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Definable Functions

Grzegorz Bancerek
Warsow University
Białystok

Summary. The article is contituation of [6] and [5]. It deals with concepts of variables occuring in a formula and free variables, replacement of variables in a formula and definable functions. The goal is to create a base of facts which are neccesary to show that every model of ZF set theory is a good model, i.e. it is closed under fundamental set-theoretical operations (union, intersection, Cartesian product ect.). The base includes the facts concerning the composition and conditional sum of two definable functions.

MML Identifier: ZFMODEL2.

The notation and terminology used here are introduced in the following articles: [12], [1], [11], [8], [7], [10], [4], [9], [2], [3], [5], and [6]. For simplicity we follow a convention: x, y, z, x_1, x_2, x_3, x_4 will denote variables, M will denote a non-empty set, i, j will denote natural numbers, m, m_1, m_2, m_3, m_4 will denote elements of M, H, H_1, H_2 will denote ZF-formulae, and v, v_1, v_2 will denote functions from VAR into M. One can prove the following propositions:

(1) $\mathrm{Free}(H(\frac{x}{y})) \subseteq (\mathrm{Free}\,H \setminus \{x\}) \cup \{y\}$.

(2) If $y \notin \mathrm{Var}_H$, then if $x \in \mathrm{Free}\,H$, then $\mathrm{Free}(H(\frac{x}{y})) = (\mathrm{Free}\,H \setminus \{x\}) \cup \{y\}$ but if $x \notin \mathrm{Free}\,H$, then $\mathrm{Free}(H(\frac{x}{y})) = \mathrm{Free}\,H$.

(3) Var_H is finite.

(4) There exists i such that for every j such that $x_j \in \mathrm{Var}_H$ holds $j < i$ and there exists x such that $x \notin \mathrm{Var}_H$.

(5) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ if and only if $M, v \models \forall_x H$.

(6) If $x \notin \mathrm{Var}_H$, then $M, v \models H$ if and only if $M, v(\frac{x}{m}) \models H$.

(7) Suppose $x \neq y$ and $y \neq z$ and $z \neq x$. Then $((v(\frac{x}{m_1}))(\frac{y}{m_2}))(\frac{z}{m_3}) = ((v(\frac{z}{m_3}))(\frac{y}{m_2}))(\frac{x}{m_1})$ and $((v(\frac{x}{m_1}))(\frac{y}{m_2}))(\frac{z}{m_3}) = ((v(\frac{y}{m_2}))(\frac{z}{m_3}))(\frac{x}{m_1})$.

(8) Suppose $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then

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(i) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m_1})$,
(ii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m_1}))(\frac{x_2}{m_2})$,
(iii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}) = (((v(\frac{x_4}{m_4}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m_1})$.

(9) (i) $((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_1}{m}) = (v(\frac{x_2}{m_2}))(\frac{x_1}{m})$,
(ii) $(((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m}) = ((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_1}{m})$,
(iii) $((((v(\frac{x_1}{m_1}))(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m}) = (((v(\frac{x_2}{m_2}))(\frac{x_3}{m_3}))(\frac{x_4}{m_4}))(\frac{x_1}{m})$.

(10) If $x \notin \mathrm{Free}\, H$, then $M, v \models H$ if and only if $M, v(\frac{x}{m}) \models H$.

(11) Suppose $x_0 \notin \mathrm{Free}\, H$ and $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Then for all m_1, m_2 holds $\mathrm{f}_H[v](m_1) = m_2$ if and only if $M, (v(\frac{x_3}{m_1}))(\frac{x_4}{m_2}) \models H$.

(12) If $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $M \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$, then $\mathrm{f}_H[v] = \mathrm{f}_H[M]$.

(13) If $x \notin \mathrm{Var}_H$, then $M, v \models H(\frac{y}{x})$ if and only if $M, v(\frac{y}{v(x)}) \models H$.

(14) If $x \notin \mathrm{Var}_H$ and $M, v \models H$, then $M, v(\frac{x}{v(y)}) \models H(\frac{y}{x})$.

(15) Suppose that
(i) $x_0 \notin \mathrm{Free}\, H$,
(ii) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$,
(iii) $x \notin \mathrm{Var}_H$,
(iv) $y \neq x_3$,
(v) $y \neq x_4$,
(vi) $y \notin \mathrm{Free}\, H$,
(vii) $x \neq x_0$,
(viii) $x \neq x_3$,
(ix) $x \neq x_4$.
Then
(x) $x_0 \notin \mathrm{Free}(H(\frac{y}{x}))$,
(xi) $M, v(\frac{x}{v(y)}) \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4}(H(\frac{y}{x})) \Leftrightarrow x_4{=}x_0))$,
(xii) $\mathrm{f}_H[v] = \mathrm{f}_{H(\frac{y}{x})}[v(\frac{x}{v(y)})]$.

(16) If $x \notin \mathrm{Var}_H$, then $M \models H(\frac{y}{x})$ if and only if $M \models H$.

(17) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$. Then there exist H_2, v_2 such that for every j such that $j < i$ and $x_j \in \mathrm{Var}_{H_2}$ holds $j = 3$ or $j = 4$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$ and $\mathrm{f}_{H_1}[v_1] = \mathrm{f}_{H_2}[v_2]$.

(18) Suppose $x_0 \notin \mathrm{Free}\, H_1$ and $M, v_1 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$. Then there exist H_2, v_2 such that $\mathrm{Free}\, H_1 \cap \mathrm{Free}\, H_2 \subseteq \{x_3, x_4\}$ and $x_0 \notin \mathrm{Free}\, H_2$ and $M, v_2 \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$ and $\mathrm{f}_{H_1}[v_1] = \mathrm{f}_{H_2}[v_2]$.

In the sequel F, G are functions. One can prove the following propositions:

(19) If F is definable in M and G is definable in M, then $F \cdot G$ is definable in M.

(20) If $x_0 \notin \mathrm{Free}\, H$, then $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ if and only if for every m_1 there exists m_2 such that for every m_3 holds $M, (v(\frac{x_3}{m_1}))(\frac{x_4}{m_3}) \models H$ if and only if $m_3 = m_2$.

(21) Suppose F is definable in M and G is definable in M and $\mathrm{Free}\,H \subseteq \{x_3\}$. Let F_1 be a function. Then if $\mathrm{dom}\,F_1 = M$ and for every v holds if $M, v \models H$, then $F_1(v(x_3)) = F(v(x_3))$ but if $M, v \models \neg H$, then $F_1(v(x_3)) = G(v(x_3))$, then F_1 is definable in M.

(22) If F is parametrically definable in M and G is parametrically definable in M, then $G \cdot F$ is parametrically definable in M.

(23) Suppose that

(i) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\,H_1$,
(ii) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_1 \Leftrightarrow x_4{=}x_0))$,
(iii) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\,H_2$,
(iv) $M, v \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H_2 \Leftrightarrow x_4{=}x_0))$,
(v) $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\,H$,
(vi) $x_4 \notin \mathrm{Free}\,H$.

Let F_1 be a function. Then if $\mathrm{dom}\,F_1 = M$ and for every m holds if $M, v(\frac{x_3}{m}) \models H$, then $F_1(m) = \mathrm{f}_{H_1}[v](m)$ but if $M, v(\frac{x_3}{m}) \models \neg H$, then $F_1(m) = \mathrm{f}_{H_2}[v](m)$, then F_1 is parametrically definable in M.

(24) id_M is definable in M.

(25) id_M is parametrically definable in M.

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