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Opposite Categories and Contravariant Functors

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Summary. The opposite category of a category, contravariant functors and duality functors are defined.

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The articles [6], [1], [2], [5], [4], and [3] provide the notation and terminology for this paper. In the sequel B, C, D will be categories. Let X be a set, and let C, D be non-empty sets, and let f be a function from X into C, and let g be a function from C into D. Then $g \cdot f$ is a function from X into D.

Let X, Y, Z be non-empty sets, and let f be a partial function from $[\!: X, Y :\!]$ to Z. Then $\curvearrowleft f$ is a partial function from $[\!: Y, X :\!]$ to Z.

The following proposition is true

(1) $\langle$The objects of C, the morphisms of C, the cod-map of C, the dom-map of $C, \curvearrowleft$ (the composition of C), the id-map of $C\rangle$ is a category.

Let us consider C. The functor $C^{\rm op}$ yielding a category is defined as follows:

(Def.1) $C^{\rm op} = \langle$ the objects of C, the morphisms of C, the cod-map of C, the dom-map of $C, \curvearrowleft$ (the composition of C), the id-map of $C\rangle$.

One can prove the following proposition

(2) $(C^{\rm op})^{\rm op} = C$.

Let us consider C, and let c be an object of C. The functor $c^{\rm op}$ yields an object of $C^{\rm op}$ and is defined by:

(Def.2) $c^{\rm op} = c$.

Let us consider C, and let c be an object of $C^{\rm op}$. The functor ${}^{\rm op}c$ yielding an object of C is defined by:

(Def.3) ${}^{\rm op}c = c^{\rm op}$.

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One can prove the following three propositions:

(3) For every object c of C holds $(c^{\rm op})^{\rm op} = c$.

(4) For every object c of C holds ${}^{\rm op}(c^{\rm op}) = c$.

(5) For every object c of $C^{\rm op}$ holds $({}^{\rm op}c)^{\rm op} = c$.

Let us consider C, and let f be a morphism of C. The functor $f^{\rm op}$ yields a morphism of $C^{\rm op}$ and is defined as follows:

(Def.4) $f^{\rm op} = f$.

Let us consider C, and let f be a morphism of $C^{\rm op}$. The functor ${}^{\rm op}f$ yields a morphism of C and is defined by:

(Def.5) ${}^{\rm op}f = f^{\rm op}$.

One can prove the following propositions:

(6) For every morphism f of C holds $(f^{\rm op})^{\rm op} = f$.

(7) For every morphism f of C holds ${}^{\rm op}(f^{\rm op}) = f$.

(8) For every morphism f of $C^{\rm op}$ holds $({}^{\rm op}f)^{\rm op} = f$.

(9) For every morphism f of C holds $\mathrm{dom}(f^{\rm op}) = \mathrm{cod}\, f$ and $\mathrm{cod}(f^{\rm op}) = \mathrm{dom}\, f$.

(10) For every morphism f of $C^{\rm op}$ holds $\mathrm{dom}\, {}^{\rm op}f = \mathrm{cod}\, f$ and $\mathrm{cod}\, {}^{\rm op}f = \mathrm{dom}\, f$.

(11) For every morphism f of C holds $(\mathrm{dom}\, f)^{\rm op} = \mathrm{cod}(f^{\rm op})$ and $(\mathrm{cod}\, f)^{\rm op} = \mathrm{dom}(f^{\rm op})$.

(12) For every morphism f of $C^{\rm op}$ holds ${}^{\rm op}\mathrm{dom}\, f = \mathrm{cod}\, {}^{\rm op}f$ and ${}^{\rm op}\mathrm{cod}\, f = \mathrm{dom}\, {}^{\rm op}f$.

(13) For all objects a, b of C and for every morphism f of C holds $f \in \mathrm{hom}(a, b)$ if and only if $f^{\rm op} \in \mathrm{hom}(b^{\rm op}, a^{\rm op})$.

(14) For all objects a, b of $C^{\rm op}$ and for every morphism f of $C^{\rm op}$ holds $f \in \mathrm{hom}(a, b)$ if and only if ${}^{\rm op}f \in \mathrm{hom}({}^{\rm op}b, {}^{\rm op}a)$.

(15) For all objects a, b of C and for every morphism f from a to b such that $\mathrm{hom}(a, b) \neq \emptyset$ holds $f^{\rm op}$ is a morphism from $b^{\rm op}$ to $a^{\rm op}$.

(16) For all objects a, b of $C^{\rm op}$ and for every morphism f from a to b such that $\mathrm{hom}(a, b) \neq \emptyset$ holds ${}^{\rm op}f$ is a morphism from ${}^{\rm op}b$ to ${}^{\rm op}a$.

(17) For all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(18) For all morphisms f, g of C such that $\mathrm{cod}(g^{\rm op}) = \mathrm{dom}(f^{\rm op})$ holds $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(19) For all morphisms f, g of $C^{\rm op}$ such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds ${}^{\rm op}(g \cdot f) = {}^{\rm op}f \cdot {}^{\rm op}g$.

(20) For all objects a, b, c of C and for every morphism f from a to b and for every morphism g from b to c such that $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$ holds $(g \cdot f)^{\rm op} = f^{\rm op} \cdot g^{\rm op}$.

(21) For every object a of C holds $\mathrm{id}_a^{\rm op} = \mathrm{id}_{a^{\rm op}}$.

(22) For every object a of $C^{\rm op}$ holds ${}^{\rm op}(\mathrm{id}_a) = \mathrm{id}_{({}^{\rm op}a)}$.

(23) For every morphism f of C holds $f^{\rm op}$ is monic if and only if f is epi.

(24) For every morphism f of C holds $f^{\rm op}$ is epi if and only if f is monic.

(25) For every morphism f of C holds $f^{\rm op}$ is invertible if and only if f is invertible.

(26) For every object c of C holds c is an initial object if and only if $c^{\rm op}$ is a terminal object.

(27) For every object c of C holds $c^{\rm op}$ is an initial object if and only if c is a terminal object.

Let us consider C, B, and let S be a function from the morphisms of $C^{\rm op}$ into the morphisms of B. The functor ${}_*S$ yields a function from the morphisms of C into the morphisms of B and is defined by:

(Def.6) for every morphism f of C holds $({}_*S)(f) = S(f^{\rm op})$.

One can prove the following propositions:

(28) For every function S from the morphisms of $C^{\rm op}$ into the morphisms of B and for every morphism f of $C^{\rm op}$ holds $({}_*S)({}^{\rm op}f) = S(f)$.

(29) For every functor S from $C^{\rm op}$ to B and for every object c of C holds $({\rm Obj}\,{}_*S)(c) = ({\rm Obj}\,S)(c^{\rm op})$.

(30) For every functor S from $C^{\rm op}$ to B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}_*S)({}^{\rm op}c) = ({\rm Obj}\,S)(c)$.

Let us consider C, D. A function from the morphisms of C into the morphisms of D is called a contravariant functor from C into D if it satisfies the conditions (Def.7).

(Def.7) (i) For every object c of C there exists an object d of D such that ${\rm it}({\rm id}_c) = {\rm id}_d$,

(ii) for every morphism f of C holds ${\rm it}({\rm id}_{{\rm dom}\,f}) = {\rm id}_{{\rm cod}({\rm it}(f))}$ and ${\rm it}({\rm id}_{{\rm cod}\,f}) = {\rm id}_{{\rm dom}({\rm it}(f))}$,

(iii) for all morphisms f, g of C such that ${\rm dom}\,g = {\rm cod}\,f$ holds ${\rm it}(g \cdot f) = {\rm it}(f) \cdot {\rm it}(g)$.

The following propositions are true:

(31) For every contravariant functor S from C into D and for every object c of C and for every object d of D such that $S({\rm id}_c) = {\rm id}_d$ holds $({\rm Obj}\,S)(c) = d$.

(32) For every contravariant functor S from C into D and for every object c of C holds $S({\rm id}_c) = {\rm id}_{({\rm Obj}\,S)(c)}$.

(33) For every contravariant functor S from C into D and for every morphism f of C holds $({\rm Obj}\,S)({\rm dom}\,f) = {\rm cod}(S(f))$ and $({\rm Obj}\,S)({\rm cod}\,f) = {\rm dom}(S(f))$.

(34) For every contravariant functor S from C into D and for all morphisms f, g of C such that ${\rm dom}\,g = {\rm cod}\,f$ holds ${\rm dom}(S(f)) = {\rm cod}(S(g))$.

(35) For every functor S from $C^{\rm op}$ to B holds ${}_*S$ is a contravariant functor from C into B.

(36) For every contravariant functor S_1 from C into B and for every contravariant functor S_2 from B into D holds $S_2 \cdot S_1$ is a functor from C to D.

(37) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of C holds $({\rm Obj}\,{}_*S)(c) = ({\rm Obj}\,S)(c^{\rm op})$.

(38) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}_*S)({}^{\rm op}c) = ({\rm Obj}\,S)(c)$.

(39) For every contravariant functor S from $C^{\rm op}$ into B holds ${}_*S$ is a functor from C to B.

We now define two new functors. Let us consider C, B, and let S be a function from the morphisms of C into the morphisms of B. The functor ${}^*S$ yielding a function from the morphisms of $C^{\rm op}$ into the morphisms of B is defined as follows:

(Def.8) for every morphism f of $C^{\rm op}$ holds $({}^*S)(f) = S({}^{\rm op}f)$.

The functor S^* yields a function from the morphisms of C into the morphisms of $B^{\rm op}$ and is defined by:

(Def.9) for every morphism f of C holds $S^*(f) = S(f)^{\rm op}$.

The following propositions are true:

(40) For every function S from the morphisms of C into the morphisms of B and for every morphism f of C holds $({}^*S)(f^{\rm op}) = S(f)$.

(41) For every functor S from C to B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}^*S)(c) = ({\rm Obj}\,S)({}^{\rm op}c)$.

(42) For every functor S from C to B and for every object c of C holds $({\rm Obj}\,{}^*S)(c^{\rm op}) = ({\rm Obj}\,S)(c)$.

(43) For every functor S from C to B and for every object c of C holds $({\rm Obj}(S^*))(c) = ({\rm Obj}\,S)(c)^{\rm op}$.

(44) For every contravariant functor S from C into B and for every object c of $C^{\rm op}$ holds $({\rm Obj}\,{}^*S)(c) = ({\rm Obj}\,S)({}^{\rm op}c)$.

(45) For every contravariant functor S from C into B and for every object c of C holds $({\rm Obj}\,{}^*S)(c^{\rm op}) = ({\rm Obj}\,S)(c)$.

(46) For every contravariant functor S from C into B and for every object c of C holds $({\rm Obj}(S^*))(c) = ({\rm Obj}\,S)(c)^{\rm op}$.

(47) For every function F from the morphisms of C into the morphisms of D and for every morphism f of C holds $({}^*F)^*(f^{\rm op}) = F(f)^{\rm op}$.

(48) For every function S from the morphisms of C into the morphisms of D holds ${}_*{}^*S = S$.

(49) For every function S from the morphisms of $C^{\rm op}$ into the morphisms of D holds ${}^*{}_*S = S$.

(50) For every function S from the morphisms of C into the morphisms of D holds $({}^*S)^* = {}^*(S^*)$.

(51) For every function S from the morphisms of C into the morphisms of D holds $(S^*)^* = S$.

(52) For every function S from the morphisms of C into the morphisms of D holds ${}^*({}^*S) = S$.

(53) For every function S from the morphisms of C into the morphisms of B and for every function T from the morphisms of B into the morphisms of D holds ${}^*(T \cdot S) = T \cdot {}^*S$.

(54) For every function S from the morphisms of C into the morphisms of B and for every function T from the morphisms of B into the morphisms of D holds $(T \cdot S)^* = T^* \cdot S$.

(55) For every function F_1 from the morphisms of C into the morphisms of B and for every function F_2 from the morphisms of B into the morphisms of D holds $({}^*(F_2 \cdot F_1))^* = ({}^*F_2)^* \cdot ({}^*F_1)^*$.

(56) For every contravariant functor S from C into D holds ${}^*S$ is a functor from $C^{\rm op}$ to D.

(57) For every contravariant functor S from C into D holds S^* is a functor from C to $D^{\rm op}$.

(58) For every functor S from C to D holds ${}^*S$ is a contravariant functor from $C^{\rm op}$ into D.

(59) For every functor S from C to D holds S^* is a contravariant functor from C into $D^{\rm op}$.

(60) For every contravariant functor S_1 from C into B and for every functor S_2 from B to D holds $S_2 \cdot S_1$ is a contravariant functor from C into D.

(61) For every functor S_1 from C to B and for every contravariant functor S_2 from B into D holds $S_2 \cdot S_1$ is a contravariant functor from C into D.

(62) For every functor F from C to D and for every object c of C holds $(\mathrm{Obj}(({}^*F)^*))(c^{\rm op}) = (\mathrm{Obj}\, F)(c)^{\rm op}$.

(63) For every contravariant functor F from C into D and for every object c of C holds $(\mathrm{Obj}(({}^*F)^*))(c^{\rm op}) = (\mathrm{Obj}\, F)(c)^{\rm op}$.

(64) For every functor F from C to D holds $({}^*F)^*$ is a functor from $C^{\rm op}$ to $D^{\rm op}$.

(65) For every contravariant functor F from C into D holds $({}^*F)^*$ is a contravariant functor from $C^{\rm op}$ into $D^{\rm op}$.

We now define two new functors. Let us consider C. The functor $\mathrm{id}^{\rm op}(C)$ yielding a contravariant functor from C into $C^{\rm op}$ is defined as follows:

(Def.10) $\mathrm{id}^{\rm op}(C) = \mathrm{id}_C^*$.

The functor ${}^{\rm op}\mathrm{id}(C)$ yielding a contravariant functor from $C^{\rm op}$ into C is defined as follows:

(Def.11) ${}^{\rm op}\mathrm{id}(C) = {}^*(\mathrm{id}_C)$.

One can prove the following propositions:

(66) For every morphism f of C holds $\mathrm{id}^{\rm op}(C)(f) = f^{\rm op}$.

(67) For every object c of C holds $(\mathrm{Obj}\,\mathrm{id}^{\mathrm{op}}(C))(c) = c^{\mathrm{op}}$.

(68) For every morphism f of C^{op} holds $({}^{\mathrm{op}}\mathrm{id}(C))(f) = {}^{\mathrm{op}}f$.

(69) For every object c of C^{op} holds $(\mathrm{Obj}\,{}^{\mathrm{op}}\mathrm{id}(C))(c) = {}^{\mathrm{op}}c$.

(70) For every function S from the morphisms of C into the morphisms of D holds ${}^{*}S = S \cdot {}^{\mathrm{op}}\mathrm{id}(C)$ and $S^{*} = \mathrm{id}^{\mathrm{op}}(D) \cdot S$.

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