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Semi-Affine Space [1]

Eugeniusz Kusak
Warsaw University
Białystok

Krzysztof Radziszewski
Gdańsk University

Summary. A brief survey on semi-affine geometry, which results from the classical Pappian and Desarguesian affine (dimension free) geometry by weakening the so called trapezium axiom. With the help of the relation of parallelogram in every semi-affine space we define the operation of "addition" of "vectors". Next we investigate in greater details the relation of (affine) trapezium in such spaces.

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The papers [3], [2], and [1] provide the notation and terminology for this paper. An affine structure is called a semi affine space if it satisfies the conditions (Def.1).

(Def.1) (i) For all elements a, b of the points of it holds $a,b \parallel b,a$,

(ii) for all elements a, b, c of the points of it holds $a,b \parallel c,c$,

(iii) for all elements a, b, p, q, r, s of the points of it such that $a \neq b$ and $a,b \parallel p,q$ and $a,b \parallel r,s$ holds $p,q \parallel r,s$,

(iv) for all elements a, b, c of the points of it such that $a,b \parallel a,c$ holds $b,a \parallel b,c$,

(v) there exist elements a, b, c of the points of it such that $a,b \nparallel a,c$,

(vi) for every elements a, b, p of the points of it there exists an element q of the points of it such that $a,b \parallel p,q$ and $a,p \parallel b,q$,

(vii) for every elements o, a of the points of it there exists an element p of the points of it such that for all elements b, c of the points of it holds $o,a \parallel o,p$ and there exists an element d of the points of it such that if $o,p \parallel o,b$, then $o,c \parallel o,d$ and $p,c \parallel b,d$,

(viii) for all elements o, a, a', b, b', c, c' of the points of it such that $o,a \nparallel o,b$ and $o,a \nparallel o,c$ and $o,a \parallel o,a'$ and $o,b \parallel o,b'$ and $o,c \parallel o,c'$ and $a,b \parallel a',b'$ and $a,c \parallel a',c'$ holds $b,c \parallel b',c'$,

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(ix) for all elements a, a', b, b', c, c' of the points of it such that $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$,

(x) for all elements a_1, a_2, a_3, b_1, b_2, b_3 of the points of it such that $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$ holds $a_3, b_1 \parallel a_1, b_3$,

(xi) for all elements a, b, c, d of the points of it such that $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, d \nparallel b, c$.

We adopt the following convention: S_1 will be a semi affine space and a, a', a_1, a_2, a_3, a_4, b, b', b_1, b_2, b_3, c, c', d, d', d_1, d_2, o, p, p_1, p_2, q, r, r_1, r_2, s, x, y, z will be elements of the points of S_1. The following propositions are true:

(1) $a, b \parallel b, a$.

(2) $a, b \parallel c, c$.

(3) If $a \neq b$ and $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(4) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(5) There exist a, b, c such that $a, b \nparallel a, c$.

(6) There exists q such that $a, b \parallel p, q$ and $a, p \parallel b, q$.

(7) For every o, a there exists p such that for all b, c holds $o, a \parallel o, p$ and there exists d such that if $o, p \parallel o, b$, then $o, c \parallel o, d$ and $p, c \parallel b, d$.

(8) If $o, a \nparallel o, b$ and $o, a \nparallel o, c$ and $o, a \parallel o, a'$ and $o, b \parallel o, b'$ and $o, c \parallel o, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(9) If $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(10) If $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$, then $a_3, b_1 \parallel a_1, b_3$.

(11) If $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, d \nparallel b, c$.

(12) $a, b \parallel a, b$.

(13) If $a, b \parallel c, d$, then $c, d \parallel a, b$.

(14) $a, a \parallel b, c$.

(15) If $a, b \parallel c, d$, then $b, a \parallel c, d$.

(16) If $a, b \parallel c, d$, then $a, b \parallel d, c$.

(17) If $a, b \parallel c, d$, then $b, a \parallel c, d$ and $a, b \parallel d, c$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $d, c \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel b, a$.

(18) Suppose $a, b \parallel a, c$. Then $a, c \parallel a, b$ and $b, a \parallel a, c$ and $a, b \parallel c, a$ and $a, c \parallel b, a$ and $b, a \parallel c, a$ and $c, a \parallel a, b$ and $c, a \parallel b, a$ and $b, a \parallel b, c$ and $a, b \parallel b, c$ and $b, a \parallel c, b$ and $b, c \parallel b, a$ and $a, b \parallel c, b$ and $c, b \parallel b, a$ and $b, c \parallel a, b$ and $c, b \parallel a, b$ and $c, a \parallel c, b$ and $a, c \parallel c, b$ and $c, a \parallel b, c$ and $a, c \parallel b, c$ and $c, b \parallel c, a$ and $b, c \parallel c, a$ and $c, b \parallel a, c$ and $b, c \parallel a, c$.

(19) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $a = b$ or $p, q \parallel r, s$.

(20) If $a \neq b$ and $p, q \parallel a, b$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(21) If $a, b \nparallel a, d$, then $a \neq b$ and $b \neq d$ and $d \neq a$.

(22) If $a, b \nparallel p, q$, then $a \neq b$ and $p \neq q$.

(23) If $a, b \parallel a, x$ and $b, c \parallel b, x$ and $c, a \parallel c, x$, then $a, b \parallel a, c$.

(24) If $a, b \nparallel a, c$, then $a, b \nparallel a, x$ or $b, c \nparallel b, x$ or $c, a \nparallel c, x$.

(25) If $a, b \nparallel a, c$ and $p \neq q$, then $p, q \nparallel p, a$ or $p, q \nparallel p, b$ or $p, q \nparallel p, c$.

(26) If $p \neq q$, then there exists r such that $p, q \nparallel p, r$.

(27) Suppose $a, b \nparallel c, d$. Then $a, b \nparallel d, c$ and $b, a \nparallel c, d$ and $b, a \nparallel d, c$ and $c, d \nparallel a, b$ and $c, d \nparallel b, a$ and $d, c \nparallel a, b$ and $d, c \nparallel b, a$.

(28) Suppose $a, b \nparallel a, c$. Then $a, b \nparallel c, a$ and $b, a \nparallel a, c$ and $b, a \nparallel c, a$ and $a, c \nparallel a, b$ and $a, c \nparallel b, a$ and $c, a \nparallel a, b$ and $c, a \nparallel b, a$ and $b, a \nparallel b, c$ and $b, a \nparallel c, b$ and $a, b \nparallel b, c$ and $a, b \nparallel c, b$ and $b, c \nparallel b, a$ and $b, c \nparallel a, b$ and $c, b \nparallel a, b$ and $c, b \nparallel b, a$ and $c, b \nparallel c, a$ and $c, b \nparallel a, c$ and $b, c \nparallel c, a$ and $b, c \nparallel a, c$ and $c, a \nparallel c, b$ and $c, a \nparallel b, c$ and $a, c \nparallel b, c$ and $a, c \nparallel c, b$.

(29) If $a, b \nparallel c, d$ and $a, b \parallel p, q$ and $c, d \parallel r, s$ and $p \neq q$ and $r \neq s$, then $p, q \nparallel r, s$.

(30) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $b, c \parallel q, r$ and $p \neq q$, then $p, q \nparallel p, r$.

(31) If $a, b \nparallel a, c$ and $a, c \parallel p, r$ and $b, c \parallel p, r$, then $p = r$.

We now state four propositions:

(32) If $p, q \nparallel p, r_1$ and $p, r_1 \parallel p, r_2$ and $q, r_1 \parallel q, r_2$, then $r_1 = r_2$.

(33) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r_1$ and $a, c \parallel p, r_2$ and $b, c \parallel q, r_1$ and $b, c \parallel q, r_2$, then $r_1 = r_2$.

(34) If $a = b$ or $c = d$ or $a = c$ and $b = d$ or $a = d$ and $b = c$, then $a, b \parallel c, d$.

(35) If $a = b$ or $a = c$ or $b = c$, then $a, b \parallel a, c$.

Let us consider S_1, a, b, c. We say that a, b and c are collinear if and only if:

(Def.2) $a, b \parallel a, c$.

We now state a number of propositions:

(37)[2] If a_1, a_2 and a_3 are collinear, then a_1, a_3 and a_2 are collinear and a_2, a_1 and a_3 are collinear and a_2, a_3 and a_1 are collinear and a_3, a_1 and a_2 are collinear and a_3, a_2 and a_1 are collinear.

(38) If a_1, a_2 and a_3 are not collinear, then a_1, a_3 and a_2 are not collinear and a_2, a_1 and a_3 are not collinear and a_2, a_3 and a_1 are not collinear and a_3, a_1 and a_2 are not collinear and a_3, a_2 and a_1 are not collinear.

(39) If a, b and c are not collinear and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $p \neq q$ and $p \neq r$, then p, q and r are not collinear.

(40) If $a = b$ or $b = c$ or $c = a$, then a, b and c are collinear.

(41) If $p \neq q$, then there exists r such that p, q and r are not collinear.

(42) If a, b and c are collinear and a, b and d are collinear, then $a, b \parallel c, d$.

(43) If a, b and c are not collinear and $a, b \parallel c, d$, then a, b and d are not collinear.

[2]The proposition (36) was either repeated or obvious.

(44) If a, b and c are not collinear and $a, b \parallel c, d$ and $c \neq d$ and c, d and x are collinear, then a, b and x are not collinear.

(45) If o, a and b are not collinear and o, a and x are collinear and o, b and x are collinear, then $o = x$.

(46) If $o \neq a$ and $o \neq b$ and o, a and b are collinear and o, a and a' are collinear and o, b and b' are collinear, then $a, b \parallel a', b'$.

(48)[3] If $a, b \nparallel c, d$ and a, b and p_1 are collinear and a, b and p_2 are collinear and c, d and p_1 are collinear and c, d and p_2 are collinear, then $p_1 = p_2$.

(49) If $a \neq b$ and a, b and c are collinear and $a, b \parallel c, d$, then $a, c \parallel b, d$.

(50) If $a \neq b$ and a, b and c are collinear and $a, b \parallel c, d$, then $c, b \parallel c, d$.

(51) If o, a and c are not collinear and o, a and b are collinear and o, c and d_1 are collinear and o, c and d_2 are collinear and $a, c \parallel b, d_1$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

(52) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

Let us consider S_1, a, b, c, d. We say that a, b, c, d form a parallelogram if and only if:

(Def.3) a, b and c are not collinear and $a, b \parallel c, d$ and $a, c \parallel b, d$.

We now state a number of propositions:

(54)[4] If a, b, c, d form a parallelogram, then $a \neq b$ and $a \neq c$ and $c \neq b$ and $a \neq d$ and $b \neq d$ and $c \neq d$.

(55) If a, b, c, d form a parallelogram, then a, b and c are not collinear and b, a and d are not collinear and c, d and a are not collinear and d, c and b are not collinear.

(56) Suppose a_1, a_2, a_3, a_4 form a parallelogram. Then a_1, a_2 and a_3 are not collinear and a_1, a_3 and a_2 are not collinear and a_1, a_2 and a_4 are not collinear and a_1, a_4 and a_2 are not collinear and a_1, a_3 and a_4 are not collinear and a_1, a_4 and a_3 are not collinear and a_2, a_1 and a_3 are not collinear and a_2, a_3 and a_1 are not collinear and a_2, a_1 and a_4 are not collinear and a_2, a_4 and a_1 are not collinear and a_2, a_3 and a_4 are not collinear and a_2, a_4 and a_3 are not collinear and a_3, a_1 and a_2 are not collinear and a_3, a_2 and a_1 are not collinear and a_3, a_1 and a_4 are not collinear and a_3, a_4 and a_1 are not collinear and a_3, a_2 and a_4 are not collinear and a_3, a_4 and a_2 are not collinear and a_4, a_1 and a_2 are not collinear and a_4, a_2 and a_1 are not collinear and a_4, a_1 and a_3 are not collinear and a_4, a_3 and a_1 are not collinear and a_4, a_2 and a_3 are not collinear and a_4, a_3 and a_2 are not collinear.

(57) If a, b, c, d form a parallelogram, then a, b and x are not collinear or c, d and x are not collinear.

(58) If a, b, c, d form a parallelogram, then a, c, b, d form a parallelogram.

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[3] The proposition (47) was either repeated or obvious.
[4] The proposition (53) was either repeated or obvious.

(59) If a, b, c, d form a parallelogram, then c, d, a, b form a parallelogram.

(60) If a, b, c, d form a parallelogram, then b, a, d, c form a parallelogram.

(61) If a, b, c, d form a parallelogram, then a, c, b, d form a parallelogram and c, d, a, b form a parallelogram and b, a, d, c form a parallelogram and c, a, d, b form a parallelogram and d, b, c, a form a parallelogram and b, d, a, c form a parallelogram.

(62) If a, b and c are not collinear, then there exists d such that a, b, c, d form a parallelogram.

(63) If a, b, c, d_1 form a parallelogram and a, b, c, d_2 form a parallelogram, then $d_1 = d_2$.

(64) If a, b, c, d form a parallelogram, then $a, d \nparallel b, c$.

(65) If a, b, c, d form a parallelogram, then a, b, d, c do not form a parallelogram.

(66) If $a \neq b$, then there exists c such that a, b and c are collinear and $c \neq a$ and $c \neq b$.

(67) If a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then $b, c \parallel b', c'$.

(68) If b, b' and c are not collinear and a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then b, b', c, c' form a parallelogram.

(69) If a, b and c are collinear and $b \neq c$ and a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then b, b', c, c' form a parallelogram.

(70) If a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram and b, b', d, d' form a parallelogram, then $c, d \parallel c', d'$.

(71) If $a \neq d$, then there exist b, c such that a, b, c, d form a parallelogram.

Let us consider S_1, a, b, r, s. We say that a, b are congruent to r, s if and only if:

(Def.4) $a = b$ and $r = s$ or there exist p, q such that p, q, a, b form a parallelogram and p, q, r, s form a parallelogram.

Next we state a number of propositions:

(73)[5] If a, a are congruent to b, c, then $b = c$.

(74) If a, b are congruent to c, c, then $a = b$.

(75) If a, b are congruent to b, a, then $a = b$.

(76) If a, b are congruent to c, d, then $a, b \parallel c, d$.

(77) If a, b are congruent to c, d, then $a, c \parallel b, d$.

(78) If a, b are congruent to c, d and a, b and c are not collinear, then a, b, c, d form a parallelogram.

(79) If a, b, c, d form a parallelogram, then a, b are congruent to c, d.

(80) If a, b are congruent to c, d and a, b and c are collinear and r, s, a, b form a parallelogram, then r, s, c, d form a parallelogram.

[5]The proposition (72) was either repeated or obvious.

(81) If a, b are congruent to c, x and a, b are congruent to c, y, then $x = y$.

(82) There exists d such that a, b are congruent to c, d.

(83) a, a are congruent to b, b.

(84) a, b are congruent to a, b.

(85) If r, s are congruent to a, b and r, s are congruent to c, d, then a, b are congruent to c, d.

(86) If a, b are congruent to c, d, then c, d are congruent to a, b.

(87) If a, b are congruent to c, d, then b, a are congruent to d, c.

(88) If a, b are congruent to c, d, then a, c are congruent to b, d.

(89) If a, b are congruent to c, d, then c, d are congruent to a, b and b, a are congruent to d, c and a, c are congruent to b, d and d, c are congruent to b, a and b, d are congruent to a, c and c, a are congruent to d, b and d, b are congruent to c, a.

(90) If a, b are congruent to p, q and b, c are congruent to q, s, then a, c are congruent to p, s.

(91) If b, a are congruent to p, q and c, a are congruent to p, r, then b, c are congruent to r, q.

(92) If a, o are congruent to o, p and b, o are congruent to o, q, then a, b are congruent to q, p.

(93) If b, a are congruent to p, q and c, a are congruent to p, r, then $b, c \parallel q, r$.

(94) If a, o are congruent to o, p and b, o are congruent to o, q, then $a, b \parallel p, q$.

Let us consider S_1, a, b, o. The functor $\mathrm{sum}_o(a, b)$ yielding an element of the points of S_1 is defined as follows:

(Def.5) o, a are congruent to b, $\mathrm{sum}_o(a, b)$.

Next we state the proposition

(95) $\mathrm{sum}_o(a, b) = c$ if and only if o, a are congruent to b, c.

Let us consider S_1, a, o. The functor $\mathrm{opposite}_o(a)$ yields an element of the points of S_1 and is defined as follows:

(Def.6) $\mathrm{sum}_o(a, \mathrm{opposite}_o(a)) = o$.

We now state the proposition

(96) $\mathrm{opposite}_o(a) = b$ if and only if $\mathrm{sum}_o(a, b) = o$.

Let us consider S_1, a, b, o. The functor $\mathrm{diff}_o(a, b)$ yielding an element of the points of S_1 is defined as follows:

(Def.7) $\mathrm{diff}_o(a, b) = \mathrm{sum}_o(a, \mathrm{opposite}_o(b))$.

Next we state a number of propositions:

(97) $\mathrm{diff}_o(a, b) = \mathrm{sum}_o(a, \mathrm{opposite}_o(b))$.

(98) o, a are congruent to b, $\mathrm{sum}_o(a, b)$.

(99) $\mathrm{sum}_o(a, o) = a$.

(100) There exists x such that $\mathrm{sum}_o(a, x) = o$.

(101) $\mathrm{sum}_o(\mathrm{sum}_o(a, b), c) = \mathrm{sum}_o(a, \mathrm{sum}_o(b, c))$.

(102) $\mathrm{sum}_o(a, b) = \mathrm{sum}_o(b, a)$.

(103) If $\mathrm{sum}_o(a, a) = o$, then $a = o$.

(104) If $\mathrm{sum}_o(a, x) = \mathrm{sum}_o(a, y)$, then $x = y$.

(105) $\mathrm{sum}_o(a, \mathrm{opposite}_o(a)) = o$.

(106) a, o are congruent to o, $\mathrm{opposite}_o(a)$.

(107) If $\mathrm{opposite}_o(a) = \mathrm{opposite}_o(b)$, then $a = b$.

(108) $a, b \parallel \mathrm{opposite}_o(a), \mathrm{opposite}_o(b)$.

(109) $\mathrm{opposite}_o(o) = o$.

(110) $p, q \parallel \mathrm{sum}_o(p, r), \mathrm{sum}_o(q, r)$.

(111) If $p, q \parallel r, s$, then $p, q \parallel \mathrm{sum}_o(p, r), \mathrm{sum}_o(q, s)$.

(113)[6] $\mathrm{diff}_o(a, b) = o$ if and only if $a = b$.

(114) $o, \mathrm{diff}_o(b, a) \parallel a, b$.

(115) o, $\mathrm{diff}_o(b, a)$ and $\mathrm{diff}_o(d, c)$ are collinear if and only if $a, b \parallel c, d$.

Let us consider S_1, a, b, c, d, o. We say that a, b, c, d form a trapezium with vertex o if and only if:

(Def.8) o, a and c are not collinear and o, a and b are collinear and o, c and d are collinear and $a, c \parallel b, d$.

Let us consider S_1, o, p. We say that there are trapeziums through p with vertex o if and only if:

(Def.9) for every b, c there exists d such that if o, p and b are collinear, then o, c and d are collinear and $p, c \parallel b, d$.

One can prove the following propositions:

(118)[7] If a, b, c, d form a trapezium with vertex o, then $o \neq a$ and $a \neq c$ and $c \neq o$.

(119) If a, b, c, x form a trapezium with vertex o and a, b, c, y form a trapezium with vertex o, then $x = y$.

(120) If o, a and b are not collinear, then a, o, b, o form a trapezium with vertex o.

(121) If a, b, c, d form a trapezium with vertex o, then c, d, a, b form a trapezium with vertex o.

(122) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then $o \neq d$.

(123) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then o, b and d are not collinear.

(124) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then b, a, d, c form a trapezium with vertex o.

(125) If $o = b$ or $o = d$ but a, b, c, d form a trapezium with vertex o, then $o = b$ and $o = d$.

[6]The proposition (112) was either repeated or obvious.
[7]The propositions (116)–(117) were either repeated or obvious.

(126) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o, then $b, c \parallel q, r$.

(127) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o and o, b and c are not collinear, then b, q, c, r form a trapezium with vertex o.

(128) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o and b, q, d, s form a trapezium with vertex o, then $c, d \parallel r, s$.

(129) For every o, a there exists p such that o, a and p are collinear and there are trapeziums through p with vertex o.

(130) There exist x, y, z such that $x \neq y$ and $y \neq z$ and $z \neq x$.

(131) If there are trapeziums through p with vertex o, then $o \neq p$.

(132) If there are trapeziums through p with vertex o, then there exists q such that o, p and q are not collinear and there are trapeziums through q with vertex o.

(133) If o, p and c are not collinear and o, p and b are collinear and there are trapeziums through p with vertex o, then there exists d such that p, b, c, d form a trapezium with vertex o.

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