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Projective Spaces - part II

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Summary. Distinction is made among several types of many dimensional projective spaces - at least three dimensional and exactly three-dimensional projective structures. We prove that analytical projective spaces defined over appropiate real linear spaces may serve as examples of the introduced classes of projective spaces. Corresponding subclasses of Fano projective structures are distinguished. Note that in projective geometry the axiom which assures that the dimension is not greater than three can be formulated as the statement: there exists a plane which intersects every line.

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The terminology and notation used in this paper have been introduced in the following articles: [1], [4], [2], and [3]. We follow a convention: V will be a real linear space, p, q, r, s, u, v, w, y, u_1, v_1 will be vectors of V, and a, b, c, d, a_1, b_1, c_1 will be real numbers. The following two propositions are true:

(1) Suppose that

(i) for every w there exist a, b, c, d such that $w = ((a \cdot p + b \cdot q) + c \cdot r) + d \cdot s$,

(ii) for all a, b, c, d such that $((a \cdot p + b \cdot q) + c \cdot r) + d \cdot s = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

Then for all u, v such that u is a proper vector and v is a proper vector there exist y, w such that u, v and w are lineary dependent and q, r and y are lineary dependent and p, w and y are lineary dependent and y is a proper vector and w is a proper vector.

(2) Suppose for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then for no y holds y is a proper vector and u, v and y are lineary dependent and u_1, v_1 and y are lineary dependent.

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We adopt the following rules: V will be a non-trivial real linear space, u, v, w, y, w_1 will be vectors of V, and p, p_1, q, q_1, q_2, q_3, r, r_1, r_2, r_3 will be elements of the points of the projective space over V. We now state two propositions:

(3) If there exist p, q, r such that p, q and r are not collinear, then for all p, q such that $p \neq q$ there exists r such that p, q and r are not collinear.

(4) Suppose that

(i) there exist y, u, v, w such that for every w_1 there exist a, b, a_1, b_1 such that $w_1 = ((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w$ and for all a, b, a_1, b_1 such that $((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then there exist p, q_1, q_2 such that p, q_1 and q_2 are not collinear and for every r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

Next we state the proposition

(5) Suppose that

(i) there exist p, q, r such that p, q and r are not collinear,

(ii) for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iii) there exist p, q_1, q_2 such that p, q_1 and q_2 are not collinear and for every r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

Then for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

In the sequel u, v, w, y, u_1, v_1, w_1 will be vectors of V. Next we state three propositions:

(6) Suppose that

(i) there exist y, u, v, w such that for every w_1 there exist a, b, c, c_1 such that $w_1 = ((a \cdot y + b \cdot u) + c \cdot v) + c_1 \cdot w$ and for all a, b, a_1, b_1 such that $((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(7) Suppose there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then there exist p, p_1, q, q_1 such that for no r holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(8) Suppose that

(i) there exist u, v, u_1, v_1 such that for every w there exist a, b, a_1, b_1 such that $w = ((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1$ and for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then

(ii) for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(iii) for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist p, q, r such that p, q and r are not collinear,

(v) there exist p, p_1, q, q_1 such that for no r holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A projective space defined in terms of collinearity is called an at least 3 dimensional projective space defined in terms of collinearity if:

(Def.1) there exist elements p, p_1, q, q_1 of the points of it such that for no element r of the points of it holds p, p_1 and r are collinear and q, q_1 and r are collinear.

We now state three propositions:

(9) For every projective space C_1 defined in terms of collinearity holds C_1 is an at least 3 dimensional projective space defined in terms of collinearity if and only if there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(10) If there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$, then the projective space over V is an at least 3 dimensional projective space defined in terms of collinearity.

(11) Let C_1 be a collinearity structure. Then C_1 is an at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

An at least 3 dimensional projective space defined in terms of collinearity is said to be a Fanoian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.2) Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

One can prove the following propositions:

(12) Let C_1 be an at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(13) If there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$, then the projective space over V is a Fanoian at least 3 dimensional projective space defined in terms of collinearity.

(14) Let C_1 be a collinearity structure. Then C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(15) For every C_1 being a collinearity structure holds C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and

only if C_1 is a Fanoian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

An at least 3 dimensional projective space defined in terms of collinearity is called a 3 dimensional projective space defined in terms of collinearity if:

(Def.3) for every elements p, p_1, q, q_1, r_2 of the points of it there exist elements r, r_1 of the points of it such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

The following propositions are true:

(16) For every at least 3 dimensional projective space C_1 defined in terms of collinearity holds C_1 is a 3 dimensional projective space defined in terms of collinearity if and only if for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(17) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a 3 dimensional projective space defined in terms of collinearity.

(18) Let C_1 be a collinearity structure. Then C_1 is a 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

A 3 dimensional projective space defined in terms of collinearity is called a Fanoian 3 dimensional projective space defined in terms of collinearity if:

(Def.4) Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

We now state four propositions:

(19) Let C_1 be a 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(20) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a Fanoian 3 dimensional projective space defined in terms of collinearity.

(21) Let C_1 be a collinearity structure. Then C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(22) For every C_1 being a collinearity structure holds C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

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