FORMALIZED MATHEMATICS
Vol.2, No.**2**, March–April 1991
Université Catholique de Louvain

Algebra of Normal Forms

Andrzej Trybulec
Warsaw University
Białystok

Summary. We mean by a normal form a finite set of ordered pairs of subsets of a fixed set that fulfils two conditions: elements of it consist of disjoint sets and elements of it are incomparable w.r.t. inclusion. The underlying set corresponds to a set of propositional variables but is arbitrary. The correspodents to a normal form of a formula, e.g. a disjunctive normal form, is as follows. The normal form is the set of disjuncts and a disjunct is an ordered pair consisting of the sets of propostional variables that occur in the non-negated and negated disjunct. The requirement that the element of a normal form consists of disjoint sets means that contradictory disjuncts have been removed, and the second condition means that the absorption law has been used to shorten the normal form. We construct a lattice $\langle \mathbb{N}, \sqcup, \sqcap \rangle$, where $a \sqcup b = \mu(a \cup b)$ and $a \sqcap b = \mu c$, c being the set of all pairs $\langle X_1 \cup Y_1, X_2 \cup Y_2 \rangle$, $\langle X_1, X_2 \rangle \in a$ and $\langle Y_1, Y_2 \rangle \in b$, which consist of disjoint sets. μa denotes here the set of all minimal, w.r.t. inclusion, elements of a. We prove that the lattice of normal forms over a set defined in this way is distributive and that $\emptyset$ is the minimal element of it.

MML Identifier: `NORMFORM`.

The terminology and notation used here have been introduced in the following articles: [8], [9], [3], [4], [1], [5], [2], [6], [10], [7], and [11]. In the sequel A, B, C, D will be sets. We now state two propositions:

(1) If $A \subseteq B$ and $C \subseteq D$ and B misses D, then A misses C.

(2) If $A \setminus B \subseteq C$, then $A \subseteq B \cup C$.

In the sequel A, B will denote Boolean domains and x, y will denote elements of $[\!: A, B :\!]$. We now define five new constructions. Let us consider A, B, x, y. The predicate $x \subseteq y$ is defined by:

(Def.1) $x_{\mathbf{1}} \subseteq y_{\mathbf{1}}$ and $x_{\mathbf{2}} \subseteq y_{\mathbf{2}}$.

The functor $x \cup y$ yielding an element of $[\!: A, B :\!]$ is defined as follows:

(Def.2) $x \cup y = \langle x_{\mathbf{1}} \cup y_{\mathbf{1}}, x_{\mathbf{2}} \cup y_{\mathbf{2}} \rangle$.

The functor $x \cap y$ yielding an element of $[\!: A, B :\!]$ is defined as follows:

ISSN 0777–4028

(Def.3) $x \cap y = \langle x_{\mathbf{1}} \cap y_{\mathbf{1}}, x_{\mathbf{2}} \cap y_{\mathbf{2}} \rangle$.

The functor $x \setminus y$ yields an element of $[\!: A, B :\!]$ and is defined as follows:

(Def.4) $x \setminus y = \langle x_{\mathbf{1}} \setminus y_{\mathbf{1}}, x_{\mathbf{2}} \setminus y_{\mathbf{2}} \rangle$.

The functor $x \dot{-} y$ yields an element of $[\!: A, B :\!]$ and is defined as follows:

(Def.5) $x \dot{-} y = \langle x_{\mathbf{1}} \dot{-} y_{\mathbf{1}}, x_{\mathbf{2}} \dot{-} y_{\mathbf{2}} \rangle$.

In the sequel X will be a set and a, b, c will be elements of $[\!: A, B :\!]$. We now state a number of propositions:

(3) $a \subseteq a$.

(4) If $a \subseteq b$ and $b \subseteq a$, then $a = b$.

(5) If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$.

(6) $a \cup b = \langle a_{\mathbf{1}} \cup b_{\mathbf{1}}, a_{\mathbf{2}} \cup b_{\mathbf{2}} \rangle$.

(7) $a \cap b = \langle a_{\mathbf{1}} \cap b_{\mathbf{1}}, a_{\mathbf{2}} \cap b_{\mathbf{2}} \rangle$.

(8) $a \setminus b = \langle a_{\mathbf{1}} \setminus b_{\mathbf{1}}, a_{\mathbf{2}} \setminus b_{\mathbf{2}} \rangle$.

(9) $a \dot{-} b = \langle a_{\mathbf{1}} \dot{-} b_{\mathbf{1}}, a_{\mathbf{2}} \dot{-} b_{\mathbf{2}} \rangle$.

(10) $(a \cup b)_{\mathbf{1}} = a_{\mathbf{1}} \cup b_{\mathbf{1}}$ and $(a \cup b)_{\mathbf{2}} = a_{\mathbf{2}} \cup b_{\mathbf{2}}$.

(11) $(a \cap b)_{\mathbf{1}} = a_{\mathbf{1}} \cap b_{\mathbf{1}}$ and $(a \cap b)_{\mathbf{2}} = a_{\mathbf{2}} \cap b_{\mathbf{2}}$.

(12) $(a \setminus b)_{\mathbf{1}} = a_{\mathbf{1}} \setminus b_{\mathbf{1}}$ and $(a \setminus b)_{\mathbf{2}} = a_{\mathbf{2}} \setminus b_{\mathbf{2}}$.

(13) $(a \dot{-} b)_{\mathbf{1}} = a_{\mathbf{1}} \dot{-} b_{\mathbf{1}}$ and $(a \dot{-} b)_{\mathbf{2}} = a_{\mathbf{2}} \dot{-} b_{\mathbf{2}}$.

(14) $a \cup a = a$.

(15) $a \cup b = b \cup a$.

(16) $a \cup b \cup c = a \cup (b \cup c)$.

(17) $a \cap a = a$.

(18) $a \cap b = b \cap a$.

(19) $a \cap b \cap c = a \cap (b \cap c)$.

(20) $a \cap (b \cup c) = a \cap b \cup a \cap c$.

(21) $a \cup b \cap a = a$.

(22) $a \cap (b \cup a) = a$.

(24)[1] $a \cup b \cap c = (a \cup b) \cap (a \cup c)$.

(25) If $a \subseteq c$ and $b \subseteq c$, then $a \cup b \subseteq c$.

(26) $a \subseteq a \cup b$ and $b \subseteq a \cup b$.

(27) If $a = a \cup b$, then $b \subseteq a$.

(28) If $a \subseteq b$, then $c \cup a \subseteq c \cup b$ and $a \cup c \subseteq b \cup c$.

(29) $(a \setminus b) \cup b = a \cup b$.

(30) If $a \setminus b \subseteq c$, then $a \subseteq b \cup c$.

(31) If $a \subseteq b \cup c$, then $a \setminus c \subseteq b$.

In the sequel a will be an element of $[\!: \mathrm{Fin}\, X, \mathrm{Fin}\, X :\!]$. Let A be a set. The functor $\mathrm{FinUnion}_A$ yields a binary operation on $[\!: \mathrm{Fin}\, A, \mathrm{Fin}\, A :\!]$ and is defined by:

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[1] The proposition (23) was either repeated or obvious.

(Def.6) for all elements x, y of $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ holds $\mathrm{FinUnion}_A(x, y) = x \cup y$.

In the sequel A will denote a set. Let X be a non-empty set, and let A be a set, and let B be an element of $\mathrm{Fin}\,X$, and let f be a function from X into $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$. The functor $\mathrm{FinUnion}(B, f)$ yields an element of $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ and is defined as follows:

(Def.7) $\mathrm{FinUnion}(B, f) = \mathrm{FinUnion}_A \text{-} \sum_B f$.

The following propositions are true:

(32) $\mathrm{FinUnion}_A$ is idempotent.

(33) $\mathrm{FinUnion}_A$ is commutative.

(34) $\mathrm{FinUnion}_A$ is associative.

(35) For every non-empty set X and for every function f from X into $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ and for every element B of $\mathrm{Fin}\,X$ and for every element x of X such that $x \in B$ holds $f(x) \subseteq \mathrm{FinUnion}(B, f)$.

(36) $\langle 0_A, 0_A \rangle$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(37) $\mathrm{FinUnion}_A$ has a unity.

(38) $\mathbf{1}_{\mathrm{FinUnion}_A} = \langle 0_A, 0_A \rangle$.

(39) For every element x of $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ holds $\mathbf{1}_{\mathrm{FinUnion}_A} \subseteq x$.

(40) For every non-empty set X and for every function f from X into $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ and for every element B of $\mathrm{Fin}\,X$ and for every element c of $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ such that for every element x of X such that $x \in B$ holds $f(x) \subseteq c$ holds $\mathrm{FinUnion}(B, f) \subseteq c$.

(41) For every non-empty set X and for every element B of $\mathrm{Fin}\,X$ and for all functions f, g from X into $[:\mathrm{Fin}\,A, \mathrm{Fin}\,A:]$ such that $f \restriction B = g \restriction B$ holds $\mathrm{FinUnion}(B, f) = \mathrm{FinUnion}(B, g)$.

Let us consider X. The functor $\mathrm{DP}(X)$ yields a non-empty subset of $[:\mathrm{Fin}\,X, \mathrm{Fin}\,X:]$ and is defined as follows:

(Def.8) $\mathrm{DP}(X) = \{a : a_{\mathbf{1}} \text{ misses } a_{\mathbf{2}}\}$.

The following proposition is true

(42) For every element y of $[:\mathrm{Fin}\,X, \mathrm{Fin}\,X:]$ holds $y \in \mathrm{DP}(X)$ if and only if $y_{\mathbf{1}} \cap y_{\mathbf{2}} = \emptyset$.

In the sequel x, y will denote elements of $[:\mathrm{Fin}\,X, \mathrm{Fin}\,X:]$ and a, b will denote elements of $\mathrm{DP}(X)$. We now state several propositions:

(43) If $y \in \mathrm{DP}(X)$ and $x \in \mathrm{DP}(X)$, then $y \cup x \in \mathrm{DP}(X)$ if and only if $y_{\mathbf{1}} \cap x_{\mathbf{2}} \cup x_{\mathbf{1}} \cap y_{\mathbf{2}} = \emptyset$.

(44) $a_{\mathbf{1}} \cap a_{\mathbf{2}} = \emptyset$.

(45) If $x \subseteq b$, then x is an element of $\mathrm{DP}(X)$.

(46) For no arbitrary x holds $x \in a_{\mathbf{1}}$ and $x \in a_{\mathbf{2}}$.

(47) If $a \cup b \notin \mathrm{DP}(X)$, then there exists an element p of X such that $p \in a_{\mathbf{1}}$ and $p \in b_{\mathbf{2}}$ or $p \in b_{\mathbf{1}}$ and $p \in a_{\mathbf{2}}$.

(48) $a_{\mathbf{1}}$ misses $a_{\mathbf{2}}$.

(49) If $x_{\mathbf{1}}$ misses $x_{\mathbf{2}}$, then x is an element of $\mathrm{DP}(X)$.

(50) For all sets V, W such that $V \subseteq a_{\mathbf{1}}$ and $W \subseteq a_{\mathbf{2}}$ holds $\langle V, W \rangle$ is an element of $\mathrm{DP}(X)$.

In this article we present several logical schemes. The scheme *LambdaX* concerns a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a non-empty subset $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$ and states that:

there exists a function f from $\mathcal{B}$ into $\mathcal{C}$ such that for every element x of $\mathcal{B}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The scheme *BinOpLambdaX* deals with a non-empty set $\mathcal{A}$, a non-empty subset $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a binary operation o on $\mathcal{B}$ such that for all elements a, b of $\mathcal{B}$ holds $o(a, b) = \mathcal{F}(a, b)$

for all values of the parameters.

For simplicity we follow a convention: A will be a set, x will be an element of $[\!:\mathrm{Fin}\,A, \mathrm{Fin}\,A:\!]$, a, b, c, s, t will be elements of $\mathrm{DP}(A)$, and B, C, D will be elements of $\mathrm{Fin}\,\mathrm{DP}(A)$. Let us consider A. The normal forms over A yields a non-empty subset of $\mathrm{Fin}\,\mathrm{DP}(A)$ and is defined as follows:

(Def.9) the normal forms over $A = \{B : a \in B \wedge b \in B \wedge a \subseteq b \Rightarrow a = b\}$.

In the sequel K, L, M are elements of the normal forms over A. Next we state three propositions:

(51) $\emptyset \in$ the normal forms over A.

(52) If $B \in$ the normal forms over A and $a \in B$ and $b \in B$ and $a \subseteq b$, then $a = b$.

(53) If for all a, b such that $a \in B$ and $b \in B$ and $a \subseteq b$ holds $a = b$, then $B \in$ the normal forms over A.

We now define two new functors. Let us consider A, B. The functor μB yielding an element of the normal forms over A is defined by:

(Def.10) $\mu B = \{t : s \in B \wedge s \subseteq t \Leftrightarrow s = t\}$.

Let us consider C. The functor $B \frown C$ yielding an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ is defined as follows:

(Def.11) $B \frown C = \mathrm{DP}(A) \cap \{s \cup t : s \in B \wedge t \in C\}$.

The following propositions are true:

(54) $B \frown C = \mathrm{DP}(A) \cap \{s \cup t : s \in B \wedge t \in C\}$.

(55) If $x \in B \frown C$, then there exist b, c such that $b \in B$ and $c \in C$ and $x = b \cup c$.

(56) If $b \in B$ and $c \in C$ and $b \cup c \in \mathrm{DP}(A)$, then $b \cup c \in B \frown C$.

(57) If $b \in B$ and $c \in C$ and $a = b \cup c$, then $a \in B \frown C$.

(58) If $a \in \mu B$, then $a \in B$ but if $b \in B$ and $b \subseteq a$, then $b = a$.

(59) If $a \in \mu B$, then $a \in B$.

(60) If $a \in \mu B$ and $b \in B$ and $b \subseteq a$, then $b = a$.

(61) If $a \in B$ and for every b such that $b \in B$ and $b \subseteq a$ holds $b = a$, then $a \in \mu B$.

We now define two new functors. Let us consider A. The functor $\sqcup_A$ yields a binary operation on the normal forms over A and is defined by:

(Def.12) $\sqcup_A(K, L) = \mu(K \cup L)$.

The functor $\sqcap_A$ yielding a binary operation on the normal forms over A is defined by:

(Def.13) $\sqcap_A(K, L) = \mu(K \frown L)$.

One can prove the following propositions:

(62) $\sqcup_A(K, L) = \mu(K \cup L)$.

(63) $\sqcap_A(K, L) = \mu(K \frown L)$.

Let A be a non-empty set, and let B be a non-empty subset of A, and let O be a binary operation on B, and let a, b be elements of B. Then $O(a, b)$ is an element of B.

One can prove the following propositions:

(64) $\mu B \subseteq B$.

(65) If $b \in B$, then there exists c such that $c \subseteq b$ and $c \in \mu B$.

(66) $\mu K = K$.

(67) $\mu(B \cup C) \subseteq \mu B \cup C$.

(68) $\mu(\mu B \cup C) = \mu(B \cup C)$.

(69) $\mu(B \cup \mu C) = \mu(B \cup C)$.

(70) If $B \subseteq C$, then $B \frown D \subseteq C \frown D$.

(71) $\mu(B \frown C) \subseteq \mu B \frown C$.

(72) $B \frown C = C \frown B$.

(73) If $B \subseteq C$, then $D \frown B \subseteq D \frown C$.

(74) $\mu(\mu B \frown C) = \mu(B \frown C)$.

(75) $\mu(B \frown \mu C) = \mu(B \frown C)$.

(76) $K \frown (L \frown M) = K \frown L \frown M$.

(77) $K \frown (L \cup M) = K \frown L \cup K \frown M$.

(78) $B \subseteq B \frown B$.

(79) $\mu(K \frown K) = \mu K$.

Let us consider A. The lattice of normal forms over A yields a lower bound lattice and is defined as follows:

(Def.14) the lattice of normal forms over $A = \langle$the normal forms over $A, \sqcup_A, \sqcap_A\rangle$.

The following propositions are true:

(80) The lattice of normal forms over $A = \langle$the normal forms over $A, \sqcup_A, \sqcap_A\rangle$.

(81) The lattice of normal forms over A is a distributive lattice.

(82) The carrier of the lattice of normal forms over $A =$ the normal forms over A.

(83) The join operation of the lattice of normal forms over $A = \sqcup_A$.

(84) The meet operation of the lattice of normal forms over $A = \sqcap_A$.

(85) $\emptyset$ is an element of the carrier of the lattice of normal forms over A.

(86) $\perp_{\text{The lattice of normal forms over } A} = \emptyset$.

(87) The join operation of the lattice of normal forms over A has a unity.

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Received October 5, 1990