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Operations on Subspaces in Vector Space

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Summary. Sum, direct sum and intersection of subspaces are introduced. We prove some theorems concerning those notions and the decomposition of vector onto two subspaces. Linear complement of a subspace is also defined. We prove theorem that belong rather to [3].

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The papers [2], [8], [9], [5], [3], [4], [6], [1], and [7] provide the terminology and notation for this paper. For simplicity we adopt the following rules: G_1 will denote a field, V will denote a vector space over G_1, W, W_1, W_2, W_3 will denote subspaces of V, u, u_1, u_2, v, v_1, v_2 will denote vectors of V, and x will be arbitrary. Let us consider G_1, V, W_1, W_2. The functor $W_1 + W_2$ yields a subspace of V and is defined by:

(Def.1) the carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider G_1, V, W_1, W_2. The functor $W_1 \cap W_2$ yields a subspace of V and is defined by:

(Def.2) the carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

We now state a number of propositions:

(1) The carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

(2) If the carrier of the carrier of $W = \{v + u : v \in W_1 \wedge u \in W_2\}$, then $W = W_1 + W_2$.

(3) The carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

(4) If the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2), then $W = W_1 \cap W_2$.

(5) $x \in W_1 + W_2$ if and only if there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

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(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(8) $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(13) $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

(15) $\Omega_V + W = V$ and $W + \Omega_V = V$.

(16) $\Omega_V + \Omega_V = V$.

(17) $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21) W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(22) If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(23) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(24) If W_1 is a subspace of W_2 and W_1 is a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(25) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(26) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(27) $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(28) $\Omega_V \cap \Omega_V = V$.

(29) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(30) $W_1 \cap W_2 + W_2 = W_2$.

(31) $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If W_1 is a subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) $W_1 + W_2 = W_2$ if and only if $W_1 \cap W_2 = W_1$.

(38) If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(39) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(40) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(41) There exists W such that the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cup$ (the carrier of the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

Let us consider G_1, V. The functor Subspaces V yielding a non-empty set is defined as follows:

(Def.3) for every x holds $x \in$ Subspaces V if and only if x is a subspace of V.

In the sequel D denotes a non-empty set. The following three propositions are true:

(42) If for every x holds $x \in D$ if and only if x is a subspace of V, then $D =$ Subspaces V.

(43) $x \in$ Subspaces V if and only if x is a subspace of V.

(44) $V \in$ Subspaces V.

Let us consider G_1, V, W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def.4) $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let us consider G_1, V, W. A subspace of V is said to be a linear complement of W if:

(Def.5) V is the direct sum of it and W.

We now state three propositions:

(45) V is the direct sum of W_1 and W_2 if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

(46) If V is the direct sum of W_1 and W_2, then W_1 is a linear complement of W_2.

(47) If V is the direct sum of W_1 and W_2, then W_2 is a linear complement of W_1.

In the sequel L denotes a linear complement of W. The following propositions are true:

(48) V is the direct sum of L and W and V is the direct sum of W and L.

(49) $W + L = V$ and $L + W = V$.

(50) $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(51) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(52) V is the direct sum of $\mathbf{0}_V$ and Ω_V and V is the direct sum of Ω_V and $\mathbf{0}_V$.

(53) W is a linear complement of L.

(54) $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C_1 is a coset of W_1 and C_2 is a coset of W_2. We now state several propositions:

(55) If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(56) V is the direct sum of W_1 and W_2 if and only if for every C_1, C_2 there exists v such that $C_1 \cap C_2 = \{v\}$.

(57) $W_1 + W_2 = V$ if and only if for every v there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(58) If V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(59) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

In the sequel t will denote an element of [: the carrier of the carrier of V, the carrier of the carrier of V :]. Let us consider G_1, V, t. Then $t_{\mathbf{1}}$ is a vector of V. Then $t_{\mathbf{2}}$ is a vector of V.

Let us consider G_1, V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v \lhd (W_1, W_2)$ yielding an element of [: the carrier of the carrier of V, the carrier of the carrier of V :] is defined by:

(Def.6) $v = (v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}}$ and $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$ and $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

Next we state a number of propositions:

(60) If V is the direct sum of W_1 and W_2 and $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W_1$ and $t_{\mathbf{2}} \in W_2$, then $t = v \lhd (W_1, W_2)$.

(61) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}} = v$.

(62) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$.

(63) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

(64) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} = (v \lhd (W_2, W_1))_{\mathbf{2}}$.

(65) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} = (v \lhd (W_2, W_1))_{\mathbf{1}}$.

(66) If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v \lhd (W, L)$.

(67) $(v \lhd (W, L))_{\mathbf{1}} + (v \lhd (W, L))_{\mathbf{2}} = v$.

(68) $(v \lhd (W, L))_{\mathbf{1}} \in W$ and $(v \lhd (W, L))_{\mathbf{2}} \in L$.

(69) $(v \lhd (W, L))_{\mathbf{1}} = (v \lhd (L, W))_{\mathbf{2}}$.

(70) $(v \lhd (W, L))_{\mathbf{2}} = (v \lhd (L, W))_{\mathbf{1}}$.

In the sequel A_1, A_2 will be elements of Subspaces V. Let us consider G_1, V. The functor SubJoin V yields a binary operation on Subspaces V and is defined by:

(Def.7) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, V)(A_1, A_2) = W_1 + W_2$.

Let us consider G_1, V. The functor SubMeet V yielding a binary operation on Subspaces V is defined by:

(Def.8) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, V)(A_1, A_2) = W_1 \cap W_2$.

In the sequel o denotes a binary operation on Subspaces V. One can prove the following propositions:

(71) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubJoin}\, V(A_1, A_2) = W_1 + W_2$.

(72) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 + W_2$, then $o = \mathrm{SubJoin}\, V$.

(73) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubMeet}\, V(A_1, A_2) = W_1 \cap W_2$.

(74) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 \cap W_2$, then $o = \mathrm{SubMeet}\, V$.

(75) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lattice.

(76) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lower bound lattice.

(77) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is an upper bound lattice.

(78) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a bound lattice.

(79) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a modular lattice.

(80) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a complemented lattice.

(81) $v = v_1 + v_2$ if and only if $v_1 = v - v_2$.

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