg} \operatorname{len}\langle x, y, z\rangle$ and $2 \in \operatorname{Seg} \operatorname{len}\langle x, y, z\rangle$ and $3 \in \operatorname{Seg} \operatorname{len}\langle x, y, z\rangle$.

(7) $(p_1 + p_2)_{\mathbf{1}} = p_{1\mathbf{1}} + p_{2\mathbf{1}}$ and $(p_1 + p_2)_{\mathbf{2}} = p_{1\mathbf{2}} + p_{2\mathbf{2}}$.

(8) $(p_1 - p_2)_{\mathbf{1}} = p_{1\mathbf{1}} - p_{2\mathbf{1}}$ and $(p_1 - p_2)_{\mathbf{2}} = p_{1\mathbf{2}} - p_{2\mathbf{2}}$.

(9) $(r \cdot p)_{\mathbf{1}} = r \cdot p_{\mathbf{1}}$ and $(r \cdot p)_{\mathbf{2}} = r \cdot p_{\mathbf{2}}$.

(10) If $p_1 = \langle r_1, s_1\rangle$ and $p_2 = \langle r_2, s_2\rangle$, then $p_1 + p_2 = \langle r_1 + r_2, s_1 + s_2\rangle$ and $p_1 - p_2 = \langle r_1 - r_2, s_1 - s_2\rangle$.

(11) $p = q$ if and only if $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$.

(12) If $u_1 = p_1$ and $u_2 = p_2$, then $\rho^2(u_1, u_2) = \sqrt{(p_{1\mathbf{1}} - p_{2\mathbf{1}})^{\mathbf{2}} + (p_{1\mathbf{2}} - p_{2\mathbf{2}})^{\mathbf{2}}}$.

(13) The carrier of $\mathcal{E}^n_{\mathrm{T}}$ = the carrier of $\mathcal{E}^n$.

(14) x is a point of $\mathcal{E}^2$ if and only if x is a point of $\mathcal{E}^2_{\mathrm{T}}$.

(15) If $r_1 < s_1$, then $\{p_1 : p_{1\mathbf{1}} = r \wedge r_1 \leq p_{1\mathbf{2}} \wedge p_{1\mathbf{2}} \leq s_1\} = \mathcal{L}([r, r_1], [r, s_1])$.

(16) If $r_1 < s_1$, then $\{p_1 : p_{1\mathbf{2}} = r \wedge r_1 \leq p_{1\mathbf{1}} \wedge p_{1\mathbf{1}} \leq s_1\} = \mathcal{L}([r_1, r], [s_1, r])$.

(17) If $p \in \mathcal{L}([r, r_1], [r, s_1])$, then $p_{\mathbf{1}} = r$.

(18) If $p \in \mathcal{L}([r_1, r], [s_1, r])$, then $p_{\mathbf{2}} = r$.

(19) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}] \in \mathcal{L}(p, q)$.

(20) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}] \in \mathcal{L}(p, q)$.

(21) If $f = \langle p, p_1, q\rangle$ and $i \neq 0$ and $j - i > 1$, then $\mathcal{L}(f, j, j+1) = \emptyset$.

(22) If $i = 0$, then $\mathcal{L}(f, i, i+1) = \emptyset$.

(23) If $f = \langle p_1, p_2, p_3\rangle$, then $\widetilde{\mathcal{L}}(f) = \mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, p_3)$.

(24) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom}(f \upharpoonright i)$ and $k \in \operatorname{dom}(f \upharpoonright i)$, then $\mathcal{L}(f, j, k) = \mathcal{L}(f \upharpoonright i, j, k)$.

(25) If $j \in \operatorname{dom} f$ and $i \in \operatorname{dom} f$, then $\mathcal{L}(f \frown h, j, i) = \mathcal{L}(f, j, i)$.

(26) $\mathcal{L}(f, i, i+1) \subseteq \widetilde{\mathcal{L}}(f)$.

(27) $\widetilde{\mathcal{L}}(f \upharpoonright i) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) For all r, p_1, p_2, u such that $r > 0$ and $p_1 \in \operatorname{Ball}(u, r)$ and $p_2 \in \operatorname{Ball}(u, r)$ holds $\mathcal{L}(p_1, p_2) \subseteq \operatorname{Ball}(u, r)$.

(29) If $u = p_1$ and $p_1 = [r_1, s_1]$ and $p_2 = [r_2, s_2]$ and $p = [r_2, s_1]$ and $p_2 \in \operatorname{Ball}(u, r)$, then $p \in \operatorname{Ball}(u, r)$.

(30) If $r_1 \neq s_1$ and $r > 0$ and $[s, r_1] \in \operatorname{Ball}(u, r)$ and $[s, s_1] \in \operatorname{Ball}(u, r)$, then $[s, \frac{r_1+s_1}{2}] \in \operatorname{Ball}(u, r)$.

(31) If $r_1 \neq s_1$ and $r > 0$ and $[r_1, s] \in \operatorname{Ball}(u, r)$ and $[s_1, s] \in \operatorname{Ball}(u, r)$, then $[\frac{r_1+s_1}{2}, s] \in \operatorname{Ball}(u, r)$.

(32) If $r_1 \neq s_1$ and $s_2 \neq r_2$ and $r > 0$ and $[r_1, r_2] \in \operatorname{Ball}(u, r)$ and $[s_1, s_2] \in \operatorname{Ball}(u, r)$, then $[r_1, s_2] \in \operatorname{Ball}(u, r)$ or $[s_1, r_2] \in \operatorname{Ball}(u, r)$.

(33) Suppose that

(i) $f(1) \notin \operatorname{Ball}(u, r)$,

(ii) $1 \leq m$,
(iii) $m \leq \operatorname{len} f - 1$,
(iv) $\mathcal{L}(f, m, m+1) \cap \operatorname{Ball}(u, r) \neq \emptyset$,
(v) for every i such that $1 \leq i$ and $i \leq \operatorname{len} f - 1$ and $\mathcal{L}(f, i, i+1) \cap \operatorname{Ball}(u, r) \neq \emptyset$ holds $m \leq i$.

Then $f(m) \notin \operatorname{Ball}(u, r)$.

(34) For all q, p_2, p such that $q_{\mathbf{2}} = p_{2\mathbf{2}}$ and $p_{\mathbf{2}} \neq p_{2\mathbf{2}}$ holds $(\mathcal{L}(p_2, [p_{2\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([p_{2\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \mathcal{L}(q, p_2) = \{p_2\}$.

(35) For all q, p_2, p such that $q_{\mathbf{1}} = p_{2\mathbf{1}}$ and $p_{\mathbf{1}} \neq p_{2\mathbf{1}}$ holds $(\mathcal{L}(p_2, [p_{\mathbf{1}}, p_{2\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, p_{2\mathbf{2}}], p)) \cap \mathcal{L}(q, p_2) = \{p_2\}$.

(36) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [p_{\mathbf{1}}, q_{\mathbf{2}}]) \cap \mathcal{L}([p_{\mathbf{1}}, q_{\mathbf{2}}], q) = \{[p_{\mathbf{1}}, q_{\mathbf{2}}]\}$.

One can prove the following propositions:

(37) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [q_{\mathbf{1}}, p_{\mathbf{2}}]) \cap \mathcal{L}([q_{\mathbf{1}}, p_{\mathbf{2}}], q) = \{[q_{\mathbf{1}}, p_{\mathbf{2}}]\}$.

(38) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]) \cap \mathcal{L}([p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q) = \{[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]\}$.

(39) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]) \cap \mathcal{L}([\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q) = \{[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]\}$.

(40) If $i > 2$ and $i \in \operatorname{dom} f$ and f is a special sequence, then $f \restriction i$ is a special sequence.

(41) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(42) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(43) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(44) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q \rangle$, then $f(1) = p$ and $f(\operatorname{len} f) = q$ and f is a special sequence.

(45) If $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $f(i) = p$ and $f(i+1) = q$, then $\widetilde{\mathcal{L}}(f \restriction (i+1)) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(p, q)$.

(46) If $\operatorname{len} f \geq 2$ and $p \notin \widetilde{\mathcal{L}}(f)$, then for every n such that $1 \leq n$ and $n \leq \operatorname{len} f$ holds $f(n) \neq p$.

(47) If $q \neq p$ and $\mathcal{L}(q, p) \cap \widetilde{\mathcal{L}}(f) = \{q\}$, then $p \notin \widetilde{\mathcal{L}}(f)$.

(48) Suppose that
(i) f is a special sequence,
(ii) $f(1) = p$,
(iii) $f(\operatorname{len} f) = q$,
(iv) $p \notin \operatorname{Ball}(u, r)$,
(v) $q \in \operatorname{Ball}(u, r)$,
(vi) $q \in \mathcal{L}(f, m, m+1)$,
(vii) $1 \leq m$,
(viii) $m \leq \operatorname{len} f - 1$,

(ix) $\mathcal{L}(f, m, m+1) \cap \mathrm{Ball}(u, r) \neq \emptyset$.
Then $m = \mathrm{len}\, f - 1$.

(49) Suppose that
(i) $r > 0$,
(ii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iii) $q \in \mathrm{Ball}(u, r)$,
(iv) $p \in \mathrm{Ball}(u, r)$,
(v) $p \notin \mathcal{L}(p_1, q)$,
(vi) $q_{\mathbf{1}} = p_{\mathbf{1}}$ and $q_{\mathbf{2}} \neq p_{\mathbf{2}}$ or $q_{\mathbf{1}} \neq p_{\mathbf{1}}$ and $q_{\mathbf{2}} = p_{\mathbf{2}}$,
(vii) $p_{1\mathbf{1}} = q_{\mathbf{1}}$ or $p_{1\mathbf{2}} = q_{\mathbf{2}}$.
Then $\mathcal{L}(p_1, q) \cap \mathcal{L}(q, p) = \{q\}$.

(50) Suppose that
(i) $r > 0$,
(ii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iii) $p \in \mathrm{Ball}(u, r)$,
(iv) $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
(v) $q \in \mathrm{Ball}(u, r)$,
(vi) $[p_{\mathbf{1}}, q_{\mathbf{2}}] \notin \mathcal{L}(p_1, p)$,
(vii) $p_{1\mathbf{1}} = p_{\mathbf{1}}$,
(viii) $p_{\mathbf{1}} \neq q_{\mathbf{1}}$,
(ix) $p_{\mathbf{2}} \neq q_{\mathbf{2}}$.
Then $(\mathcal{L}(p, [p_{\mathbf{1}}, q_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, q_{\mathbf{2}}], q)) \cap \mathcal{L}(p_1, p) = \{p\}$.

(51) Suppose that
(i) $r > 0$,
(ii) $p_1 \notin \mathrm{Ball}(u, r)$,
(iii) $p \in \mathrm{Ball}(u, r)$,
(iv) $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
(v) $q \in \mathrm{Ball}(u, r)$,
(vi) $[q_{\mathbf{1}}, p_{\mathbf{2}}] \notin \mathcal{L}(p_1, p)$,
(vii) $p_{1\mathbf{2}} = p_{\mathbf{2}}$,
(viii) $p_{\mathbf{1}} \neq q_{\mathbf{1}}$,
(ix) $p_{\mathbf{2}} \neq q_{\mathbf{2}}$.
Then $(\mathcal{L}(p, [q_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([q_{\mathbf{1}}, p_{\mathbf{2}}], q)) \cap \mathcal{L}(p_1, p) = \{p\}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.
[10] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[11] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\rm T}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Connectedness Conditions Using Polygonal Arcs

Yatsuka Nakamura
Shinshu University
Nagano

Jarosław Kotowicz[1]
Warsaw University
Białystok

Summary. A concept of special polygonal arc joining two different points is defined. Any two points in a ball can be connected by this kind of arc, and that is also true for any region in $\mathcal{E}^2_{\rm T}$.

MML Identifier: `TOPREAL4`.

The notation and terminology used here have been introduced in the following articles: [13], [9], [1], [4], [2], [12], [11], [14], [10], [5], [3], [6], [7], and [8]. For simplicity we follow a convention: P, P_1, P_2, R will denote subsets of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q will denote points of $\mathcal{E}^2_{\rm T}$, f, h will denote finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r will denote a real number, u will denote a point of $\mathcal{E}^2$, and n, i will denote natural numbers. We now define three new predicates. Let us consider P, p, q. We say that P is a special polygonal arc joining p and q if and only if:

(Def.1) there exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and $p = f(1)$ and $q = f(\mathrm{len}\, f)$.

Let us consider P. We say that P is a special polygon if and only if the conditions (Def.2) is satisfied.

(Def.2) (i) There exist p_1, p_2 such that $p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$,

(ii) for all p, q such that $p \in P$ and $q \in P$ and $p \neq q$ there exist P_1, P_2 such that P_1 is a special polygonal arc joining p and q and P_2 is a special polygonal arc joining p and q and $P_1 \cap P_2 = \{p, q\}$ and $P = P_1 \cup P_2$.

We say that P is a region if and only if:

(Def.3) P is open and P is connected.

The following propositions are true:

[1]The article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

(1) If P is a special polygonal arc joining p and q, then P is a special polygonal arc.

(2) If P is a special polygonal arc joining p and q, then P is an arc from p to q.

(3) If P is a special polygonal arc joining p and q, then $p \in P$ and $q \in P$.

(4) If P is a special polygonal arc joining p and q, then $p \neq q$.

(5) If P is a special polygon, then P is a simple closed curve.

(6) Suppose $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u, r)$.

(7) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u, r)$.

(8) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u, r)$.

(9) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f(1) = p$ and $f(\mathrm{len}\, f) = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u, r)$.

(10) If $r > 0$ and $p \neq q$ and $p \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq \mathrm{Ball}(u, r)$.

(11) Suppose $p \neq p_1$ and $p_{1\mathbf{2}} = p_{\mathbf{2}}$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \in \mathcal{L}(f, 1, 2)$ and $h = \langle p_1, [\frac{p_{1\mathbf{1}}+p_{\mathbf{1}}}{2}, p_{1\mathbf{2}}], p\rangle$. Then h is a special sequence and $h(1) = p_1$ and $h(\mathrm{len}\, h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$.

(12) Suppose $p \neq p_1$ and $p_{1\mathbf{1}} = p_{\mathbf{1}}$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \in \mathcal{L}(f, 1, 2)$ and $h = \langle p_1, [p_{1\mathbf{1}}, \frac{p_{1\mathbf{2}}+p_{\mathbf{2}}}{2}], p\rangle$. Then h is a special sequence and $h(1) = p_1$ and $h(\mathrm{len}\, h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$.

(13) Suppose that

(i) $p \neq p_1$,

(ii) f is a special sequence,

(iii) $f(1) = p_1$,

(iv) $f(\mathrm{len}\, f) = p_2$,

(v) $i \in \mathrm{dom}\, f$,

(vi) $i+1 \in \operatorname{dom} f$,
(vii) $i > 1$,
(viii) $p \in \mathcal{L}(f, i, i+1)$,
(ix) $p \neq f(i)$,
(x) $p \neq f(i+1)$,
(xi) $h = (f \restriction i) ^\frown \langle p \rangle$,
(xii) $q = f(i)$.

Then h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(q, p)$.

(14) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(2) = p$ and $p_{\mathbf{2}} = p_{1\mathbf{2}}$ and $h = \langle p_1, [\frac{p_{1\mathbf{1}} + p_{\mathbf{1}}}{2}, p_{1\mathbf{2}}], p \rangle$. Then

(i) h is a special sequence,
(ii) $h(1) = p_1$,
(iii) $h(\operatorname{len} h) = p$,
(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p,
(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$,
(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$,
(vii) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 2) \cup \mathcal{L}(p, p)$.

(15) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(2) = p$ and $p_{\mathbf{1}} = p_{1\mathbf{1}}$ and $h = \langle p_1, [p_{1\mathbf{1}}, \frac{p_{1\mathbf{2}} + p_{\mathbf{2}}}{2}], p \rangle$. Then

(i) h is a special sequence,
(ii) $h(1) = p_1$,
(iii) $h(\operatorname{len} h) = p$,
(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p,
(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$,
(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(p_1, p)$,
(vii) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 2) \cup \mathcal{L}(p, p)$.

(16) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $f(i) = p$ and $i > 2$ and $i \in \operatorname{dom} f$ and $h = f \restriction i$. Then h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(p, p)$.

(17) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $p \in \mathcal{L}(f, n, n+1)$ and $q = f(n)$. Then there exists h such that h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction n) \cup \mathcal{L}(q, p)$.

(18) Suppose $p \neq p_1$ and f is a special sequence and $f(1) = p_1$ and $f(\operatorname{len} f) = p_2$ and $p \in \widetilde{\mathcal{L}}(f)$. Then there exists h such that h is a special sequence and $h(1) = p_1$ and $h(\operatorname{len} h) = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$.

(19) Suppose that
- (i) $p_{\mathbf{1}} = p2_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$ or $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$ and $p_{\mathbf{2}} = p2_{\mathbf{2}}$,
- (ii) $r > 0$,
- (iii) $p_1 \notin \mathrm{Ball}(u, r)$,
- (iv) $p_2 \in \mathrm{Ball}(u, r)$,
- (v) $p \in \mathrm{Ball}(u, r)$,
- (vi) f is a special sequence,
- (vii) $f(1) = p_1$,
- (viii) $f(\mathrm{len}\, f) = p_2$,
- (ix) $\mathcal{L}(p_2, p) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$,
- (x) $h = f \frown \langle p \rangle$.

Then h is a special sequence and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(20) Suppose that
- (i) $r > 0$,
- (ii) $p_1 \notin \mathrm{Ball}(u, r)$,
- (iii) $p_2 \in \mathrm{Ball}(u, r)$,
- (iv) $p \in \mathrm{Ball}(u, r)$,
- (v) $[p_{\mathbf{1}}, p2_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
- (vi) f is a special sequence,
- (vii) $f(1) = p_1$,
- (viii) $f(\mathrm{len}\, f) = p_2$,
- (ix) $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$,
- (x) $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$,
- (xi) $h = f \frown \langle [p_{\mathbf{1}}, p2_{\mathbf{2}}], p \rangle$,
- (xii) $(\mathcal{L}(p_2, [p_{\mathbf{1}}, p2_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, p2_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$.

Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(21) Suppose that
- (i) $r > 0$,
- (ii) $p_1 \notin \mathrm{Ball}(u, r)$,
- (iii) $p_2 \in \mathrm{Ball}(u, r)$,
- (iv) $p \in \mathrm{Ball}(u, r)$,
- (v) $[p2_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$,
- (vi) f is a special sequence,
- (vii) $f(1) = p_1$,
- (viii) $f(\mathrm{len}\, f) = p_2$,
- (ix) $p_{\mathbf{1}} \neq p2_{\mathbf{1}}$,
- (x) $p_{\mathbf{2}} \neq p2_{\mathbf{2}}$,
- (xi) $h = f \frown \langle [p2_{\mathbf{1}}, p_{\mathbf{2}}], p \rangle$,
- (xii) $(\mathcal{L}(p_2, [p2_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([p2_{\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{p_2\}$.

Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(22) Suppose $r > 0$ and $p_1 \notin \mathrm{Ball}(u, r)$ and $p_2 \in \mathrm{Ball}(u, r)$ and $p \in \mathrm{Ball}(u, r)$ and f is a special sequence and $f(1) = p_1$ and $f(\mathrm{len}\, f) = p_2$ and $p \notin \widetilde{\mathcal{L}}(f)$. Then there exists h such that $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining p_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u, r)$.

(23) Given R, p, p_1, p_2, P, r, u. Then if $p \neq p_1$ and P is a special polygonal arc joining p_1 and p_2 and $P \subseteq R$ and $r > 0$ and $p \in \mathrm{Ball}(u, r)$ and $p_2 \in \mathrm{Ball}(u, r)$ and $\mathrm{Ball}(u, r) \subseteq R$, then there exists P_1 such that P_1 is a special polygonal arc joining p_1 and p and $P_1 \subseteq R$.

(24) For every p such that R is a region and $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$ holds P is open.

(25) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then P is open.

(26) If $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $P \subseteq R$.

(27) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $R \subseteq P$.

(28) If R is a region and $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1} [P_1$ is a special polygonal arc joining p and $q \wedge P_1 \subseteq R]\}$, then $R = P$.

(29) If R is a region and $p \in R$ and $q \in R$ and $p \neq q$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq R$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.
[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[8] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[11] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Introduction to Go-Board - Part I

Jarosław Kotowicz[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. In the article we introduce Go-board as some kinds of matrix which elements belong to topological space $\mathcal{E}^2_{\mathrm{T}}$. We define the functor of delaying column in Go-board and relation between Go-board and finite sequence of point from $\mathcal{E}^2_{\mathrm{T}}$. Basic facts about those notations are proved. The concept of the article is based on [16].

MML Identifier: GOBOARD1.

The notation and terminology used here have been introduced in the following papers: [17], [11], [2], [6], [3], [9], [7], [14], [15], [1], [18], [5], [12], [4], [8], [10], and [13].

1. Real Numbers Preliminaries

For simplicity we follow the rules: p denotes a point of $\mathcal{E}^2_{\mathrm{T}}$, f, f_1, f_2, g denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, v denotes a finite sequence of elements of $\mathbb{R}$, r, s denote real numbers, n, m, i, j, k denote natural numbers, and x is arbitrary. One can prove the following three propositions:

(1) $|r-s|=1$ if and only if $r>s$ and $r=s+1$ or $r<s$ and $s=r+1$.

(2) $|i-j|+|n-m|=1$ if and only if $|i-j|=1$ and $n=m$ or $|n-m|=1$ and $i=j$.

(3) $n>1$ if and only if there exists m such that $n=m+1$ and $m>0$.

[1]This article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

2. Finite Sequences Preliminaries

The scheme *FinSeqDChoice* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a finite sequence f of elements of $\mathcal{A}$ such that $\operatorname{len} f = \mathcal{B}$ and for every n such that $n \in \operatorname{Seg} \mathcal{B}$ holds $\mathcal{P}[n, f(n)]$

provided the parameters have the following property:

- for every n such that $n \in \operatorname{Seg} \mathcal{B}$ there exists an element d of $\mathcal{A}$ such that $\mathcal{P}[n, d]$.

One can prove the following propositions:

(4) If $n = m + 1$ and $i \in \operatorname{Seg} n$, then $\operatorname{len} \operatorname{Sgm}(\operatorname{Seg} n \setminus \{i\}) = m$.

(5) Suppose $n = m + 1$ and $k \in \operatorname{Seg} n$ and $i \in \operatorname{Seg} m$. Then if $1 \leq i$ and $i < k$, then $(\operatorname{Sgm}(\operatorname{Seg} n \setminus \{k\}))(i) = i$ but if $k \leq i$ and $i \leq m$, then $(\operatorname{Sgm}(\operatorname{Seg} n \setminus \{k\}))(i) = i + 1$.

(6) For every finite sequence f and for all n, m such that $\operatorname{len} f = m+1$ and $n \in \operatorname{Seg} \operatorname{len} f$ holds $\operatorname{len}(f_{\restriction n}) = m$.

(7) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $k \in \operatorname{Seg} m$ holds $f_{\restriction n}(k) = f(k)$ or $f_{\restriction n}(k) = f(k+1)$.

(8) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $1 \leq k$ and $k < n$ holds $f_{\restriction n}(k) = f(k)$.

(9) For every finite sequence f and for all n, m, k such that $\operatorname{len} f = m + 1$ and $n \in \operatorname{Seg} \operatorname{len} f$ and $n \leq k$ and $k \leq m$ holds $f_{\restriction n}(k) = f(k + 1)$.

(10) If $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} n$, then $(f \restriction n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

We now define four new constructions. A finite sequence of elements of $\mathbb{R}$ is increasing if:

(Def.1) for all n, m such that $n \in \operatorname{dom} \mathrm{it}$ and $m \in \operatorname{dom} \mathrm{it}$ and $n < m$ and for all r, s such that $r = \mathrm{it}(n)$ and $s = \mathrm{it}(m)$ holds $r < s$.

A finite sequence is constant if:

(Def.2) for all n, m such that $n \in \operatorname{dom} \mathrm{it}$ and $m \in \operatorname{dom} \mathrm{it}$ holds $\mathrm{it}(n) = \mathrm{it}(m)$.

Let us observe that there exists a finite sequence of elements of $\mathbb{R}$ which is increasing. Note also that there exists a finite sequence of elements of $\mathbb{R}$ which is constant.

Let us consider f. The functor $\mathbf{X}$-coordinate(f) yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.3) $\operatorname{len} \mathbf{X}\text{-coordinate}(f) = \operatorname{len} f$
and for every n such that $n \in \operatorname{dom} \mathbf{X}\text{-coordinate}(f)$ and for every p such that $p = f(n)$ holds $(\mathbf{X}\text{-coordinate}(f))(n) = p_{\mathbf{1}}$.

The functor $\mathbf{Y}$-coordinate(f) yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def.4) $\operatorname{len} \mathbf{Y}\text{-coordinate}(f) = \operatorname{len} f$
and for every n such that $n \in \operatorname{dom} \mathbf{Y}\text{-coordinate}(f)$ and for every p such that $p = f(n)$ holds $(\mathbf{Y}\text{-coordinate}(f))(n) = p_{\mathbf{2}}$.

One can prove the following propositions:

(11) Suppose that
(i) $v \neq \varepsilon$,
(ii) $\operatorname{rng} v \subseteq \operatorname{Seg} n$,
(iii) $v(\operatorname{len} v) = n$,
(iv) for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r - s| = 1$ or $r = s$,
(v) $i \in \operatorname{Seg} n$,
(vi) $i + 1 \in \operatorname{Seg} n$,
(vii) $m \in \operatorname{dom} v$,
(viii) $v(m) = i$,
(ix) for every k such that $k \in \operatorname{dom} v$ and $v(k) = i$ holds $k \leq m$.
Then $m + 1 \in \operatorname{dom} v$ and $v(m+1) = i + 1$.

(12) Suppose that
(i) $v \neq \varepsilon$,
(ii) $\operatorname{rng} v \subseteq \operatorname{Seg} n$,
(iii) $v(1) = 1$,
(iv) $v(\operatorname{len} v) = n$,
(v) for every k such that $1 \leq k$ and $k \leq \operatorname{len} v - 1$ and for all r, s such that $r = v(k)$ and $s = v(k+1)$ holds $|r - s| = 1$ or $r = s$.
Then
(vi) for every i such that $i \in \operatorname{Seg} n$ there exists k such that $k \in \operatorname{dom} v$ and $v(k) = i$,
(vii) for all m, k, i, r such that $m \in \operatorname{dom} v$ and $v(m) = i$ and for every j such that $j \in \operatorname{dom} v$ and $v(j) = i$ holds $j \leq m$ and $m < k$ and $k \in \operatorname{dom} v$ and $r = v(k)$ holds $i < r$.

(13) If $i \in \operatorname{dom} f$ and $2 \leq \operatorname{len} f$, then $f(i) \in \widetilde{\mathcal{L}}(f)$.

3. Matrix Preliminaries

Next we state two propositions:

(14) For every non-empty set D and for every matrix M over D and for all i, j such that $j \in \operatorname{Seg} \operatorname{len} M$ and $i \in \operatorname{Seg} \operatorname{width} M$ holds $M_{\square,i}(j) = \operatorname{Line}(M, j)(i)$.

(15) For every non-empty set D and for every matrix M over D and for every k such that $k \in \operatorname{Seg} \operatorname{len} M$ holds $M(k) = \operatorname{Line}(M, k)$.

We now define several new constructions. Let T be a topological space. A matrix over T is a matrix over the carrier of T.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is non-trivial if:

(Def.5) $0 < \operatorname{len} \mathrm{it}$ and $0 < \operatorname{width} \mathrm{it}$.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is line $\mathbf{X}$-constant if:

(Def.6) for every n such that $n \in \operatorname{Seg} \operatorname{len} \mathrm{it}$ holds $\mathbf{X}$-coordinate(Line(it, n)) is constant.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is column $\mathbf{Y}$-constant if:

(Def.7) for every n such that $n \in \operatorname{Seg} \operatorname{width} \mathrm{it}$ holds $\mathbf{Y}$-coordinate($\mathrm{it}_{\square,n}$) is constant.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is line $\mathbf{Y}$-increasing if:

(Def.8) for every n such that $n \in \operatorname{Seg} \operatorname{len} \mathrm{it}$ holds $\mathbf{Y}$-coordinate(Line(it, n)) is increasing.

A matrix over $\mathcal{E}^2_{\mathrm{T}}$ is column $\mathbf{X}$-increasing if:

(Def.9) for every n such that $n \in \operatorname{Seg} \operatorname{width} \mathrm{it}$ holds $\mathbf{X}$-coordinate($\mathrm{it}_{\square,n}$) is increasing.

One can readily verify that there exists a matrix over $\mathcal{E}^2_{\mathrm{T}}$ which is non-trivial, line $\mathbf{X}$-constant, column $\mathbf{Y}$-constant, line $\mathbf{Y}$-increasing and column $\mathbf{X}$-increasing.

We now state two propositions:

(16) For every column $\mathbf{X}$-increasing line $\mathbf{X}$-constant matrix M over $\mathcal{E}^2_{\mathrm{T}}$ and for all x, n, m such that $x \in \operatorname{rng} \operatorname{Line}(M, n)$ and $x \in \operatorname{rng} \operatorname{Line}(M, m)$ and $n \in \operatorname{Seg} \operatorname{len} M$ and $m \in \operatorname{Seg} \operatorname{len} M$ holds $n = m$.

(17) For every line $\mathbf{Y}$-increasing column $\mathbf{Y}$-constant matrix M over $\mathcal{E}^2_{\mathrm{T}}$ and for all x, n, m such that $x \in \operatorname{rng}(M_{\square,n})$ and $x \in \operatorname{rng}(M_{\square,m})$ and $n \in \operatorname{Seg} \operatorname{width} M$ and $m \in \operatorname{Seg} \operatorname{width} M$ holds $n = m$.

4. Basic Go-Board‘s Notation

A Go-board is a non-trivial line $\mathbf{X}$-constant column $\mathbf{Y}$-constant line $\mathbf{Y}$-increasing column $\mathbf{X}$-increasing matrix over $\mathcal{E}^2_{\mathrm{T}}$.

In the sequel G denotes a Go-board. The following four propositions are true:

(18) If $x = G_{m,k}$ and $x = G_{i,j}$ and $\langle m, k \rangle \in$ the indices of G and $\langle i, j \rangle \in$ the indices of G, then $m = i$ and $k = j$.

(19) If $m \in \operatorname{dom} f$ and $f(1) \in \operatorname{rng}(G_{\square,1})$, then $(f \restriction m)(1) \in \operatorname{rng}(G_{\square,1})$.

(20) If $m \in \operatorname{dom} f$ and $f(m) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$, then $(f \restriction m)(\operatorname{len}(f \restriction m)) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.

(21) If $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,i}) = \emptyset$ and $f(n) = G_{m,k}$ and $n \in \operatorname{dom} f$ and $m \in \operatorname{Seg} \operatorname{len} G$, then $i \neq k$.

Let us consider G, i. Let us assume that $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$. The deleting of i-column in G yielding a Go-board is defined by:

(Def.10) len(the deleting of i-column in G) $= \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} G$ holds (the deleting of i-column in G)(k) $= \operatorname{Line}(G, k)_{\restriction i}$.

One can prove the following propositions:

(22) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$ and $k \in \operatorname{Seg} \operatorname{len} G$, then Line(the deleting of i-column in $G, k) = \operatorname{Line}(G, k)_{\restriction i}$.

(23) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$, then width(the deleting of i-column in G) $= m$.

(24) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$, then $\operatorname{width} G =$ width(the deleting of i-column in G) $+ 1$.

(25) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G > 1$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $m \in$ Seg width(the deleting of i-column in G), then (the deleting of i-column in $G)_{n,m} = \operatorname{Line}(G, n)_{\restriction i}(m)$.

(26) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m+1$ and $m > 0$ and $1 \leq k$ and $k < i$, then (the deleting of i-column in $G)_{\square,k} = G_{\square,k}$ and $k \in$ Seg width(the deleting of i-column in G) and $k \in \operatorname{Seg} \operatorname{width} G$.

(27) Suppose $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in$ Seg width(the deleting of i-column in G) and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(28) If $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $1 \leq k$ and $k < i$, then (the deleting of i-column in $G)_{n,k} = G_{n,k}$ and $k \in \operatorname{Seg} \operatorname{width} G$.

(29) Suppose $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{width} G = m + 1$ and $m > 0$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $i \leq k$ and $k \leq m$. Then (the deleting of i-column in $G)_{n,k} = G_{n,k+1}$ and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(30) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$, then (the deleting of 1-column in $G)_{\square,k} = G_{\square,k+1}$ and $k \in$ Seg width(the deleting of 1-column in G) and $k + 1 \in \operatorname{Seg} \operatorname{width} G$.

(31) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{Seg} \operatorname{len} G$, then (the deleting of 1-column in $G)_{n,k} = G_{n,k+1}$ and $1 \in \operatorname{Seg} \operatorname{width} G$.

(32) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$, then (the deleting of $\operatorname{width} G$-column in $G)_{\square,k} = G_{\square,k}$ and $k \in$ Seg width(the deleting of $\operatorname{width} G$-column in G).

(33) If $\operatorname{width} G = m + 1$ and $m > 0$ and $k \in \operatorname{Seg} m$ and $n \in \operatorname{Seg} \operatorname{len} G$, then $k \in \operatorname{Seg} \operatorname{width} G$ and (the deleting of $\operatorname{width} G$-column in $G)_{n,k} = G_{n,k}$ and $\operatorname{width} G \in \operatorname{Seg} \operatorname{width} G$.

(34) Suppose $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,i}) = \emptyset$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, m)$ and $n \in \operatorname{dom} f$ and $i \in \operatorname{Seg} \operatorname{width} G$ and $m \in \operatorname{Seg} \operatorname{len} G$ and $\operatorname{width} G > 1$. Then $f(n) \in$ rng Line(the deleting of i-column in G, m).

Let us consider f, G. We say that f is a sequence which elements belong to G if and only if the conditions (Def.11) is satisfied.

(Def.11) (i) For every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{i,j}$,

(ii) for every n such that $n \in \operatorname{dom} f$ and $n + 1 \in \operatorname{dom} f$ and for all m, k, i, j such that $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{m,k}$ and $f(n + 1) = G_{i,j}$ holds $|m - i| + |k - j| = 1$.

One can prove the following propositions:

(35) If f is a sequence which elements belong to G and $m \in \operatorname{dom} f$, then $1 \leq \operatorname{len}(f \upharpoonright m)$ and $f \upharpoonright m$ is a sequence which elements belong to G.

(36) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f_1(n) = G_{i,j}$,

(ii) for every n such that $n \in \operatorname{dom} f_2$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f_2(n) = G_{i,j}$.

Then for every n such that $n \in \operatorname{dom}(f_1 \frown f_2)$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $(f_1 \frown f_2)(n) = G_{i,j}$.

(37) Suppose that

(i) for every n such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ and for all m, k, i, j such that $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G and $f_1(n) = G_{m,k}$ and $f_1(n+1) = G_{i,j}$ holds $|m-i| + |k-j| = 1$,

(ii) for every n such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ and for all m, k, i, j such that $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G and $f_2(n) = G_{m,k}$ and $f_2(n+1) = G_{i,j}$ holds $|m-i| + |k-j| = 1$,

(iii) for all m, k, i, j such that $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G and $f_1(\operatorname{len} f_1) = G_{m,k}$ and $f_2(1) = G_{i,j}$ and $\operatorname{len} f_1 \in \operatorname{dom} f_1$ and $1 \in \operatorname{dom} f_2$ holds $|m-i| + |k-j| = 1$.

Given n. Suppose $n \in \operatorname{dom}(f_1 \frown f_2)$ and $n+1 \in \operatorname{dom}(f_1 \frown f_2)$. Given m, k, i, j. Then if $\langle m, k\rangle \in$ the indices of G and $\langle i, j\rangle \in$ the indices of G and $(f_1 \frown f_2)(n) = G_{m,k}$ and $(f_1 \frown f_2)(n+1) = G_{i,j}$, then $|m-i| + |k-j| = 1$.

(38) If f is a sequence which elements belong to G and $i \in \operatorname{Seg} \operatorname{width} G$ and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,i}) = \emptyset$ and $\operatorname{width} G > 1$, then f is a sequence which elements belong to the deleting of i-column in G.

(39) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{Seg} \operatorname{len} G$ and $f(i) \in \operatorname{rng} \operatorname{Line}(G, n)$.

(40) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $n \in \operatorname{Seg} \operatorname{len} G$ and $f(i) \in \operatorname{rng} \operatorname{Line}(G, n)$. Then $f(i+1) \in \operatorname{rng} \operatorname{Line}(G, n)$ or for every k such that $f(i+1) \in \operatorname{rng} \operatorname{Line}(G, k)$ and $k \in \operatorname{Seg} \operatorname{len} G$ holds $|n-k| = 1$.

(41) Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) $f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,

(iii) f is a sequence which elements belong to G,

(iv) $i \in \operatorname{Seg} \operatorname{len} G$,

(v) $i+1 \in \operatorname{Seg} \operatorname{len} G$,

(vi) $m \in \operatorname{dom} f$,

(vii) $f(m) \in \operatorname{rng} \operatorname{Line}(G, i)$,

(viii) for every k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $k \leq m$.

Then $m+1 \in \operatorname{dom} f$ and $f(m+1) \in \operatorname{rng} \operatorname{Line}(G, i+1)$.

(42) Suppose $1 \leq \operatorname{len} f$ and $f(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$ and
$f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$
and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ there exists k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$,

(ii) for every i such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $2 \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(f) \cap \operatorname{rng} \operatorname{Line}(G, i) \neq \emptyset$,

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \operatorname{len} G$ and $1 \leq j$ and $j \leq \operatorname{len} G$ and $k \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng} \operatorname{Line}(G, i)$ and for every n such that $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, i)$ holds $n \leq k$ and $k < m$ and $f(m) \in \operatorname{rng} \operatorname{Line}(G, j)$ holds $i < j$.

(43) If f is a sequence which elements belong to G and $i \in \operatorname{dom} f$, then there exists n such that $n \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,n})$.

(44) Suppose f is a sequence which elements belong to G and $i \in \operatorname{dom} f$ and $i+1 \in \operatorname{dom} f$ and $n \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,n})$. Then $f(i+1) \in \operatorname{rng}(G_{\square,n})$ or for every k such that $f(i+1) \in \operatorname{rng}(G_{\square,k})$ and $k \in \operatorname{Seg} \operatorname{width} G$ holds $|n-k| = 1$.

(45) Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$,

(iii) f is a sequence which elements belong to G,

(iv) $i \in \operatorname{Seg} \operatorname{width} G$,

(v) $i+1 \in \operatorname{Seg} \operatorname{width} G$,

(vi) $m \in \operatorname{dom} f$,

(vii) $f(m) \in \operatorname{rng}(G_{\square,i})$,

(viii) for every k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$ holds $k \leq m$.

Then $m+1 \in \operatorname{dom} f$ and $f(m+1) \in \operatorname{rng}(G_{\square,i+1})$.

(46) Suppose $1 \leq \operatorname{len} f$ and $f(1) \in \operatorname{rng}(G_{\square,1})$ and $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and f is a sequence which elements belong to G. Then

(i) for every i such that $1 \leq i$ and $i \leq \operatorname{width} G$ there exists k such that $k \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$,

(ii) for every i such that $1 \leq i$ and $i \leq \operatorname{width} G$ and $2 \leq \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(f) \cap \operatorname{rng}(G_{\square,i}) \neq \emptyset$,

(iii) for all i, j, k, m such that $1 \leq i$ and $i \leq \operatorname{width} G$ and $1 \leq j$ and $j \leq \operatorname{width} G$ and $k \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $f(k) \in \operatorname{rng}(G_{\square,i})$ and for every n such that $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng}(G_{\square,i})$ holds $n \leq k$ and $k < m$ and $f(m) \in \operatorname{rng}(G_{\square,j})$ holds $i < j$.

(47) Suppose that

(i) $n \in \operatorname{dom} f$,

(ii) $f(n) \in \operatorname{rng}(G_{\square,k})$,

(iii) $k \in \operatorname{Seg} \operatorname{width} G$,

(iv) $f(1) \in \operatorname{rng}(G_{\square,1})$,

(v) f is a sequence which elements belong to G,

(vi) for every i such that $i \in \operatorname{dom} f$ and $f(i) \in \operatorname{rng}(G_{\square,k})$ holds $n \leq i$.

Then for every i such that $i \in \operatorname{dom} f$ and $i \leq n$ and for every m such that $m \in \operatorname{Seg} \operatorname{width} G$ and $f(i) \in \operatorname{rng}(G_{\square,m})$ holds $m \leq k$.

(48) Suppose f is a sequence which elements belong to G and $f(1) \in \operatorname{rng}(G_{\square,1})$ and $f(\operatorname{len} f) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and $\operatorname{width} G > 1$ and $1 \leq \operatorname{len} f$. Then there exists g such that $g(1) \in \operatorname{rng}((\text{the deleting of } \operatorname{width} G\text{-column in } G)_{\square,1})$ and $g(\operatorname{len} g) \in \operatorname{rng}((\text{the deleting of } \operatorname{width} G\text{-column in } G)_{\square,\operatorname{width}(\text{the deleting of } \operatorname{width} G-\text{column in } G)})$ and $1 \leq \operatorname{len} g$ and g is a sequence which elements belong to the deleting of $\operatorname{width} G$-column in G and $\operatorname{rng} g \subseteq \operatorname{rng} f$.

(49) Suppose f is a sequence which elements belong to G and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,1}) \neq \emptyset$ and $\operatorname{rng} f \cap \operatorname{rng}(G_{\square,\operatorname{width} G}) \neq \emptyset$. Then there exists g such that $\operatorname{rng} g \subseteq \operatorname{rng} f$ and $g(1) \in \operatorname{rng}(G_{\square,1})$ and $g(\operatorname{len} g) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$ and $1 \leq \operatorname{len} g$ and g is a sequence which elements belong to G.

(50) Suppose $k \in \operatorname{Seg} \operatorname{len} G$ and f is a sequence which elements belong to G and $f(\operatorname{len} f) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$ and $n \in \operatorname{dom} f$ and $f(n) \in \operatorname{rng} \operatorname{Line}(G, k)$. Then
 (i) for every i such that $k \leq i$ and $i \leq \operatorname{len} G$ there exists j such that $j \in \operatorname{dom} f$ and $n \leq j$ and $f(j) \in \operatorname{rng} \operatorname{Line}(G, i)$,
 (ii) for every i such that $k < i$ and $i \leq \operatorname{len} G$ there exists j such that $j \in \operatorname{dom} f$ and $n < j$ and $f(j) \in \operatorname{rng} \operatorname{Line}(G, i)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[12] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[13] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[15] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[16] Yukio Takeuchi and Yatsuka Nakamura. *On the Jordan curve theorem.* Technical Report 19804, Dept. of Information Eng., Shinshu University, 500 Wakasato, Nagano city, Japan, April 1980.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Introduction to Go-Board - Part II

Jarosław Kotowicz[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. In article we define Go-board determined by finite sequence of points from topological space $\mathcal{E}^2_{\mathrm{T}}$. A few facts about this notation are proved.

MML Identifier: GOBOARD2.

The papers [17], [10], [2], [6], [3], [8], [15], [16], [1], [18], [13], [5], [12], [11], [4], [7], [9], and [14] provide the notation and terminology for this paper.

1. Real Numbers Preliminaries

For simplicity we follow the rules: p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$, f, f_1, f_2, g denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, R denotes a subset of $\mathbb{R}$, r, s denote real numbers, v, v_1, v_2 denote finite sequences of elements of $\mathbb{R}$, n, m, i, j, k denote natural numbers, and G denotes a Go-board. We now state the proposition

(1) If R is finite and $R \neq \emptyset$, then R is upper bounded and $\sup R \in R$ and R is lower bounded and $\inf R \in R$.

2. Properties of Finite Sequences of Points from $\mathcal{E}^2_{\mathrm{T}}$

One can prove the following propositions:

(2) For every finite sequence f holds f is one-to-one if and only if for all n, m such that $n \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $n \neq m$ holds $f(n) \neq f(m)$.

(3) For every n holds $1 \leq n$ and $n \leq \operatorname{len} f - 1$ if and only if $n \in \operatorname{dom} f$ and $n + 1 \in \operatorname{dom} f$.

[1]This article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

(4) For every n holds $1 \leq n$ and $n \leq \operatorname{len} f - 2$ if and only if $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $n+2 \in \operatorname{dom} f$.

(5) The following conditions are equivalent:

(i) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$,

(ii) for all n, m such that $n - m > 1$ or $m - n > 1$ but $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $m \in \operatorname{dom} f$ and $m+1 \in \operatorname{dom} f$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$.

(6) Suppose that

(i) for every n such that $1 \leq n$ and $n \leq \operatorname{len} f - 2$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, n+1, n+2) = \{f(n+1)\}$,

(ii) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$,

(iii) f is one-to-one,

(iv) $f(\operatorname{len} f) \in \mathcal{L}(f, i, i+1)$,

(v) $i \in \operatorname{dom} f$,

(vi) $i + 1 \in \operatorname{dom} f$.

Then $i + 1 = \operatorname{len} f$.

(7) If $k \neq 0$ and $\operatorname{len} f = k + 1$, then $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(f \restriction k) \cup \mathcal{L}(f, k, k+1)$.

(8) Suppose that

(i) $1 < k$,

(ii) $\operatorname{len} f = k + 1$,

(iii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} f - 2$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, n+1, n+2) = \{f(n+1)\}$,

(iv) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$.

Then $\widetilde{\mathcal{L}}(f \restriction k) \cap \mathcal{L}(f, k, k+1) = \{f(k)\}$.

(9) If $\operatorname{len} f_1 < n$ and $n \leq \operatorname{len}(f_1 \frown f_2) - 1$ and $m = n - \operatorname{len} f_1$, then $\mathcal{L}(f_1 \frown f_2, n, n+1) = \mathcal{L}(f_2, m, m+1)$.

(10) $\widetilde{\mathcal{L}}(f) \subseteq \widetilde{\mathcal{L}}(f \frown g)$.

(11) Suppose for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$. Then for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f \restriction i, n, n+1) \cap \mathcal{L}(f \restriction i, m, m+1) = \emptyset$.

(12) Suppose that

(i) for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} f_1 - 1$ and $f_1(n) = p$ and $f_1(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(ii) for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} f_2 - 1$ and $f_2(n) = p$ and $f_2(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(iii) for all p, q such that $f_1(\operatorname{len} f_1) = p$ and $f_2(1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$.

Then for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len}(f_1 \frown f_2) - 1$ and $(f_1 \frown f_2)(n) = p$ and $(f_1 \frown f_2)(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$.

(13) If $f \neq \varepsilon$, then $\mathbf{X}$-coordinate$(f) \neq \varepsilon$.

(14) If $f \neq \varepsilon$, then $\mathbf{Y}$-coordinate$(f) \neq \varepsilon$.

(15) Suppose for all n, p, q such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $f(n) = p$ and $f(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$. Given n. Suppose $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$. Then for all i, j, m, k such that $\langle i, j\rangle \in$ the indices of G and $\langle m, k\rangle \in$ the indices of G and $f(n) = G_{i,j}$ and $f(n+1) = G_{m,k}$ holds $i = m$ or $k = j$.

(16) Suppose that

(i) for every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{i,j}$,

(ii) for all n, p, q such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ and $f(n) = p$ and $f(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,

(iii) for every n such that $n \in \operatorname{dom} f$ and $n+1 \in \operatorname{dom} f$ holds $f(n) \neq f(n+1)$.

Then there exists g such that g is a sequence which elements belong to G and $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(g)$ and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$ and $\operatorname{len} f \leq \operatorname{len} g$.

(17) If v is increasing, then for all n, m such that $n \in \operatorname{dom} v$ and $m \in \operatorname{dom} v$ and $n \leq m$ and for all r, s such that $r = v(n)$ and $s = v(m)$ holds $r \leq s$.

(18) If v is increasing, then for all n, m such that $n \in \operatorname{dom} v$ and $m \in \operatorname{dom} v$ and $n \neq m$ holds $v(n) \neq v(m)$.

(19) If v is increasing and $v_1 = v \restriction \operatorname{Seg} n$, then v_1 is increasing.

(20) For every v there exists v_1 such that $\operatorname{rng} v_1 = \operatorname{rng} v$ and $\operatorname{len} v_1 = \operatorname{card} \operatorname{rng} v$ and v_1 is increasing.

(21) For all v_1, v_2 such that $\operatorname{len} v_1 = \operatorname{len} v_2$ and $\operatorname{rng} v_1 = \operatorname{rng} v_2$ and v_1 is increasing and v_2 is increasing holds $v_1 = v_2$.

3. Go-Board Determined by Finite Sequence

We now define three new functors. Let v_1, v_2 be increasing finite sequences of elements of $\mathbb{R}$. Let us assume that $v_1 \neq \varepsilon$ and $v_2 \neq \varepsilon$. The Go-board of v_1, v_2 yields a Go-board and is defined by:

(Def.1) len the Go-board of v_1, $v_2 = \operatorname{len} v_1$ and width the Go-board of v_1, $v_2 = \operatorname{len} v_2$ and for all n, m such that $\langle n, m\rangle \in$ the indices of the Go-board of v_1, v_2 and for all r, s such that $v_1(n) = r$ and $v_2(m) = s$ holds (the Go-board of v_1, $v_2)_{n,m} = [r, s]$.

Let us consider v. The functor $\operatorname{Inc}(v)$ yielding an increasing finite sequence of elements of $\mathbb{R}$ is defined by:

(Def.2) $\operatorname{rng} \operatorname{Inc}(v) = \operatorname{rng} v$ and $\operatorname{len} \operatorname{Inc}(v) = \operatorname{card} \operatorname{rng} v$.

Let us consider f. Let us assume that $f \neq \varepsilon$. The Go-board of f yielding a Go-board is defined by:

(Def.3) the Go-board of $f =$ the Go-board of $\operatorname{Inc}(\mathbf{X}$-coordinate$(f))$, $\operatorname{Inc}(\mathbf{Y}$-coordinate$(f))$.

One can prove the following propositions:

(22) If $v \neq \varepsilon$, then $\mathrm{Inc}(v) \neq \varepsilon$.

(23) If $f \neq \varepsilon$, then $\mathrm{len}\,\text{the Go-board of } f = \mathrm{card}\,\mathrm{rng}\,\mathbf{X}\text{-coordinate}(f)$ and $\mathrm{width}\,\text{the Go-board of } f = \mathrm{card}\,\mathrm{rng}\,\mathbf{Y}\text{-coordinate}(f)$.

(24) If $f \neq \varepsilon$, then for every n such that $n \in \mathrm{dom}\, f$ there exist i, j such that $\langle i, j \rangle \in$ the indices of the Go-board of f and $f(n) = (\text{the Go-board of } f)_{i,j}$.

(25) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{X}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{X}\text{-coordinate}(f))(m)$ holds $r \leq s$, then $f(n) \in \mathrm{rng}\,\mathrm{Line}(\text{the Go-board of } f, 1)$.

(26) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{X}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{X}\text{-coordinate}(f))(m)$ holds $s \leq r$, then $f(n) \in \mathrm{rng}\,\mathrm{Line}(\text{the Go-board of } f, \mathrm{len}\,\text{the Go-board of } f)$.

(27) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{Y}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{Y}\text{-coordinate}(f))(m)$ holds $r \leq s$, then $f(n) \in \mathrm{rng}((\text{the Go-board of } f)_{\square,1})$.

(28) If $f \neq \varepsilon$ and $n \in \mathrm{dom}\, f$ and $r = (\mathbf{Y}\text{-coordinate}(f))(n)$ and for every m such that $m \in \mathrm{dom}\, f$ and for every s such that $s = (\mathbf{Y}\text{-coordinate}(f))(m)$ holds $s \leq r$, then $f(n) \in \mathrm{rng}((\text{the Go-board of } f)_{\square,\mathrm{width\,the\,Go-board\,of}\,f})$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[12] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[13] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[14] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[16] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Properties of Go-Board - Part III

Jarosław Kotowicz[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. Two useful facts about Go-board are proved.

MML Identifier: GOBOARD3.

The terminology and notation used in this paper have been introduced in the following articles: [16], [8], [1], [5], [2], [14], [15], [17], [4], [10], [9], [3], [6], [7], [13], [11], and [12]. For simplicity we follow the rules: p, q are points of $\mathcal{E}^2_{\rm T}$, f, g are finite sequences of elements of $\mathcal{E}^2_{\rm T}$, n, m, i, j are natural numbers, and G is a Go-board. One can prove the following two propositions:

(1) Suppose that

(i) for every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{i,j}$,

(ii) f is one-to-one,

(iii) for every n such that $1 \leq n$ and $n \leq \operatorname{len} f - 2$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, n+1, n+2) = \{f(n+1)\}$,

(iv) for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(f, n, n+1) \cap \mathcal{L}(f, m, m+1) = \emptyset$,

(v) for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} f - 1$ and $f(n) = p$ and $f(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$.

Then there exists g such that g is a sequence which elements belong to G and g is one-to-one and for every n such that $1 \leq n$ and $n \leq \operatorname{len} g - 2$ holds $\mathcal{L}(g, n, n+1) \cap \mathcal{L}(g, n+1, n+2) = \{g(n+1)\}$ and for all n, m such that $n - m > 1$ or $m - n > 1$ holds $\mathcal{L}(g, n, n+1) \cap \mathcal{L}(g, m, m+1) = \emptyset$ and for all n, p, q such that $1 \leq n$ and $n \leq \operatorname{len} g - 1$ and $g(n) = p$ and $g(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(g)$ and $f(1) = g(1)$ and $f(\operatorname{len} f) = g(\operatorname{len} g)$ and $\operatorname{len} f \leq \operatorname{len} g$.

[1]This article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

(2) Suppose for every n such that $n \in \operatorname{dom} f$ there exist i, j such that $\langle i, j\rangle \in$ the indices of G and $f(n) = G_{i,j}$ and f is a special sequence. Then there exists g such that g is a sequence which elements belong to G and g is a special sequence and $\widetilde{\mathcal{L}}(f) = \widetilde{\mathcal{L}}(g)$ and $f(1) = g(1)$ and $f(\operatorname{len} f) = g(\operatorname{len} g)$ and $\operatorname{len} f \leq \operatorname{len} g$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[10] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[11] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[12] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part II. *Formalized Mathematics*, 3(**1**):117–121, 1992.

[13] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Formalized Mathematics*, 3(**1**):101–106, 1992.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[15] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Go-Board Theorem

Jarosław Kotowicz[1]
Warsaw University
Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. We prove the Go-board theorem which is a special case of Hex Theorem. The article is based on [15].

MML Identifier: `GOBOARD4`.

The terminology and notation used in this paper are introduced in the following articles: [16], [7], [1], [4], [2], [13], [14], [17], [3], [8], [5], [6], [9], [12], [10], and [11]. For simplicity we adopt the following convention: p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\rm T}$, P_1, P_2 will be subsets of $\mathcal{E}^2_{\rm T}$, f_1, f_2 will be finite sequences of elements of $\mathcal{E}^2_{\rm T}$, r, s will be real numbers, n will be a natural number, and G will be a Go-board. We now state several propositions:

(1) Given G, f_1, f_2. Suppose that
(i) $1 \leq \mathrm{len}\, f_1$,
(ii) $1 \leq \mathrm{len}\, f_2$,
(iii) f_1 is a sequence which elements belong to G,
(iv) f_2 is a sequence which elements belong to G,
(v) $f_1(1) \in \mathrm{rng}\,\mathrm{Line}(G, 1)$,
(vi) $f_1(\mathrm{len}\, f_1) \in \mathrm{rng}\,\mathrm{Line}(G, \mathrm{len}\, G)$,
(vii) $f_2(1) \in \mathrm{rng}(G_{\square,1})$,
(viii) $f_2(\mathrm{len}\, f_2) \in \mathrm{rng}(G_{\square,\mathrm{width}\, G})$.
Then $\mathrm{rng}\, f_1 \cap \mathrm{rng}\, f_2 \neq \emptyset$.

(2) Given G, f_1, f_2. Suppose that
(i) $2 \leq \mathrm{len}\, f_1$,
(ii) $2 \leq \mathrm{len}\, f_2$,
(iii) f_1 is a sequence which elements belong to G,
(iv) f_2 is a sequence which elements belong to G,
(v) $f_1(1) \in \mathrm{rng}\,\mathrm{Line}(G, 1)$,
(vi) $f_1(\mathrm{len}\, f_1) \in \mathrm{rng}\,\mathrm{Line}(G, \mathrm{len}\, G)$,

[1]This article was written during my visit at Shinshu University in 1992.

ISSN 0777–4028

(vii) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(viii) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(3) Given G, f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) f_2 is a special sequence,
(iii) f_1 is a sequence which elements belong to G,
(iv) f_2 is a sequence which elements belong to G,
(v) $f_1(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$,
(vi) $f_1(\operatorname{len} f_1) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,
(vii) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(viii) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(4) Given f_1, f_2. Suppose that
(i) $2 \leq \operatorname{len} f_1$,
(ii) $2 \leq \operatorname{len} f_2$,
(iii) for all n, p, q such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ and $f_1(n) = p$ and $f_1(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,
(iv) for all n, p, q such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ and $f_2(n) = p$ and $f_2(n+1) = q$ holds $p_{\mathbf{1}} = q_{\mathbf{1}}$ or $p_{\mathbf{2}} = q_{\mathbf{2}}$,
(v) for every n such that $n \in \operatorname{dom} f_1$ and $n+1 \in \operatorname{dom} f_1$ holds $f_1(n) \neq f_1(n+1)$,
(vi) for every n such that $n \in \operatorname{dom} f_2$ and $n+1 \in \operatorname{dom} f_2$ holds $f_2(n) \neq f_2(n+1)$,
(vii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(viii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(ix) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(x) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,
(xi) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(xii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(xiii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(xiv) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.
Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(5) Given f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) f_2 is a special sequence,

(iii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(iv) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(v) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(vi) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,
(vii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(viii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(ix) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(x) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.

Then $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) \neq \emptyset$.

(6) Given P_1, P_2. Suppose P_1 is a special polygonal arc and P_2 is a special polygonal arc. Given G, f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) $P_1 = \widetilde{\mathcal{L}}(f_1)$,
(iii) f_2 is a special sequence,
(iv) $P_2 = \widetilde{\mathcal{L}}(f_2)$,
(v) f_1 is a sequence which elements belong to G,
(vi) f_2 is a sequence which elements belong to G,
(vii) $f_1(1) \in \operatorname{rng} \operatorname{Line}(G, 1)$,
(viii) $f_1(\operatorname{len} f_1) \in \operatorname{rng} \operatorname{Line}(G, \operatorname{len} G)$,
(ix) $f_2(1) \in \operatorname{rng}(G_{\square,1})$,
(x) $f_2(\operatorname{len} f_2) \in \operatorname{rng}(G_{\square,\operatorname{width} G})$.

Then $P_1 \cap P_2 \neq \emptyset$.

(7) Given P_1, P_2. Suppose P_1 is a special polygonal arc and P_2 is a special polygonal arc. Given f_1, f_2. Suppose that
(i) f_1 is a special sequence,
(ii) $P_1 = \widetilde{\mathcal{L}}(f_1)$,
(iii) f_2 is a special sequence,
(iv) $P_2 = \widetilde{\mathcal{L}}(f_2)$,
(v) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,
(vi) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,
(vii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{X}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,
(viii) for every r such that $r = (\mathbf{X}\text{-coordinate}(f_1))(\operatorname{len} f_1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{X}\text{-coordinate}(f_2))(n)$ holds $s \leq r$,

(ix) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $r \leq s$,

(x) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(1)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $r \leq s$,

(xi) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_1$ and $s = (\mathbf{Y}\text{-coordinate}(f_1))(n)$ holds $s \leq r$,

(xii) for every r such that $r = (\mathbf{Y}\text{-coordinate}(f_2))(\operatorname{len} f_2)$ and for all n, s such that $n \in \operatorname{dom} f_2$ and $s = (\mathbf{Y}\text{-coordinate}(f_2))(n)$ holds $s \leq r$.

Then $P_1 \cap P_2 \neq \emptyset$.

(8) Given P_1, P_2, p_1, p_2, q_1, q_2. Suppose that

(i) P_1 is a special polygonal arc joining p_1 and q_1,

(ii) P_2 is a special polygonal arc joining p_2 and q_2,

(iii) for every p such that $p \in P_1 \cup P_2$ holds $p_{1\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq q_{1\mathbf{1}}$,

(iv) for every p such that $p \in P_1 \cup P_2$ holds $p_{2\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq q_{2\mathbf{2}}$.

Then $P_1 \cap P_2 \neq \emptyset$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[6] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[9] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.

[10] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[11] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part II. *Formalized Mathematics*, 3(**1**):117–121, 1992.

[12] Yatsuka Nakamura and Jarosław Kotowicz. Connectedness conditions using polygonal arcs. *Formalized Mathematics*, 3(**1**):101–106, 1992.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[15] Yukio Takeuchi and Yatsuka Nakamura. *On the Jordan curve theorem.* Technical Report 19804, Dept. of Information Eng., Shinshu University, 500 Wakasato, Nagano city, Japan, April 1980.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received August 24, 1992

FORMALIZED MATHEMATICS
Volume 3, Number 1, 1992
Université Catholique de Louvain

Some Properties of Binary Relations

Waldemar Korczyński
Pedagogical University Kielce

Summary. The article contains some theorems on binary relations, which are used in papers [2], [3], [1], and other.

MML Identifier: `SYSREL`.

The articles [5], [6], [7], and [4] provide the terminology and notation for this paper. We adopt the following rules: x, y are arbitrary, X, Y, Z, W are sets, and R, S, T are binary relations. We now state a number of propositions:

(1) If $X \cap Y = \emptyset$ and $x \in X \cup Y$, then $x \in X$ and $x \notin Y$ or $x \in Y$ and $x \notin X$.

(2) $(X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z)$.

(3) $X \cup (X \cup Y) = X \cup Y$.

(4) If $X \subseteq Y \cap Z$, then $X \subseteq Y$ and $X \subseteq Z$.

(5) $\varnothing = \emptyset$.

(6) $\varnothing \setminus R = \varnothing$.

(7) $R \subseteq S$ if and only if $R \setminus S = \varnothing$.

(8) $R \cap S = \varnothing$ if and only if $R \setminus S = R$.

(9) $R \setminus R = \varnothing$.

(10) If $R \subseteq \varnothing$, then $R = \varnothing$.

(11) $\varnothing \cup R = R$ and $R \cup \varnothing = R$ and $\varnothing \cap R = \varnothing$ and $R \cap \varnothing = \varnothing$.

Let us consider X, Y. Then $[\!:X, Y:\!]$ is a binary relation.

Next we state several propositions:

(12) If $X \neq \emptyset$ and $Y \neq \emptyset$, then $\mathrm{dom}[\!:X, Y:\!] = X$ and $\mathrm{rng}[\!:X, Y:\!] = Y$.

(13) $\mathrm{dom}(R \cap [\!:X, Y:\!]) \subseteq X$ and $\mathrm{rng}(R \cap [\!:X, Y:\!]) \subseteq Y$.

(14) If $X \cap Y = \emptyset$, then $\mathrm{dom}(R \cap [\!:X, Y:\!]) \cap \mathrm{rng}(R \cap [\!:X, Y:\!]) = \emptyset$ and $\mathrm{dom}(R^{\smile} \cap [\!:X, Y:\!]) \cap \mathrm{rng}(R^{\smile} \cap [\!:X, Y:\!]) = \emptyset$.

(15) If $R \subseteq [\!:X, Y:\!]$, then $\mathrm{dom}\, R \subseteq X$ and $\mathrm{rng}\, R \subseteq Y$.

(16) If $R \subseteq [\!:X, Y:\!]$, then $R^{\smile} \subseteq [\!:Y, X:\!]$.

ISSN 0777–4028

(17) If $X \cap Y = \emptyset$, then $[\!: X, Y :\!] \cap [\!: Y, X :\!] = \emptyset$.

(18) $[\!: X, Y :\!]^\smile = [\!: Y, X :\!]$.

Next we state a number of propositions:

(19) $(R \cup S) \cdot T = R \cdot T \cup S \cdot T$ and $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$.

(20) If $R \subseteq [\!: X, Y :\!]$ and $\langle x, y\rangle \in R$, then $x \in X$ and $y \in Y$.

(21) (i) If $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y\rangle \in R$ and $x \in X$, then $x \notin Y$ and $y \notin X$ and $y \in Y$,

(ii) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y\rangle \in R$ and $y \in Y$, then $y \notin X$ and $x \notin Y$ and $x \in X$,

(iii) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y\rangle \in R$ and $x \in Y$, then $x \notin X$ and $y \notin Y$ and $y \in X$,

(iv) if $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$ and $\langle x, y\rangle \in R$ and $y \in X$, then $x \notin X$ and $y \notin Y$ and $x \in Y$.

(22) If $\operatorname{rng} R \cap \operatorname{dom} S = \emptyset$ or $\operatorname{dom} S \cap \operatorname{rng} R = \emptyset$, then $R \cdot S = \varnothing$.

(23) If $R \subseteq [\!: X, Y :\!]$ and $Z \subseteq X$, then $R \restriction Z = R \cap [\!: Z, Y :\!]$ but if $R \subseteq [\!: X, Y :\!]$ and $Z \subseteq Y$, then $Z \restriction R = R \cap [\!: X, Z :\!]$.

(24) If $R \subseteq [\!: X, Y :\!]$ and $X = Z \cup W$, then $R = R \restriction Z \cup R \restriction W$.

(25) If $X \cap Y = \emptyset$ and $R \subseteq [\!: X, Y :\!] \cup [\!: Y, X :\!]$, then $R \restriction X \subseteq [\!: X, Y :\!]$.

(26) If $R \subseteq S$, then $R^\smile \subseteq S^\smile$.

(27) $\triangle_X \subseteq [\!: X, X :\!]$.

(28) $\triangle_X \cdot \triangle_X = \triangle_X$.

(29) $\triangle_{\{x\}} = \{\langle x, x\rangle\}$.

(30) $\langle x, y\rangle \in \triangle_X$ if and only if $\langle y, x\rangle \in \triangle_X$.

(31) $\triangle_{X \cup Y} = \triangle_X \cup \triangle_Y$ and $\triangle_{X \cap Y} = \triangle_X \cap \triangle_Y$ and $\triangle_{X \setminus Y} = \triangle_X \setminus \triangle_Y$.

(32) If $X \subseteq Y$, then $\triangle_X \subseteq \triangle_Y$.

(33) $\triangle_{X \setminus Y} \setminus \triangle_X = \varnothing$.

(34) If $R \subseteq \triangle_{\operatorname{dom} R}$, then $R = \triangle_{\operatorname{dom} R}$.

(35) If $\triangle_X \subseteq R \cup R^\smile$, then $\triangle_X \subseteq R$ and $\triangle_X \subseteq R^\smile$.

(36) If $\triangle_X \subseteq R$, then $\triangle_X \subseteq R^\smile$.

(37) If $R \subseteq [\!: X, X :\!]$, then $R \setminus \triangle_{\operatorname{dom} R} = R \setminus \triangle_X$ and $R \setminus \triangle_{\operatorname{rng} R} = R \setminus \triangle_X$.

(38) If $\triangle_X \cdot (R \setminus \triangle_X) = \varnothing$, then $\operatorname{dom}(R \setminus \triangle_X) = \operatorname{dom} R \setminus \operatorname{dom}(\triangle_X)$ but if $(R \setminus \triangle_X) \cdot \triangle_X = \varnothing$, then $\operatorname{rng}(R \setminus \triangle_X) = \operatorname{rng} R \setminus \operatorname{rng}(\triangle_X)$.

(39) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$, then $\triangle_{\operatorname{rng} R} \subseteq R$ but if $R \subseteq R \cdot R$ and $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$, then $\triangle_{\operatorname{dom} R} \subseteq R$.

(40) (i) If $R \subseteq R \cdot R$ and $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$, then $R \cap \triangle_{\operatorname{rng} R} = \triangle_{\operatorname{rng} R}$,

(ii) if $R \subseteq R \cdot R$ and $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$, then $R \cap \triangle_{\operatorname{dom} R} = \triangle_{\operatorname{dom} R}$.

(41) If $R \cdot (R \setminus \triangle_X) = \varnothing$ and $\operatorname{rng} R \subseteq X$, then $R \cdot (R \setminus \triangle_{\operatorname{rng} R}) = \varnothing$ but if $(R \setminus \triangle_X) \cdot R = \varnothing$ and $\operatorname{dom} R \subseteq X$, then $(R \setminus \triangle_{\operatorname{dom} R}) \cdot R = \varnothing$.

Let us consider R. The functor $\mathrm{CL}(R)$ yielding a binary relation is defined as follows:

(Def.1) $\mathrm{CL}(R) = R \cap \triangle_{\operatorname{dom} R}$.

One can prove the following propositions:

(42) $\mathrm{CL}(R) \subseteq R$ and $\mathrm{CL}(R) \subseteq \triangle_{\mathrm{dom}\,R}$.

(43) If $\langle x, y\rangle \in \mathrm{CL}(R)$, then $x \in \mathrm{dom}\,\mathrm{CL}(R)$ and $x = y$.

(44) $\mathrm{dom}\,\mathrm{CL}(R) = \mathrm{rng}\,\mathrm{CL}(R)$.

(45) (i) $x \in \mathrm{dom}\,\mathrm{CL}(R)$ if and only if $x \in \mathrm{dom}\,R$ and $\langle x, x\rangle \in R$,
(ii) $x \in \mathrm{rng}\,\mathrm{CL}(R)$ if and only if $x \in \mathrm{dom}\,R$ and $\langle x, x\rangle \in R$,
(iii) $x \in \mathrm{rng}\,\mathrm{CL}(R)$ if and only if $x \in \mathrm{rng}\,R$ and $\langle x, x\rangle \in R$,
(iv) $x \in \mathrm{dom}\,\mathrm{CL}(R)$ if and only if $x \in \mathrm{rng}\,R$ and $\langle x, x\rangle \in R$.

(46) $\mathrm{CL}(R) = \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}$.

(47) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \mathrm{CL}(R)) = \varnothing$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\,R \setminus \mathrm{dom}\,\mathrm{CL}(R)$ and $y \in \mathrm{dom}\,\mathrm{CL}(R)$,
(ii) if $R \cdot R = R$ and $(R \setminus \mathrm{CL}(R)) \cdot R = \varnothing$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\,R \setminus \mathrm{dom}\,\mathrm{CL}(R)$ and $x \in \mathrm{dom}\,\mathrm{CL}(R)$.

(48) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $x \in \mathrm{dom}\,R \setminus \mathrm{dom}\,\mathrm{CL}(R)$ and $y \in \mathrm{dom}\,\mathrm{CL}(R)$,
(ii) if $R \cdot R = R$ and $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$ and $\langle x, y\rangle \in R$ and $x \neq y$, then $y \in \mathrm{rng}\,R \setminus \mathrm{dom}\,\mathrm{CL}(R)$ and $x \in \mathrm{dom}\,\mathrm{CL}(R)$.

(49) (i) If $R \cdot R = R$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$, then $\mathrm{dom}\,\mathrm{CL}(R) = \mathrm{rng}\,R$ and $\mathrm{rng}\,\mathrm{CL}(R) = \mathrm{rng}\,R$,
(ii) if $R \cdot R = R$ and $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$, then $\mathrm{dom}\,\mathrm{CL}(R) = \mathrm{dom}\,R$ and $\mathrm{rng}\,\mathrm{CL}(R) = \mathrm{dom}\,R$.

(50) $\mathrm{dom}\,\mathrm{CL}(R) \subseteq \mathrm{dom}\,R$ and $\mathrm{rng}\,\mathrm{CL}(R) \subseteq \mathrm{rng}\,R$ and $\mathrm{rng}\,\mathrm{CL}(R) \subseteq \mathrm{dom}\,R$ and $\mathrm{dom}\,\mathrm{CL}(R) \subseteq \mathrm{rng}\,R$.

(51) $\triangle_{\mathrm{dom}\,\mathrm{CL}(R)} \subseteq \triangle_{\mathrm{dom}\,R}$ and $\triangle_{\mathrm{rng}\,\mathrm{CL}(R)} \subseteq \triangle_{\mathrm{dom}\,R}$.

(52) $\triangle_{\mathrm{dom}\,\mathrm{CL}(R)} \subseteq R$ and $\triangle_{\mathrm{rng}\,\mathrm{CL}(R)} \subseteq R$.

(53) If $\triangle_X \subseteq R$ and $\triangle_X \cdot (R \setminus \triangle_X) = \varnothing$, then $R \restriction X = \triangle_X$ but if $\triangle_X \subseteq R$ and $(R \setminus \triangle_X) \cdot \triangle_X = \varnothing$, then $X \restriction R = \triangle_X$.

(54) (i) If $\triangle_{\mathrm{dom}\,\mathrm{CL}(R)} \cdot (R \setminus \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}) = \varnothing$, then $R \restriction \mathrm{dom}\,\mathrm{CL}(R) = \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}$ and $R \restriction \mathrm{rng}\,\mathrm{CL}(R) = \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}$,
(ii) if $(R \setminus \triangle_{\mathrm{rng}\,\mathrm{CL}(R)}) \cdot \triangle_{\mathrm{rng}\,\mathrm{CL}(R)} = \varnothing$, then $\mathrm{dom}\,\mathrm{CL}(R) \restriction R = \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}$ and $\mathrm{rng}\,\mathrm{CL}(R) \restriction R = \triangle_{\mathrm{rng}\,\mathrm{CL}(R)}$.

(55) If $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$, then $\triangle_{\mathrm{dom}\,\mathrm{CL}(R)} \cdot (R \setminus \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}) = \varnothing$ but if $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$, then $(R \setminus \triangle_{\mathrm{dom}\,\mathrm{CL}(R)}) \cdot \triangle_{\mathrm{dom}\,\mathrm{CL}(R)} = \varnothing$.

(56) (i) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$, then $S \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$,
(ii) if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$, then $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot S = \varnothing$.

(57) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$ but if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$, then $\mathrm{CL}(S) \subseteq \mathrm{CL}(R)$.

(58) (i) If $S \cdot R = S$ and $R \cdot (R \setminus \triangle_{\mathrm{dom}\,R}) = \varnothing$ and $R \cdot S = R$ and $S \cdot (S \setminus \triangle_{\mathrm{dom}\,S}) = \varnothing$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$,
(ii) if $R \cdot S = S$ and $(R \setminus \triangle_{\mathrm{dom}\,R}) \cdot R = \varnothing$ and $S \cdot R = R$ and $(S \setminus \triangle_{\mathrm{dom}\,S}) \cdot S = \varnothing$, then $\mathrm{CL}(S) = \mathrm{CL}(R)$.

References

[1] Waldemar Korczyński. Definitions of Petri net - part III - `S_SIEC`. Main Mizar Library, 1992.
[2] Waldemar Korczyński. Definitons of Petri net - part I - `FF_SIEC`. Main Mizar Library, 1992.
[3] Waldemar Korczyński. Definitons of Petri net - part II - `E_SIEC`. Main Mizar Library, 1992.
[4] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[6] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[7] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received January 17, 1992

alalalalalal alalalalalalal

Index of MML Identifiers

Contents

Form. Math. 3 (**1**)

Continued on inside back cover