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Construction of Finite Sequences over Ring and Left-, Right-, and Bi-Modules over a Ring [1]

Michał Muzalewski
Warsaw University
Białystok

Lesław W. Szczerba
Siedlce University

Summary. This text includes definitions of finite sequences over rings and left-, right-, and bi-module over a ring, treated as functions defined for *all* natural numbers, but almost everywhere equal to zero. Some elementary theorems are proved, in particular for each category of sequences the schema of existence is proved. In all four cases, *i.e* for rings, left-, right-, and bi-modules are almost exactly the same, hovewer we do not know how to do the job in Mizar in a different way. The paper is mostly based on [2]. In particular the notion of initial segment of natural numbers is introduced which differs from that of [2] by starting with zero. This proved to be more convenient for algebraic purposes.

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The notation and terminology used in this paper are introduced in the following papers: [8], [3], [5], [1], [4], [6], and [7]. We adopt the following rules: i, k, l, m, n will be natural numbers and x will be arbitrary. We now state four propositions:

(2)[2] If $m < n+1$, then $m < n$ or $m = n$.

(4)[3] If $k < 2$, then $k = 0$ or $k = 1$.

(5) For every real number x holds $x < x+1$.

(7)[4] If $k < l$ and $l \leq k+1$, then $l = k+1$.

Let us consider n. The functor PSeg n yields a set and is defined by:

(Def.1) $\mathrm{PSeg}\, n = \{k : k < n\}$.

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[2]The proposition (1) was either repeated or obvious.
[3]The proposition (3) was either repeated or obvious.
[4]The proposition (6) was either repeated or obvious.

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Let us consider n. Then $\mathrm{PSeg}\, n$ is sets of natural numbers.

We now state a number of propositions:

(8) $\mathrm{PSeg}\, n = \{k : k < n\}$.

(9) If $x \in \mathrm{PSeg}\, n$, then x is a natural number.

(10) $k \in \mathrm{PSeg}\, n$ if and only if $k < n$.

(11) $\mathrm{PSeg}\, 0 = \emptyset$ and $\mathrm{PSeg}\, 1 = \{0\}$ and $\mathrm{PSeg}\, 2 = \{0, 1\}$.

(12) $n \in \mathrm{PSeg}(n + 1)$.

(13) $n \leq m$ if and only if $\mathrm{PSeg}\, n \subseteq \mathrm{PSeg}\, m$.

(14) If $\mathrm{PSeg}\, n = \mathrm{PSeg}\, m$, then $n = m$.

(15) If $k \leq n$, then $\mathrm{PSeg}\, k = \mathrm{PSeg}\, k \cap \mathrm{PSeg}\, n$ and $\mathrm{PSeg}\, k = \mathrm{PSeg}\, n \cap \mathrm{PSeg}\, k$.

(16) If $\mathrm{PSeg}\, k = \mathrm{PSeg}\, k \cap \mathrm{PSeg}\, n$ or $\mathrm{PSeg}\, k = \mathrm{PSeg}\, n \cap \mathrm{PSeg}\, k$, then $k \leq n$.

(17) $\mathrm{PSeg}\, n \cup \{n\} = \mathrm{PSeg}(n + 1)$.

In the sequel R is a field structure and x is a scalar of R. Let us consider R. A function from $\mathbb{N}$ into the carrier of R is said to be an algebraic sequence of R if:

(Def.2) there exists n such that for every i such that $i \geq n$ holds $\mathrm{it}(i) = 0_R$.

In the sequel p, q denote algebraic sequences of R. Next we state the proposition

(19)[5] $\mathrm{dom}\, p = \mathbb{N}$.

Let us consider R, p, k. We say that the length of p is at most k if and only if:

(Def.3) for every i such that $i \geq k$ holds $p(i) = 0_R$.

We now state the proposition

(20) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = 0_R$.

Let us consider R, p. The functor $\mathrm{len}\, p$ yielding a natural number is defined as follows:

(Def.4) the length of p is at most $\mathrm{len}\, p$ and for every m such that the length of p is at most m holds $\mathrm{len}\, p \leq m$.

We now state several propositions:

(21) $i = \mathrm{len}\, p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(22) For every i such that $i \geq \mathrm{len}\, p$ holds $p(i) = 0_R$.

(23) If $p(k) \neq 0_R$, then $\mathrm{len}\, p > k$.

(24) If for every i such that $i < k$ holds $p(i) \neq 0_R$, then $\mathrm{len}\, p \geq k$.

(25) If $\mathrm{len}\, p = k + 1$, then $p(k) \neq 0_R$.

Let us consider R, p. The functor $\mathrm{support}\, p$ yields sets of natural numbers and is defined as follows:

[5] The proposition (18) was either repeated or obvious.

(Def.5) $\operatorname{support} p = \operatorname{PSeg}(\operatorname{len} p)$.

Next we state two propositions:

(26) For every y being sets of natural numbers holds $y = \operatorname{support} p$ if and only if $y = \operatorname{PSeg}(\operatorname{len} p)$.

(27) $k = \operatorname{len} p$ if and only if $\operatorname{PSeg} k = \operatorname{support} p$.

The scheme *AlgSeqLambdaF* concerns field structure $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a scalar of $\mathcal{A}$ and states that:

there exists an algebraic sequence p of $\mathcal{A}$ such that $\operatorname{len} p \leq \mathcal{B}$ and for every k such that $k < \mathcal{B}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

One can prove the following proposition

(28) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

The following proposition is true

(29) For every R such that the carrier of $R \neq \{0_R\}$ for every k there exists an algebraic sequence p of R such that $\operatorname{len} p = k$.

Let us consider R, x. The functor $\langle x \rangle$ yielding an algebraic sequence of R is defined by:

(Def.6) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

One can prove the following propositions:

(30) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(31) $p = \langle 0_R \rangle$ if and only if $\operatorname{len} p = 0$.

(32) $p = \langle 0_R \rangle$ if and only if $\operatorname{support} p = \emptyset$.

(33) $\langle 0_R \rangle(i) = 0_R$.

(34) $p = \langle 0_R \rangle$ if and only if $\operatorname{rng} p = \{0_R\}$.

In the sequel R will be an associative ring and V will be a left module over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined by:

(Def.7) $\Theta_V = 0_{\text{the carrier of } V}$.

One can prove the following proposition

(35) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x denotes a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is said to be an algebraic sequence of V if:

(Def.8) there exists n such that for every i such that $i \geq n$ holds $\operatorname{it}(i) = \Theta_V$.

In the sequel p, q will denote algebraic sequences of V. The following proposition is true

(37)[6] $\operatorname{dom} p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

[6]The proposition (36) was either repeated or obvious.

(Def.9) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

We now state the proposition

(38) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\operatorname{len} p$ yields a natural number and is defined as follows:

(Def.10) the length of p is at most $\operatorname{len} p$ and for every m such that the length of p is at most m holds $\operatorname{len} p \leq m$.

One can prove the following propositions:

(39) $i = \operatorname{len} p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(40) For every i such that $i \geq \operatorname{len} p$ holds $p(i) = \Theta_V$.

(41) If $p(k) \neq \Theta_V$, then $\operatorname{len} p > k$.

(42) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\operatorname{len} p \geq k$.

(43) If $\operatorname{len} p = k + 1$, then $p(k) \neq \Theta_V$.

Let us consider R, V, p. The functor $\operatorname{support} p$ yields sets of natural numbers and is defined by:

(Def.11) $\operatorname{support} p = \operatorname{PSeg}(\operatorname{len} p)$.

We now state two propositions:

(44) For every y being sets of natural numbers holds $y = \operatorname{support} p$ if and only if $y = \operatorname{PSeg}(\operatorname{len} p)$.

(45) $k = \operatorname{len} p$ if and only if $\operatorname{PSeg} k = \operatorname{support} p$.

The scheme *AlgSeqLambdaLM* deals with an associative ring $\mathcal{A}$, a left module $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\operatorname{len} p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$

for all values of the parameters.

The following proposition is true

(46) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

We now state the proposition

(47) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\operatorname{len} p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yielding an algebraic sequence of V is defined as follows:

(Def.12) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

One can prove the following propositions:

(48) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(49) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{len} p = 0$.

(50) $p = \langle \Theta_V \rangle$ if and only if support $p = \emptyset$.

(51) $\langle \Theta_V \rangle(i) = \Theta_V$.

(52) $p = \langle \Theta_V \rangle$ if and only if $\mathrm{rng}\, p = \{\Theta_V\}$.

In the sequel V will denote a right module over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.13) $\Theta_V = 0_{\text{the carrier of } V}$.

The following proposition is true

(53) $\Theta_V = 0_{\text{the carrier of } V}$.

Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.14) $\Theta_V = 0_{\text{the carrier of } V}$.

The following proposition is true

(54) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x will denote a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is called an algebraic sequence of V if:

(Def.15) there exists n such that for every i such that $i \geq n$ holds $\mathrm{it}(i) = \Theta_V$.

In the sequel p, q will be algebraic sequences of V. We now state the proposition

(56)[7] $\mathrm{dom}\, p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

(Def.16) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Next we state the proposition

(57) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\mathrm{len}\, p$ yields a natural number and is defined by:

(Def.17) the length of p is at most $\mathrm{len}\, p$ and for every m such that the length of p is at most m holds $\mathrm{len}\, p \leq m$.

Next we state several propositions:

(58) $i = \mathrm{len}\, p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(59) For every i such that $i \geq \mathrm{len}\, p$ holds $p(i) = \Theta_V$.

(60) If $p(k) \neq \Theta_V$, then $\mathrm{len}\, p > k$.

(61) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\mathrm{len}\, p \geq k$.

(62) If $\mathrm{len}\, p = k + 1$, then $p(k) \neq \Theta_V$.

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[7]The proposition (55) was either repeated or obvious.

Let us consider R, V, p. The functor $\operatorname{support} p$ yielding sets of natural numbers is defined by:

(Def.18) $\operatorname{support} p = \operatorname{PSeg}(\operatorname{len} p)$.

The following propositions are true:

(63) For every y being sets of natural numbers holds $y = \operatorname{support} p$ if and only if $y = \operatorname{PSeg}(\operatorname{len} p)$.

(64) $k = \operatorname{len} p$ if and only if $\operatorname{PSeg} k = \operatorname{support} p$.

The scheme *AlgSeqLambdaRM* deals with an associative ring $\mathcal{A}$, a right module $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\operatorname{len} p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

The following proposition is true

(65) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

One can prove the following proposition

(66) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\operatorname{len} p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yielding an algebraic sequence of V is defined by:

(Def.19) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

We now state several propositions:

(67) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(68) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{len} p = 0$.

(69) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{support} p = \emptyset$.

(70) $\langle \Theta_V \rangle(i) = \Theta_V$.

(71) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{rng} p = \{\Theta_V\}$.

In the sequel V is a bimodule over R. Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.20) $\Theta_V = 0_{\text{the carrier of } V}$.

One can prove the following proposition

(72) $\Theta_V = 0_{\text{the carrier of } V}$.

Let us consider R, V. The functor Θ_V yields a vector of V and is defined as follows:

(Def.21) $\Theta_V = 0_{\text{the carrier of } V}$.

We now state the proposition

(73) $\Theta_V = 0_{\text{the carrier of } V}$.

In the sequel x will denote a vector of V. Let us consider R, V. A function from $\mathbb{N}$ into the carrier of the carrier of V is said to be an algebraic sequence of V if:

(Def.22) there exists n such that for every i such that $i \geq n$ holds $\mathrm{it}(i) = \Theta_V$.

In the sequel p, q will be algebraic sequences of V. We now state the proposition

(75)[8] $\mathrm{dom}\, p = \mathbb{N}$.

Let us consider R, V, p, k. We say that the length of p is at most k if and only if:

(Def.23) for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Next we state the proposition

(76) the length of p is at most k if and only if for every i such that $i \geq k$ holds $p(i) = \Theta_V$.

Let us consider R, V, p. The functor $\mathrm{len}\, p$ yielding a natural number is defined by:

(Def.24) the length of p is at most $\mathrm{len}\, p$ and for every m such that the length of p is at most m holds $\mathrm{len}\, p \leq m$.

One can prove the following propositions:

(77) $i = \mathrm{len}\, p$ if and only if the length of p is at most i and for every m such that the length of p is at most m holds $i \leq m$.

(78) For every i such that $i \geq \mathrm{len}\, p$ holds $p(i) = \Theta_V$.

(79) If $p(k) \neq \Theta_V$, then $\mathrm{len}\, p > k$.

(80) If for every i such that $i < k$ holds $p(i) \neq \Theta_V$, then $\mathrm{len}\, p \geq k$.

(81) If $\mathrm{len}\, p = k + 1$, then $p(k) \neq \Theta_V$.

Let us consider R, V, p. The functor $\mathrm{support}\, p$ yielding sets of natural numbers is defined by:

(Def.25) $\mathrm{support}\, p = \mathrm{PSeg}(\mathrm{len}\, p)$.

We now state two propositions:

(82) For every y being sets of natural numbers holds $y = \mathrm{support}\, p$ if and only if $y = \mathrm{PSeg}(\mathrm{len}\, p)$.

(83) $k = \mathrm{len}\, p$ if and only if $\mathrm{PSeg}\, k = \mathrm{support}\, p$.

The scheme *AlgSeqLambdaBM* concerns an associative ring $\mathcal{A}$, a bimodule $\mathcal{B}$ over $\mathcal{A}$, a natural number $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a vector of $\mathcal{B}$ and states that:

there exists an algebraic sequence p of $\mathcal{B}$ such that $\mathrm{len}\, p \leq \mathcal{C}$ and for every k such that $k < \mathcal{C}$ holds $p(k) = \mathcal{F}(k)$
for all values of the parameters.

We now state the proposition

[8] The proposition (74) was either repeated or obvious.

(84) If $\operatorname{len} p = \operatorname{len} q$ and for every k such that $k < \operatorname{len} p$ holds $p(k) = q(k)$, then $p = q$.

The following proposition is true

(85) For all R, V such that the carrier of the carrier of $V \neq \{\Theta_V\}$ for every k there exists an algebraic sequence p of V such that $\operatorname{len} p = k$.

Let us consider R, V, x. The functor $\langle x \rangle$ yields an algebraic sequence of V and is defined by:

(Def.26) $\operatorname{len}\langle x \rangle \leq 1$ and $\langle x \rangle(0) = x$.

Next we state several propositions:

(86) $p = \langle x \rangle$ if and only if $\operatorname{len} p \leq 1$ and $p(0) = x$.

(87) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{len} p = 0$.

(88) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{support} p = \emptyset$.

(89) $\langle \Theta_V \rangle(i) = \Theta_V$.

(90) $p = \langle \Theta_V \rangle$ if and only if $\operatorname{rng} p = \{\Theta_V\}$.

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