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# Serieses [1]

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**Summary.** The article contains definitions and properties of convergent serieses.

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The articles [12], [2], [10], [1], [7], [6], [4], [3], [5], [11], [8], and [9] provide the notation and terminology for this paper. We follow the rules: $n$, $m$ will denote natural numbers, $a$, $p$, $r$ will denote real numbers, and $s$, $s_1$, $s_2$ will denote sequences of real numbers. We now state three propositions:

(1) If $0 < a$ and $a < 1$ and for every $n$ holds $s(n) = a^{n+1}$, then $s$ is convergent and $\lim s = 0$.

(2) If $a \neq 0$, then $|a|^n = |a^n|$.

(3) If $|a| < 1$ and for every $n$ holds $s(n) = a^{n+1}$, then $s$ is convergent and $\lim s = 0$.

Let us consider $s$. The functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ yielding a sequence of real numbers is defined by:

(Def.1) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(0) = s(0)$ and for every $n$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) + s(n+1)$.

The following proposition is true

(4) For all $s$, $s_1$ holds $s_1 = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ if and only if $s_1(0) = s(0)$ and for every $n$ holds $s_1(n+1) = s_1(n) + s(n+1)$.

Let us consider $s$. We say that $s$ is summable if and only if:

(Def.2) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is convergent.

Let us consider $s$. Let us assume that $s$ is summable. The functor $\sum s$ yields a real number and is defined as follows:

(Def.3) $\sum s = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$.

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The following propositions are true:

(6)[2] For all $s$, $r$ such that $s$ is summable holds $r = \sum s$ if and only if $r = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})$.

(7) If $s$ is summable, then $s$ is convergent and $\lim s = 0$.

(8) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1+s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(9) $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1-s_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(10) If $s_1$ is summable and $s_2$ is summable, then $s_1+s_2$ is summable and $\sum(s_1+s_2) = \sum s_1 + \sum s_2$.

(11) If $s_1$ is summable and $s_2$ is summable, then $s_1-s_2$ is summable and $\sum(s_1-s_2) = \sum s_1 - \sum s_2$.

(12) $(\sum_{\alpha=0}^{\kappa}(rs)(\alpha))_{\kappa\in\mathbb{N}} = r(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$.

(13) If $s$ is summable, then $rs$ is summable and $\sum(rs) = r \cdot \sum s$.

(14) For all $s$, $s_1$ such that for every $n$ holds $s_1(n) = s(0)$ holds $(\sum_{\alpha=0}^{\kappa}(s \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_1$.

(15) If $s$ is summable, then for every $n$ holds $s \uparrow n$ is summable.

(16) If there exists $n$ such that $s \uparrow n$ is summable, then $s$ is summable.

(17) If for every $n$ holds $s_1(n) \le s_2(n)$, then for every $n$ holds $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa\in\mathbb{N}}(n) \le (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(18) If $s$ is summable, then for every $n$ holds $\sum s = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) + \sum(s \uparrow (n+1))$.

(19) If for every $n$ holds $0 \le s(n)$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(20) If for every $n$ holds $0 \le s(n)$, then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded if and only if $s$ is summable.

(21) If $s$ is summable and for every $n$ holds $0 \le s(n)$, then $0 \le \sum s$.

(22) If for every $n$ holds $0 \le s_2(n)$ and $s_1$ is summable and there exists $m$ such that for every $n$ such that $m \le n$ holds $s_2(n) \le s_1(n)$, then $s_2$ is summable.

(23) If for every $n$ holds $0 \le s_2(n)$ and $s_2$ is not summable and there exists $m$ such that for every $n$ such that $m \le n$ holds $s_2(n) \le s_1(n)$, then $s_1$ is not summable.

(24) If for every $n$ holds $0 \le s_1(n)$ and $s_1(n) \le s_2(n)$ and $s_2$ is summable, then $s_1$ is summable and $\sum s_1 \le \sum s_2$.

(25) $s$ is summable if and only if for every $r$ such that $0 < r$ there exists $n$ such that for every $m$ such that $n \le m$ holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)| < r$.

(26) If $a \ne 1$, then $(\sum_{\alpha=0}^{\kappa}((a^{\kappa})_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1-a^{n+1}}{1-a}$.

(27) If $a \ne 1$ and for every $n$ holds $s(n+1) = a \cdot s(n)$, then for every $n$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{s(0)\cdot(1-a^{n+1})}{1-a}$.

(28) If $|a| < 1$, then $(a^{\kappa})_{\kappa\in\mathbb{N}}$ is summable and $\sum((a^{\kappa})_{\kappa\in\mathbb{N}}) = \frac{1}{1-a}$.

[2] The proposition (5) has been removed.

(29) If $|a| < 1$ and for every $n$ holds $s(n+1) = a \cdot s(n)$, then $s$ is summable and $\sum s = \frac{s(0)}{1-a}$.

(30) If for every $n$ holds $s(n) > 0$ and $s_1(n) = \frac{s(n+1)}{s(n)}$ and $s_1$ is convergent and $\lim s_1 < 1$, then $s$ is summable.

(31) If for every $n$ holds $s(n) > 0$ and there exists $m$ such that for every $n$ such that $n \geq m$ holds $\frac{s(n+1)}{s(n)} \geq 1$, then $s$ is not summable.

(32) If for every $n$ holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and $s_1$ is convergent and $\lim s_1 < 1$, then $s$ is summable.

(33) If for every $n$ holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and there exists $m$ such that for every $n$ such that $m \leq n$ holds $s_1(n) \geq 1$, then $s$ is not summable.

(34) If for every $n$ holds $s(n) \geq 0$ and $s_1(n) = \sqrt[n]{s(n)}$ and $s_1$ is convergent and $\lim s_1 > 1$, then $s$ is not summable.

Let us consider $n$. The $n$-th power of 2 yields a natural number and is defined as follows:

(Def.4) the $n$-th power of $2 = 2^n$.

One can prove the following three propositions:

(35) If $s$ is non-increasing and for every $n$ holds $s(n) \geq 0$ and $s_1(n) = 2^n \cdot s(\text{the } n\text{-th power of } 2)$, then $s$ is summable if and only if $s_1$ is summable.

(36) If $p > 1$ and for every $n$ such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then $s$ is summable.

(37) If $p \leq 1$ and for every $n$ such that $n \geq 1$ holds $s(n) = \frac{1}{n^p}$, then $s$ is not summable.

Let us consider $s$. We say that $s$ is absolutely summable if and only if:

(Def.5) $|s|$ is summable.

We now state several propositions:

(39)[3] For all $n$, $m$ such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)| \leq |(\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} |s|(\alpha))_{\kappa \in \mathbb{N}}(n)|$.

(40) If $s$ is absolutely summable, then $s$ is summable.

(41) If for every $n$ holds $0 \leq s(n)$ and $s$ is summable, then $s$ is absolutely summable.

(42) If for every $n$ holds $s(n) \neq 0$ and $s_1(n) = \frac{|s|(n+1)}{|s|(n)}$ and $s_1$ is convergent and $\lim s_1 < 1$, then $s$ is absolutely summable.

(43) If $r > 0$ and there exists $m$ such that for every $n$ such that $n \geq m$ holds $|s(n)| \geq r$, then $s$ is not convergent or $\lim s \neq 0$.

(44) If for every $n$ holds $s(n) \neq 0$ and there exists $m$ such that for every $n$ such that $n \geq m$ holds $\frac{|s|(n+1)}{|s|(n)} \geq 1$, then $s$ is not summable.

[3] The proposition (38) has been removed.

(45) If for every $n$ holds $s_1(n) = \sqrt[n]{|s|(n)}$ and $s_1$ is convergent and $\lim s_1 < 1$, then $s$ is absolutely summable.

(46) If for every $n$ holds $s_1(n) = \sqrt[n]{|s|(n)}$ and there exists $m$ such that for every $n$ such that $m \leq n$ holds $s_1(n) \geq 1$, then $s$ is not summable.

(47) If for every $n$ holds $s_1(n) = \sqrt[n]{|s|(n)}$ and $s_1$ is convergent and $\lim s_1 > 1$, then $s$ is not summable.

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