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Incidence Projective Spaces

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Summary. A basis for investigations on incidence projective spaces. With every projective space defined in terms of collinearity relation we associate the incidence structure consisting of points and lines of the given space. We introduce the general notion of projective space defined in terms of incidence and define several properties of such structures (like satisfability of the Desargues Axiom and conditions on the dimension).

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The papers [7], [8], [6], [1], [2], [3], [4], and [5] provide the notation and terminology for this paper. We consider projective incidence structures which are systems

⟨points, lines, an incidence⟩,

where the points constitute a non-empty set, the lines constitute a non-empty set, and the incidence is a relation between the points and the lines.

We see that the projective space defined in terms of collinearity is a proper collinearity space.

For simplicity we follow a convention: C_1 will be a proper collinearity space, x, y will be arbitrary, Y will be a set, and B will be an element of $2^{\text{the points of } C_1}$. Let us consider C_1. We see that the line of C_1 is an element of $2^{\text{the points of } C_1}$.

Let us consider C_1. The functor $L(C_1)$ yielding a non-empty set is defined by:

(Def.1) $L(C_1) = \{B : B \text{ is a line of } C_1\}$.

We now state two propositions:

(1) $L(C_1) = \{B : B \text{ is a line of } C_1\}$.

(2) For every x holds x is a line of C_1 if and only if x is an element of $L(C_1)$.

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Let us consider C_1. The functor $\mathbf{I}_{C_1}$ yields a relation between the points of C_1 and $L(C_1)$ and is defined by:

(Def.2) for all x, y holds $\langle x, y\rangle \in \mathbf{I}_{C_1}$ if and only if $x \in$ the points of C_1 and $y \in L(C_1)$ and there exists Y such that $y = Y$ and $x \in Y$.

Let us consider C_1. The functor Inc-ProjSp(C_1) yields a projective incidence structure and is defined by:

(Def.3) Inc-ProjSp(C_1) $= \langle$ the points of $C_1, L(C_1), \mathbf{I}_{C_1}\rangle$.

Next we state four propositions:

(3) Inc-ProjSp(C_1) $= \langle$ the points of $C_1, L(C_1), \mathbf{I}_{C_1}\rangle$.

(4) For every C_1 holds the points of Inc-ProjSp(C_1) $=$ the points of C_1 and the lines of Inc-ProjSp(C_1) $= L(C_1)$ and the incidence of Inc-ProjSp(C_1) $= \mathbf{I}_{C_1}$.

(5) For every x holds x is a line of C_1 if and only if x is an element of the lines of Inc-ProjSp(C_1).

(6) For every x holds x is an element of the points of Inc-ProjSp(C_1) if and only if x is an element of the points of C_1.

For simplicity we adopt the following rules: a, b, c, p, q, s will be elements of the points of Inc-ProjSp(C_1), P, Q, S will be elements of the lines of Inc-ProjSp(C_1), P' will be a line of C_1, and a', b', c', p' will be elements of the points of C_1. Let I_1 be a projective incidence structure, and let s be an element of the points of I_1, and let S be an element of the lines of I_1. The predicate $s \mid S$ is defined as follows:

(Def.4) $\langle s, S\rangle \in$ the incidence of I_1.

One can prove the following propositions:

(7) $s \mid S$ if and only if $\langle s, S\rangle \in \mathbf{I}_{C_1}$.

(8) If $p = p'$ and $P = P'$, then $p \mid P$ if and only if $p' \in P'$.

(9) There exist a', b', c' such that $a' \neq b'$ and $b' \neq c'$ and $c' \neq a'$.

(10) For every a' there exists b' such that $a' \neq b'$.

(11) If $p \mid P$ and $q \mid P$ and $p \mid Q$ and $q \mid Q$, then $p = q$ or $P = Q$.

(12) For every p, q there exists P such that $p \mid P$ and $q \mid P$.

(13) If $a = a'$ and $b = b'$ and $c = c'$, then a', b' and c' are collinear if and only if there exists P such that $a \mid P$ and $b \mid P$ and $c \mid P$.

(14) There exist p, P such that $p \nmid P$.

For simplicity we follow the rules: C_1 is a projective space defined in terms of collinearity, a, b, c, d, p, q are elements of the points of Inc-ProjSp(C_1), P, Q, S, M, N are elements of the lines of Inc-ProjSp(C_1), and a', b', c', d', p' are elements of the points of C_1. One can prove the following propositions:

(15) For every P there exist a, b, c such that $a \neq b$ and $b \neq c$ and $c \neq a$ and $a \mid P$ and $b \mid P$ and $c \mid P$.

(16) Suppose that

(i) $a \mid M$,

(ii) $b \mid M$,
(iii) $c \mid N$,
(iv) $d \mid N$,
(v) $p \mid M$,
(vi) $p \mid N$,
(vii) $a \mid P$,
(viii) $c \mid P$,
(ix) $b \mid Q$,
(x) $d \mid Q$,
(xi) $p \nmid P$,
(xii) $p \nmid Q$,
(xiii) $M \neq N$.

Then there exists q such that $q \mid P$ and $q \mid Q$.

(17) If for every a', b', c', d' there exists p' such that a', b' and p' are collinear and c', d' and p' are collinear, then for every M, N there exists q such that $q \mid M$ and $q \mid N$.

(18) If there exist elements p, p_1, r, r_1 of the points of C_1 such that for no element s of the points of C_1 holds p, p_1 and s are collinear and r, r_1 and s are collinear, then there exist M, N such that for no q holds $q \mid M$ and $q \mid N$.

(19) Suppose for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear. Then for every a, M, N there exist b, c, S such that $a \mid S$ and $b \mid S$ and $c \mid S$ and $b \mid M$ and $c \mid N$.

We now define two new predicates. Let x, y, z be arbitrary. We say that x, y, z are mutually different if and only if:

(Def.5) $x \neq y$ and $y \neq z$ and $z \neq x$.

Let u be arbitrary. We say that x, y, z, u are mutually different if and only if:

(Def.6) $x \neq y$ and $y \neq z$ and $z \neq x$ and $u \neq x$ and $u \neq y$ and $u \neq z$.

We now define two new predicates. Let C_2 be a projective incidence structure, and let a, b be elements of the points of C_2, and let M be an element of the lines of C_2. The predicate $a, b \mid M$ is defined as follows:

(Def.7) $a \mid M$ and $b \mid M$.

Let c be an element of the points of C_2. The predicate $a, b, c \mid M$ is defined by:

(Def.8) $a \mid M$ and $b \mid M$ and $c \mid M$.

We now state three propositions:

(20) Suppose that

(i) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1

are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

Let p, q, r, s, a, b, c be elements of the points of Inc-ProjSp(C_1). Let L, Q, R, S, A, B, C be elements of the lines of Inc-ProjSp(C_1). Suppose that

(ii) $q \nmid L$,
(iii) $r \nmid L$,
(iv) $p \nmid Q$,
(v) $s \nmid Q$,
(vi) $p \nmid R$,
(vii) $r \nmid R$,
(viii) $q \nmid S$,
(ix) $s \nmid S$,
(x) $a, p, s \mid L$,
(xi) $a, q, r \mid Q$,
(xii) $b, q, s \mid R$,
(xiii) $b, p, r \mid S$,
(xiv) $c, p, q \mid A$,
(xv) $c, r, s \mid B$,
(xvi) $a, b \mid C$.

Then $c \nmid C$.

(21) Suppose that

(i) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

Let o, b_1, a_1, b_2, a_2, b_3, a_3, r, s, t be elements of the points of Inc-ProjSp(C_1). Let C_3, C_4, C_5, A_1, A_2, A_3, B_1, B_2, B_3 be elements of the lines of Inc-ProjSp(C_1). Suppose that

(ii) $o, b_1, a_1 \mid C_3$,
(iii) $o, a_2, b_2 \mid C_4$,
(iv) $o, a_3, b_3 \mid C_5$,
(v) $a_3, a_2, t \mid A_1$,
(vi) $a_3, r, a_1 \mid A_2$,
(vii) $a_2, s, a_1 \mid A_3$,
(viii) $t, b_2, b_3 \mid B_1$,
(ix) $b_1, r, b_3 \mid B_2$,
(x) $b_1, s, b_2 \mid B_3$,
(xi) C_3, C_4, C_5 are mutually different,
(xii) $o \neq a_1$,
(xiii) $o \neq a_2$,

(xiv) $o \neq a_3$,
(xv) $o \neq b_1$,
(xvi) $o \neq b_2$,
(xvii) $o \neq b_3$,
(xviii) $a_1 \neq b_1$,
(xix) $a_2 \neq b_2$,
(xx) $a_3 \neq b_3$.

Then there exists an element O of the lines of Inc-ProjSp(C_1) such that $r, s, t \mid O$.

(22) Suppose that

(i) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

Let o, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be elements of the points of Inc-ProjSp(C_1). Let A_1, A_2, A_3, B_1, B_2, B_3, C_3, C_4, C_5 be elements of the lines of Inc-ProjSp(C_1). Suppose that

(ii) o, a_1, a_2, a_3 are mutually different,
(iii) o, b_1, b_2, b_3 are mutually different,
(iv) $A_3 \neq B_3$,
(v) $o \mid A_3$,
(vi) $o \mid B_3$,
(vii) $a_2, b_3, c_1 \mid A_1$,
(viii) $a_3, b_1, c_2 \mid B_1$,
(ix) $a_1, b_2, c_3 \mid C_3$,
(x) $a_1, b_3, c_2 \mid A_2$,
(xi) $a_3, b_2, c_1 \mid B_2$,
(xii) $a_2, b_1, c_3 \mid C_4$,
(xiii) $b_1, b_2, b_3 \mid A_3$,
(xiv) $a_1, a_2, a_3 \mid B_3$,
(xv) $c_1, c_2 \mid C_5$.

Then $c_3 \mid C_5$.

A projective incidence structure is called a projective space defined in terms of incidence if:

(Def.9) (i) for all elements p, q of the points of it and for all elements P, Q of the lines of it such that $p \mid P$ and $q \mid P$ and $p \mid Q$ and $q \mid Q$ holds $p = q$ or $P = Q$,

(ii) for every elements p, q of the points of it there exists an element P of the lines of it such that $p \mid P$ and $q \mid P$,

(iii) there exists an element p of the points of it and there exists an element P of the lines of it such that $p \nmid P$,

(iv) for every element P of the lines of it there exist elements a, b, c of the points of it such that $a \neq b$ and $b \neq c$ and $c \neq a$ and $a \mid P$ and $b \mid P$ and $c \mid P$,

(v) for all elements a, b, c, d, p, q of the points of it and for all elements M, N, P, Q of the lines of it such that $a \mid M$ and $b \mid M$ and $c \mid N$ and $d \mid N$ and $p \mid M$ and $p \mid N$ and $a \mid P$ and $c \mid P$ and $b \mid Q$ and $d \mid Q$ and $p \nmid P$ and $p \nmid Q$ and $M \neq N$ there exists an element q of the points of it such that $q \mid P$ and $q \mid Q$.

Let C_1 be a projective space defined in terms of collinearity.
Then Inc-ProjSp(C_1) is a projective space defined in terms of incidence.

A projective space defined in terms of incidence is 2-dimensional if:

(Def.10) for every elements M, N of the lines of it there exists an element q of the points of it such that $q \mid M$ and $q \mid N$.

A projective space defined in terms of incidence is at least 3-dimensional if:

(Def.11) there exist elements M, N of the lines of it such that for no element q of the points of it holds $q \mid M$ and $q \mid N$.

A projective space defined in terms of incidence is at most 3-dimensional if:

(Def.12) for every element a of the points of it and for every elements M, N of the lines of it there exist elements b, c of the points of it and there exists an element S of the lines of it such that $a \mid S$ and $b \mid S$ and $c \mid S$ and $b \mid M$ and $c \mid N$.

A projective space defined in terms of incidence is 3-dimensional if:

(Def.13) it is at most 3-dimensional and it is at least 3-dimensional.

A projective space defined in terms of incidence is Fanoian if:

(Def.14) Let p, q, r, s, a, b, c be elements of the points of it . Let L, Q, R, S, A, B, C be elements of the lines of it . Suppose that

(i) $q \nmid L$,
(ii) $r \nmid L$,
(iii) $p \nmid Q$,
(iv) $s \nmid Q$,
(v) $p \nmid R$,
(vi) $r \nmid R$,
(vii) $q \nmid S$,
(viii) $s \nmid S$,
(ix) $a, p, s \mid L$,
(x) $a, q, r \mid Q$,
(xi) $b, q, s \mid R$,
(xii) $b, p, r \mid S$,
(xiii) $c, p, q \mid A$,
(xiv) $c, r, s \mid B$,

(xv) $a, b \mid C$.
Then $c \nmid C$.

A projective space defined in terms of incidence is Desarguesian if:

(Def.15) Let o, b_1, a_1, b_2, a_2, b_3, a_3, r, s, t be elements of the points of it . Let C_3, C_4, C_5, A_1, A_2, A_3, B_1, B_2, B_3 be elements of the lines of it . Suppose that

(i) $o, b_1, a_1 \mid C_3$,
(ii) $o, a_2, b_2 \mid C_4$,
(iii) $o, a_3, b_3 \mid C_5$,
(iv) $a_3, a_2, t \mid A_1$,
(v) $a_3, r, a_1 \mid A_2$,
(vi) $a_2, s, a_1 \mid A_3$,
(vii) $t, b_2, b_3 \mid B_1$,
(viii) $b_1, r, b_3 \mid B_2$,
(ix) $b_1, s, b_2 \mid B_3$,
(x) C_3, C_4, C_5 are mutually different,
(xi) $o \neq a_1$,
(xii) $o \neq a_2$,
(xiii) $o \neq a_3$,
(xiv) $o \neq b_1$,
(xv) $o \neq b_2$,
(xvi) $o \neq b_3$,
(xvii) $a_1 \neq b_1$,
(xviii) $a_2 \neq b_2$,
(xix) $a_3 \neq b_3$.

Then there exists an element O of the lines of it such that $r, s, t \mid O$.

A projective space defined in terms of incidence is Pappian if:

(Def.16) Let o, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be elements of the points of it . Let A_1, A_2, A_3, B_1, B_2, B_3, C_3, C_4, C_5 be elements of the lines of it . Suppose that

(i) o, a_1, a_2, a_3 are mutually different,
(ii) o, b_1, b_2, b_3 are mutually different,
(iii) $A_3 \neq B_3$,
(iv) $o \mid A_3$,
(v) $o \mid B_3$,
(vi) $a_2, b_3, c_1 \mid A_1$,
(vii) $a_3, b_1, c_2 \mid B_1$,
(viii) $a_1, b_2, c_3 \mid C_3$,
(ix) $a_1, b_3, c_2 \mid A_2$,
(x) $a_3, b_2, c_1 \mid B_2$,
(xi) $a_2, b_1, c_3 \mid C_4$,
(xii) $b_1, b_2, b_3 \mid A_3$,
(xiii) $a_1, a_2, a_3 \mid B_3$,
(xiv) $c_1, c_2 \mid C_5$.

Then $c_3 \mid C_5$.

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