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The Limit of a Real Function at a Point

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Summary. We define the proper and the improper limit of a real function at a point. The main properties of the operations on the limit of a function are proved. The connection between the one-side limits and the limit of a function at a point are exposed. Equivalent Cauchy and Heine characterizations of the limit of a real function at a point are proved.

MML Identifier: `LIMFUNC3`.

The papers [17], [5], [1], [2], [3], [15], [13], [6], [8], [14], [18], [16], [4], [10], [11], [12], [7], and [9] provide the notation and terminology for this paper. For simplicity we adopt the following convention: r, r_1, r_2, g, g_1, g_2, x_0 will be real numbers, n, k will be natural numbers, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) If $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ or $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$, then $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(2) Suppose for every n holds $0 < |x_0 - s_1(n)|$ and $|x_0 - s_1(n)| < \frac{1}{n+1}$ and $s_1(n) \in \operatorname{dom} f$. Then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(3) Suppose s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$. Then for every r such that $0 < r$ there exists n such that for every k such that $n \leq k$ holds $0 < |x_0 - s_1(k)|$ and $|x_0 - s_1(k)| < r$ and $s_1(k) \in \operatorname{dom} f$.

(4) If $0 < r$, then $]x_0 - r, x_0 + r[\setminus \{x_0\} =]x_0 - r, x_0[\cup]x_0, x_0 + r[$.

(5) Suppose $0 < r_2$ and $]x_0 - r_2, x_0[\cup]x_0, x_0 + r_2[\subseteq \operatorname{dom} f$. Then for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

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(6) If for every n holds $x_0 - \frac{1}{n+1} < s_1(n)$ and $s_1(n) < x_0$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$.

(7) If s_1 is convergent and $\lim s_1 = x_0$ and $0 < g$, then there exists k such that for every n such that $k \leq n$ holds $x_0 - g < s_1(n)$ and $s_1(n) < x_0 + g$.

(8) The following conditions are equivalent:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$.

We now define three new predicates. Let us consider f, x_0. We say that f is convergent in x_0 if and only if:

(Def.1) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is divergent to $+\infty$ in x_0 if and only if:

(Def.2) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is divergent to $-\infty$ in x_0 if and only if:

(Def.3) (i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $-\infty$.

The following propositions are true:

(9) f is convergent in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(10) f is divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $+\infty$.

(11) f is divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is divergent to $-\infty$.

(12) f is convergent in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(13) f is divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every g_1 there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(14) f is divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(ii) for every g_1 there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(15) f is divergent to $+\infty$ in x_0 if and only if f is left divergent to $+\infty$ in x_0 and f is right divergent to $+\infty$ in x_0.

(16) f is divergent to $-\infty$ in x_0 if and only if f is left divergent to $-\infty$ in x_0 and f is right divergent to $-\infty$ in x_0.

(17) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) f_2 is divergent to $+\infty$ in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

Then $f_1 + f_2$ is divergent to $+\infty$ in x_0 and $f_1 f_2$ is divergent to $+\infty$ in x_0.

(18) Suppose that
(i) f_1 is divergent to $-\infty$ in x_0,
(ii) f_2 is divergent to $-\infty$ in x_0,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.
Then $f_1 + f_2$ is divergent to $-\infty$ in x_0 and $f_1 f_2$ is divergent to $+\infty$ in x_0.

(19) Suppose that
(i) f_1 is divergent to $+\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 + f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 + f_2)$,
(iii) there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0 - r, x_0[\cup]x_0, x_0 + r[$.
Then $f_1 + f_2$ is divergent to $+\infty$ in x_0.

(20) Suppose that
(i) f_1 is divergent to $+\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$,
(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \operatorname{dom} f_2 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $r_1 \leq f_2(g)$.
Then $f_1 f_2$ is divergent to $+\infty$ in x_0.

(21) (i) If f is divergent to $+\infty$ in x_0 and $r > 0$, then rf is divergent to $+\infty$ in x_0,
(ii) if f is divergent to $+\infty$ in x_0 and $r < 0$, then rf is divergent to $-\infty$ in x_0,
(iii) if f is divergent to $-\infty$ in x_0 and $r > 0$, then rf is divergent to $-\infty$ in x_0,
(iv) if f is divergent to $-\infty$ in x_0 and $r < 0$, then rf is divergent to $+\infty$ in x_0.

(22) If f is divergent to $+\infty$ in x_0 or f is divergent to $-\infty$ in x_0, then $|f|$ is divergent to $+\infty$ in x_0.

(23) Suppose that
(i) there exists r such that $0 < r$ and f is non-decreasing on $]x_0 - r, x_0[$ and f is non-increasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0, x_0 + r[$,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.
Then f is divergent to $+\infty$ in x_0.

(24) Suppose that
(i) there exists r such that $0 < r$ and f is increasing on $]x_0 - r, x_0[$ and f is decreasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0 - r, x_0[$

and f is not upper bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $+\infty$ in x_0.

(25) Suppose that

(i) there exists r such that $0 < r$ and f is non-increasing on $]x_0 - r, x_0[$ and f is non-decreasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $-\infty$ in x_0.

(26) Suppose that

(i) there exists r such that $0 < r$ and f is decreasing on $]x_0 - r, x_0[$ and f is increasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0, x_0 + r[$,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$.

Then f is divergent to $-\infty$ in x_0.

(27) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[) \subseteq \operatorname{dom} f_1 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f_1(g) \leq f(g)$.

Then f is divergent to $+\infty$ in x_0.

(28) Suppose that

(i) f_1 is divergent to $-\infty$ in x_0,

(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[) \subseteq \operatorname{dom} f_1 \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) \leq f_1(g)$.

Then f is divergent to $-\infty$ in x_0.

(29) Suppose that

(i) f_1 is divergent to $+\infty$ in x_0,

(ii) there exists r such that $0 < r$ and $]x_0 - r, x_0[\cup]x_0, x_0 + r[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]x_0 - r, x_0[\cup]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$.

Then f is divergent to $+\infty$ in x_0.

(30) Suppose that

(i) f_1 is divergent to $-\infty$ in x_0,

(ii) there exists r such that $0 < r$ and $]x_0 - r, x_0[\cup]x_0, x_0 + r[\subseteq \operatorname{dom} f \cap \operatorname{dom} f_1$ and for every g such that $g \in]x_0 - r, x_0[\cup]x_0, x_0 + r[$ holds $f(g) \leq f_1(g)$.

Then f is divergent to $-\infty$ in x_0.

Let us consider f, x_0. Let us assume that f is convergent in x_0. The functor $\lim_{x_0} f$ yields a real number and is defined by:

(Def.4) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0} f$.

The following propositions are true:

(31) If f is convergent in x_0, then $\lim_{x_0} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \setminus \{x_0\}$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(32) Suppose f is convergent in x_0. Then $\lim_{x_0} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists g_2 such that $0 < g_2$ and for every r_1 such that $0 < |x_0 - r_1|$ and $|x_0 - r_1| < g_2$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(33) If f is convergent in x_0, then f is left convergent in x_0 and f is right convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^+} f$ and $\lim_{x_0} f = \lim_{x_0^-} f$ and $\lim_{x_0} f = \lim_{x_0^+} f$.

(34) If f is left convergent in x_0 and f is right convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^+} f$, then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0^-} f$ and $\lim_{x_0} f = \lim_{x_0^+} f$.

(35) If f is convergent in x_0, then rf is convergent in x_0 and $\lim_{x_0}(rf) = r \cdot (\lim_{x_0} f)$.

(36) If f is convergent in x_0, then $-f$ is convergent in x_0 and $\lim_{x_0}(-f) = -\lim_{x_0} f$.

(37) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 + f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 + f_2)$.

Then $f_1 + f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 + f_2) = \lim_{x_0} f_1 + \lim_{x_0} f_2$.

(38) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 - f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 - f_2)$.

Then $f_1 - f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 - f_2) = \lim_{x_0} f_1 - \lim_{x_0} f_2$.

(39) If f is convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0} f \neq 0$, then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = (\lim_{x_0} f)^{-1}$.

(40) If f is convergent in x_0, then $|f|$ is convergent in x_0 and $\lim_{x_0} |f| = |\lim_{x_0} f|$.

(41) Suppose that

(i) f is convergent in x_0,

(ii) $\lim_{x_0} f \neq 0$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$.

Then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = (\lim_{x_0} f)^{-1}$.

(42) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$.

Then $f_1 f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 f_2) = (\lim_{x_0} f_1) \cdot (\lim_{x_0} f_2)$.

(43) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) $\lim_{x_0} f_2 \neq 0$,

(iv) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} \frac{f_1}{f_2}$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} \frac{f_1}{f_2}$.

Then $\frac{f_1}{f_2}$ is convergent in x_0 and $\lim_{x_0} \frac{f_1}{f_2} = \frac{\lim_{x_0} f_1}{\lim_{x_0} f_2}$.

(44) Suppose that

(i) f_1 is convergent in x_0,

(ii) $\lim_{x_0} f_1 = 0$,

(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom}(f_1 f_2)$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom}(f_1 f_2)$,

(iv) there exists r such that $0 < r$ and f_2 is bounded on $]x_0 - r, x_0[\cup]x_0, x_0 + r[$.

Then $f_1 f_2$ is convergent in x_0 and $\lim_{x_0}(f_1 f_2) = 0$.

(45) Suppose that

(i) f_1 is convergent in x_0,

(ii) f_2 is convergent in x_0,

(iii) $\lim_{x_0} f_1 = \lim_{x_0} f_2$,

(iv) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ and $\operatorname{dom} f \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ or $\operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ and $\operatorname{dom} f \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$.
Then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0} f_1$.

(46) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is convergent in x_0,
(iii) $\lim_{x_0} f_1 = \lim_{x_0} f_2$,
(iv) there exists r such that $0 < r$ and $]x_0-r,x_0[\cup]x_0,x_0+r[\subseteq (\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \cap \operatorname{dom} f$ and for every g such that $g \in]x_0-r,x_0[\cup]x_0,x_0+r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.
Then f is convergent in x_0 and $\lim_{x_0} f = \lim_{x_0} f_1$.

(47) Suppose that
(i) f_1 is convergent in x_0,
(ii) f_2 is convergent in x_0,
(iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ and for every g such that $g \in \operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[) \subseteq \operatorname{dom} f_1 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ and for every g such that $g \in \operatorname{dom} f_2 \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ holds $f_1(g) \leq f_2(g)$.
Then $\lim_{x_0} f_1 \leq \lim_{x_0} f_2$.

(48) Suppose that
(i) f is divergent to $+\infty$ in x_0 or f is divergent to $-\infty$ in x_0,
(ii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$.
Then $\frac{1}{f}$ is convergent in x_0 and $\lim_{x_0} \frac{1}{f} = 0$.

(49) Suppose that
(i) f is convergent in x_0,
(ii) $\lim_{x_0} f = 0$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and $g_2 \in \operatorname{dom} f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap (]x_0-r,x_0[\cup]x_0,x_0+r[)$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is divergent to $+\infty$ in x_0.

(50) Suppose that
(i) f is convergent in x_0,
(ii) $\lim_{x_0} f = 0$,
(iii) for all r_1, r_2 such that $r_1 < x_0$ and $x_0 < r_2$ there exist g_1, g_2 such that $r_1 < g_1$ and $g_1 < x_0$ and $g_1 \in \operatorname{dom} f$ and $g_2 < r_2$ and $x_0 < g_2$ and

$g_2 \in \mathrm{dom}\, f$ and $f(g_1) \neq 0$ and $f(g_2) \neq 0$,

(iv) there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) \leq 0$.

Then $\frac{1}{f}$ is divergent to $-\infty$ in x_0.

(51) If f is convergent in x_0 and $\lim_{x_0} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $0 < f(g)$, then $\frac{1}{f}$ is divergent to $+\infty$ in x_0.

(52) If f is convergent in x_0 and $\lim_{x_0} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap (]x_0 - r, x_0[\cup]x_0, x_0 + r[)$ holds $f(g) < 0$, then $\frac{1}{f}$ is divergent to $-\infty$ in x_0.

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