FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

The de l'Hospital Theorem

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. List of theorems concerning the de l'Hospital Theorem. We discuss the case when both functions have the zero value at a point and when the quotient of their differentials is convergent at this point.

MML Identifier: `L'HOSPIT`.

The papers [21], [4], [1], [2], [17], [15], [6], [9], [16], [3], [5], [12], [13], [20], [14], [18], [19], [8], [11], [7], and [10] provide the terminology and notation for this paper. We adopt the following rules: f, g will be partial functions from $\mathbb{R}$ to $\mathbb{R}$, r, r_1, r_2, g_1, g_2, x_0, t will be real numbers, and a will be a sequence of real numbers. Next we state a number of propositions:

(1) If f is continuous in x_0 and for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$, then f is convergent in x_0.

(2) f is right convergent in x_0 and $\lim_{x_0^+} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists t such that $t < r$ and $x_0 < t$ and $t \in \operatorname{dom} f$,

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\operatorname{rng} a \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(3) f is left convergent in x_0 and $\lim_{x_0^-} f = t$ if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists t such that $r < t$ and $t < x_0$ and $t \in \operatorname{dom} f$,

(ii) for every a such that a is convergent and $\lim a = x_0$ and $\operatorname{rng} a \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot a$ is convergent and $\lim(f \cdot a) = t$.

(4) Suppose There exists a neighbourhood N of x_0 such that $N \setminus \{x_0\} \subseteq \operatorname{dom} f$. Then for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

ISSN 0777–4028

(5) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $+\infty$ in x_0.

Then $\frac{f}{g}$ is divergent to $+\infty$ in x_0.

(6) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is divergent to $-\infty$ in x_0.

Then $\frac{f}{g}$ is divergent to $-\infty$ in x_0.

(7) Given r such that
(i) $r > 0$,
(ii) f is differentiable on $]x_0, x_0 + r[$,
(iii) g is differentiable on $]x_0, x_0 + r[$,
(iv) $]x_0, x_0 + r[\subseteq \mathrm{dom}(\frac{f}{g})$,
(v) $[x_0, x_0 + r] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}})$,
(vi) $f(x_0) = 0$,
(vii) $g(x_0) = 0$,
(viii) f is continuous in x_0,
(ix) g is continuous in x_0,
(x) $\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}}$ is right convergent in x_0.

Then $\frac{f}{g}$ is right convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0+}(\frac{f}{g}) = \lim_{x_0+}(\frac{f'_{\restriction]x_0, x_0+r[}}{g'_{\restriction]x_0, x_0+r[}})$.

(8) Given r such that
(i) $r > 0$,
(ii) f is differentiable on $]x_0 - r, x_0[$,
(iii) g is differentiable on $]x_0 - r, x_0[$,
(iv) $]x_0 - r, x_0[\subseteq \mathrm{dom}(\frac{f}{g})$,

(v) $[x_0 - r, x_0] \subseteq \mathrm{dom}(\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}})$,
(vi) $f(x_0) = 0$,
(vii) $g(x_0) = 0$,
(viii) f is continuous in x_0,
(ix) g is continuous in x_0,
(x) $\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}}$ is left convergent in x_0.

Then $\frac{f}{g}$ is left convergent in x_0 and there exists r such that $r > 0$ and $\lim_{x_0^-}(\frac{f}{g}) = \lim_{x_0^-}(\frac{f'_{\restriction]x_0-r,x_0[}}{g'_{\restriction]x_0-r,x_0[}})$.

(9) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is convergent in x_0.

Then $\frac{f}{g}$ is convergent in x_0 and there exists a neighbourhood N of x_0 such that $\lim_{x_0}(\frac{f}{g}) = \lim_{x_0}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$.

(10) Given a neighbourhood N of x_0 such that
(i) f is differentiable on N,
(ii) g is differentiable on N,
(iii) $N \setminus \{x_0\} \subseteq \mathrm{dom}(\frac{f}{g})$,
(iv) $N \subseteq \mathrm{dom}(\frac{f'_{\restriction N}}{g'_{\restriction N}})$,
(v) $f(x_0) = 0$,
(vi) $g(x_0) = 0$,
(vii) $\frac{f'_{\restriction N}}{g'_{\restriction N}}$ is continuous in x_0.

Then $\frac{f}{g}$ is convergent in x_0 and $\lim_{x_0}(\frac{f}{g}) = \frac{f'(x_0)}{g'(x_0)}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[5] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[7] Jarosław Kotowicz. The limit of a real function at a point. *Formalized Mathematics*, 2(**1**):71–80, 1991.

[8] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.

[9] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[10] Jarosław Kotowicz. Monotonic and continuous real function. *Formalized Mathematics*, 2(**3**):403–405, 1991.

[11] Jarosław Kotowicz. One-side limits of a real function at a point. *Formalized Mathematics*, 2(**1**):29–40, 1991.

[12] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.

[13] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 1(**4**):703–709, 1990.

[14] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[15] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[16] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.

[17] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[18] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[19] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[20] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received February 20, 1992