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Countable Sets and Hessenberg's Theorem

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Summary. The concept of countable sets is introduced and there are shown some facts which deal with finite and countable sets. Besides, the article includes theorems and lemmas on the sum and the product of infinite cardinals. The most important of them is Hessenberg's theorem which says that for every infinite cardinal $\mathbf{m}$ the product $\mathbf{m} \cdot \mathbf{m}$ is equal to $\mathbf{m}$.

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The papers [20], [16], [3], [11], [9], [15], [5], [8], [7], [21], [19], [2], [1], [10], [22], [12], [13], [18], [14], [17], [4], and [6] provide the terminology and notation for this paper. For simplicity we follow the rules: X, Y are sets, D is a non-empty set, m, n, n_1, n_2, n_3, m_2, m_1 are natural numbers, A, B are ordinal numbers, L, K, M, N are cardinal numbers, x is arbitrary, and f is a function. Next we state a number of propositions:

(1) X is finite if and only if $\overline{\overline{X}}$ is finite.

(2) X is finite if and only if $\overline{\overline{X}} < \aleph_{\mathbf{0}}$.

(3) If X is finite, then $\overline{\overline{X}} \in \aleph_{\mathbf{0}}$ and $\overline{\overline{X}} \in \omega$.

(4) X is finite if and only if there exists n such that $\overline{\overline{X}} = \overline{\overline{n}}$.

(5) $\operatorname{succ} A \setminus \{A\} = A$.

(6) If $A \approx \operatorname{ord}(n)$, then $A = \operatorname{ord}(n)$.

(7) A is finite if and only if $A \in \omega$.

(8) A is not finite if and only if $\omega \subseteq A$.

(9) M is finite if and only if $M \in \aleph_{\mathbf{0}}$.

(10) M is finite if and only if $M < \aleph_{\mathbf{0}}$.

(11) M is not finite if and only if $\aleph_{\mathbf{0}} \subseteq M$.

(12) M is not finite if and only if $\aleph_{\mathbf{0}} \leq M$.

(13) If N is finite and M is not finite, then $N < M$ and $N \leq M$.

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(14) X is not finite if and only if there exists Y such that $Y \subseteq X$ and $\overline{\overline{Y}} = \aleph_{\mathbf{0}}$.

(15) ω is not finite and $\mathbb{N}$ is not finite.

(16) $\aleph_{\mathbf{0}}$ is not finite.

(17) $X = \emptyset$ if and only if $\overline{\overline{X}} = \overline{\mathbf{0}}$.

(18) $M \neq \overline{\mathbf{0}}$ if and only if $\overline{\mathbf{0}} < M$.

(19) $\overline{\mathbf{0}} \leq M$.

(20) $\overline{\overline{X}} = \overline{\overline{Y}}$ if and only if $X^+ = Y^+$.

(21) $M = N$ if and only if $N^+ = M^+$.

(22) $N < M$ if and only if $N^+ \leq M$.

(23) $N < M^+$ if and only if $N \leq M$.

(24) $\overline{\mathbf{0}} < M$ if and only if $\overline{\mathbf{1}} \leq M$.

(25) $\overline{\mathbf{1}} < M$ if and only if $\overline{\mathbf{2}} \leq M$.

(26) If M is finite but $N \leq M$ or $N < M$, then N is finite.

(27) A is a limit ordinal number if and only if for all B, n such that $B \in A$ holds $B + \mathrm{ord}(n) \in A$.

(28) $A + \mathrm{succ}\,\mathrm{ord}(n) = \mathrm{succ}\,A + \mathrm{ord}(n)$ and $A + \mathrm{ord}(n+1) = \mathrm{succ}\,A + \mathrm{ord}(n)$.

(29) There exists n such that $A \cdot \mathrm{succ}\,\mathbf{1} = A + \mathrm{ord}(n)$.

(30) If A is a limit ordinal number, then $A \cdot \mathrm{succ}\,\mathbf{1} = A$.

(31) If $\omega \subseteq A$, then $\mathbf{1} + A = A$.

Next we state a number of propositions:

(32) If M is not finite, then $\mathrm{ord}(M)$ is a limit ordinal number.

(33) If M is not finite, then $M + M = M$.

(34) If M is not finite but $N \leq M$ or $N < M$, then $M + N = M$ and $N + M = M$.

(35) If X is not finite but $X \approx Y$ or $Y \approx X$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(36) If X is not finite and Y is finite, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(37) If X is not finite but $\overline{\overline{Y}} < \overline{\overline{X}}$ or $\overline{\overline{Y}} \leq \overline{\overline{X}}$, then $X \cup Y \approx X$ and $\overline{\overline{X \cup Y}} = \overline{\overline{X}}$.

(38) If M is finite and N is finite, then $M + N$ is finite.

(39) If M is not finite, then $M + N$ is not finite and $N + M$ is not finite.

(40) If M is finite and N is finite, then $M \cdot N$ is finite.

(41) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K + M \leq L + N$ and $M + K \leq L + N$.

(42) If $M < N$ or $M \leq N$, then $K + M \leq K + N$ and $K + M \leq N + K$ and $M + K \leq K + N$ and $M + K \leq N + K$.

Let us consider X. We say that X is countable if and only if:

(Def.1) $\overline{\overline{X}} \leq \aleph_{\mathbf{0}}$.

One can prove the following propositions:

(43) If X is finite, then X is countable.

(44) ω is countable and $\mathbb{N}$ is countable.

(45) X is countable if and only if there exists f such that $\operatorname{dom} f = \mathbb{N}$ and $X \subseteq \operatorname{rng} f$.

(46) If $Y \subseteq X$ and X is countable, then Y is countable.

(47) If X is countable and Y is countable, then $X \cup Y$ is countable.

(48) If X is countable, then $X \cap Y$ is countable and $Y \cap X$ is countable.

(49) If X is countable, then $X \setminus Y$ is countable.

(50) If X is countable and Y is countable, then $X \dot{-} Y$ is countable.

The scheme *Lambda2N* deals with a binary functor $\mathcal{F}$ yielding a natural number and states that:

there exists a function f from $[\!:\mathbb{N}, \mathbb{N}:\!]$ into $\mathbb{N}$ such that for all n, m holds $f(\langle n, m\rangle) = \mathcal{F}(n, m)$

for all values of the parameter.

In the sequel r will denote a real number. Next we state the proposition

(51) $r \neq 0$ or $n = 0$ if and only if $r^n \neq 0$.

Let m, n be natural numbers. Then m^n is a natural number.

One can prove the following propositions:

(52) If $2^{n_1} \cdot (2 \cdot m_1 + 1) = 2^{n_2} \cdot (2 \cdot m_2 + 1)$, then $n_1 = n_2$ and $m_1 = m_2$.

(53) $[\!:\mathbb{N}, \mathbb{N}:\!] \approx \mathbb{N}$ and $\overline{\overline{\mathbb{N}}} = \overline{\overline{[\!:\mathbb{N}, \mathbb{N}:\!]}}$.

(54) $\aleph_{\mathbf{0}} \cdot \aleph_{\mathbf{0}} = \aleph_{\mathbf{0}}$.

(55) If X is countable and Y is countable, then $[\!:X, Y:\!]$ is countable.

(56) $D^1 \approx D$ and $\overline{\overline{D^1}} = \overline{\overline{D}}$.

We now state a number of propositions:

(57) $[\!:D^n, D^m:\!] \approx D^{n+m}$ and $\overline{\overline{[\!:D^n, D^m:\!]}} = \overline{\overline{D^{n+m}}}$.

(58) If D is countable, then D^n is countable.

(59) If $\overline{\overline{\operatorname{dom} f}} \leq M$ and for every x such that $x \in \operatorname{dom} f$ holds $\overline{\overline{f(x)}} \leq N$, then $\overline{\overline{\bigcup f}} \leq M \cdot N$.

(60) If $\overline{\overline{X}} \leq M$ and for every Y such that $Y \in X$ holds $\overline{\overline{Y}} \leq N$, then $\overline{\overline{\bigcup X}} \leq M \cdot N$.

(61) For every f such that $\operatorname{dom} f$ is countable and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is countable holds $\bigcup f$ is countable.

(62) If X is countable and for every Y such that $Y \in X$ holds Y is countable, then $\bigcup X$ is countable.

(63) For every f such that $\operatorname{dom} f$ is finite and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is finite holds $\bigcup f$ is finite.

(64) If X is finite and for every Y such that $Y \in X$ holds Y is finite, then $\bigcup X$ is finite.

(65) If D is countable, then D^* is countable.

(66) $\aleph_{\mathbf{0}} \leq \overline{\overline{D^*}}$.

Now we present three schemes. The scheme *FraenCoun1* deals with a unary functor $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n) : \mathcal{P}[n]\}$ is countable

for all values of the parameters.

The scheme *FraenCoun2* concerns a binary functor $\mathcal{F}$, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2) : \mathcal{P}[n_1, n_2]\}$ is countable

for all values of the parameters.

The scheme *FraenCoun3* concerns a ternary functor $\mathcal{F}$, and a ternary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(n_1, n_2, n_3) : \mathcal{P}[n_1, n_2, n_3]\}$ is countable

for all values of the parameters.

The following propositions are true:

(67) $\aleph_{\mathbf{0}} \cdot \overline{\overline{n}} \leq \aleph_{\mathbf{0}}$ and $\overline{\overline{n}} \cdot \aleph_{\mathbf{0}} \leq \aleph_{\mathbf{0}}$.

(68) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K \cdot M \leq L \cdot N$ and $M \cdot K \leq L \cdot N$.

(69) If $M < N$ or $M \leq N$, then $K \cdot M \leq K \cdot N$ and $K \cdot M \leq N \cdot K$ and $M \cdot K \leq K \cdot N$ and $M \cdot K \leq N \cdot K$.

(70) If $K < L$ and $M < N$ or $K \leq L$ and $M < N$ or $K < L$ and $M \leq N$ or $K \leq L$ and $M \leq N$, then $K = \overline{\mathbf{0}}$ or $K^M \leq L^N$.

(71) If $M < N$ or $M \leq N$, then $K = \overline{\mathbf{0}}$ or $K^M \leq K^N$ and $M^K \leq N^K$.

(72) $M \leq M + N$ and $N \leq M + N$.

(73) If $N \neq \overline{\mathbf{0}}$, then $M \leq M \cdot N$ and $M \leq N \cdot M$.

(74) If $K < L$ and $M < N$, then $K + M < L + N$ and $M + K < L + N$.

(75) If $K + M < K + N$, then $M < N$.

(76) If $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{X}}$ and $\overline{\overline{Y}} < \overline{\overline{X}}$, then $\overline{\overline{X \setminus Y}} = \overline{\overline{X}}$.

One can prove the following propositions:

(77) If M is not finite, then $M \cdot M = M$.

(78) If M is not finite and $\overline{\mathbf{0}} < N$ but $N \leq M$ or $N < M$, then $M \cdot N = M$ and $N \cdot M = M$.

(79) If M is not finite but $N \leq M$ or $N < M$, then $M \cdot N \leq M$ and $N \cdot M \leq M$.

(80) If X is not finite, then $[\!: X, X :\!] \approx X$ and $\overline{\overline{[\!: X, X :\!]}} = \overline{\overline{X}}$.

(81) If X is not finite and Y is finite and $Y \neq \emptyset$, then $[\!: X, Y :\!] \approx X$ and $\overline{\overline{[\!: X, Y :\!]}} = \overline{\overline{X}}$.

(82) If $K < L$ and $M < N$, then $K \cdot M < L \cdot N$ and $M \cdot K < L \cdot N$.

(83) If $K \cdot M < K \cdot N$, then $M < N$.

(84) If X is not finite, then $\overline{\overline{X}} = \aleph_{\mathbf{0}} \cdot \overline{\overline{X}}$.

(85) If $X \neq \emptyset$ and X is finite and Y is not finite, then $\overline{\overline{Y}} \cdot \overline{\overline{X}} = \overline{\overline{Y}}$.

(86) If D is not finite and $n \neq 0$, then $D^n \approx D$ and $\overline{\overline{D^n}} = \overline{\overline{D}}$.

(87) If D is not finite, then $\overline{\overline{D}} = \overline{\overline{D^*}}$.

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