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# Properties of Real Functions

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**Summary.** The list of theorems concerning properties of real sequences and functions is enlarged. (See e.g. [9], [4], [8]). The monotone real functions are introduced and their properties are discussed.

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The papers [11], [3], [1], [9], [5], [6], [4], [2], [7], [10], and [8] provide the terminology and notation for this paper. For simplicity we follow a convention: $x$ is arbitrary, $X$, $X_1$, $Y$ denote sets, $g$, $r$, $r_1$, $r_2$, $p$ denote real numbers, $R$ denotes a subset of $\mathbb{R}$, $seq$, $seq_1$, $seq_2$, $seq_3$ denote sequences of real numbers, $Ns$ denotes an increasing sequence of naturals, $n$ denotes a natural number, and $h$, $h_1$, $h_2$ denote partial functions from $\mathbb{R}$ to $\mathbb{R}$. The following propositions are true:

(1) For all functions $F$, $G$ and for every $X$ such that $X \subseteq \operatorname{dom} F$ and $F^\circ X \subseteq \operatorname{dom} G$ holds $X \subseteq \operatorname{dom}(G \cdot F)$.

(2) For all functions $F$, $G$ and for every $X$ holds $G \restriction (F^\circ X) \cdot F \restriction X = (G \cdot F) \restriction X$.

(3) For all functions $F$, $G$ and for all $X$, $X_1$ holds $G \restriction X_1 \cdot F \restriction X = (G \cdot F) \restriction (X \cap F^{-1} X_1)$.

(4) For all functions $F$, $G$ and for every $X$ holds $X \subseteq \operatorname{dom}(G \cdot F)$ if and only if $X \subseteq \operatorname{dom} F$ and $F^\circ X \subseteq \operatorname{dom} G$.

(5) For every function $F$ and for every $X$ holds $(F \restriction X)^\circ X = F^\circ X$.

Let us consider $seq$. Then $\operatorname{rng} seq$ is a subset of $\mathbb{R}$.

One can prove the following propositions:

(6) $seq_1 = seq_2 - seq_3$ if and only if for every $n$ holds $seq_1(n) = seq_2(n) - seq_3(n)$.

(7) $\operatorname{rng}(seq \frown n) \subseteq \operatorname{rng} seq$.

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(8) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $seq(n) \in \operatorname{dom} h$.

(9) $x \in \operatorname{rng} seq$ if and only if there exists $n$ such that $x = seq(n)$.

(10) $seq(n) \in \operatorname{rng} seq$.

(11) If $seq_1$ is a subsequence of $seq$, then $\operatorname{rng} seq_1 \subseteq \operatorname{rng} seq$.

(12) If $seq_1$ is a subsequence of $seq$ and $seq$ is non-zero, then $seq_1$ is non-zero.

(13) $(seq_1 + seq_2) \cdot Ns = seq_1 \cdot Ns + seq_2 \cdot Ns$ and $(seq_1 - seq_2) \cdot Ns = seq_1 \cdot Ns - seq_2 \cdot Ns$ and $(seq_1 \diamond seq_2) \cdot Ns = (seq_1 \cdot Ns) \diamond (seq_2 \cdot Ns)$.

(14) $(p \diamond seq) \cdot Ns = p \diamond (seq \cdot Ns)$.

(15) $(-seq) \cdot Ns = -seq \cdot Ns$ and $|seq| \cdot Ns = |seq \cdot Ns|$.

(16) If $seq$ is non-zero, then $(seq \cdot Ns)^{-1} = seq^{-1} \cdot Ns$.

(17) If $seq$ is non-zero, then $\frac{seq_1}{seq} \cdot Ns = \frac{seq_1 \cdot Ns}{seq \cdot Ns}$.

(18) If $seq$ is convergent and for every $n$ holds $seq(n) \leq 0$, then $\lim seq \leq 0$.

(19) If for every $n$ holds $seq(n) \in Y$, then $\operatorname{rng} seq \subseteq Y$.

Let us consider $h$, $seq$. Let us assume that $\operatorname{rng} seq \subseteq \operatorname{dom} h$. The functor $h \cdot seq$ yields a sequence of real numbers and is defined by:

$h \cdot seq = (h\ \mathbf{qua}\ \text{a function}) \cdot seq$.

The following propositions are true:

(20) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $h \cdot seq = (h\ \mathbf{qua}\ \text{a function}) \cdot seq$.

(21) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq)(n) = h(seq(n))$.

(22) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq) \frown n = h \cdot (seq \frown n)$.

(23) Suppose $\operatorname{rng} seq \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$. Then $(h_1 + h_2) \cdot seq = h_1 \cdot seq + h_2 \cdot seq$ and $(h_1 - h_2) \cdot seq = h_1 \cdot seq - h_2 \cdot seq$ and $(h_1 \diamond h_2) \cdot seq = (h_1 \cdot seq) \diamond (h_2 \cdot seq)$.

(24) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(r \diamond h) \cdot seq = r \diamond (h \cdot seq)$.

(25) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $|h \cdot seq| = |h| \cdot seq$ and $-h \cdot seq = (-h) \cdot seq$.

(26) If $\operatorname{rng} seq \subseteq \operatorname{dom} \frac{1}{h}$, then $h \cdot seq$ is non-zero.

(27) If $\operatorname{rng} seq \subseteq \operatorname{dom} \frac{1}{h}$, then $\frac{1}{h} \cdot seq = (h \cdot seq)^{-1}$.

(28) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $(h \cdot seq) \cdot Ns = h \cdot (seq \cdot Ns)$.

(29) If $\operatorname{rng} seq_1 \subseteq \operatorname{dom} h$ and $seq_2$ is a subsequence of $seq_1$, then $h \cdot seq_2$ is a subsequence of $h \cdot seq_1$.

(30) If $h$ is total, then $(h \cdot seq)(n) = h(seq(n))$.

(31) If $h$ is total, then $h \cdot (seq \frown n) = (h \cdot seq) \frown n$.

(32) If $h_1$ is total and $h_2$ is total, then $(h_1 + h_2) \cdot seq = h_1 \cdot seq + h_2 \cdot seq$ and $(h_1 - h_2) \cdot seq = h_1 \cdot seq - h_2 \cdot seq$ and $(h_1 \diamond h_2) \cdot seq = (h_1 \cdot seq) \diamond (h_2 \cdot seq)$.

(33) If $h$ is total, then $(r \diamond h) \cdot seq = r \diamond (h \cdot seq)$.

(34) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$, then $h \restriction X \cdot seq = h \cdot seq$.

(35) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$ but $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction Y)$ or $X \subseteq Y$, then $h \restriction X \cdot seq = h \restriction Y \cdot seq$.

(36) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$, then $|h \restriction X \cdot seq| = |h| \restriction X \cdot seq$.

(37) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h \restriction X)$ and $h^{-1}\{0\} = \emptyset$, then $\frac{1}{h} \restriction X \cdot seq = (h \restriction X \cdot seq)^{-1}$.

(38) If $\operatorname{rng} seq \subseteq \operatorname{dom} h$, then $h^\circ \operatorname{rng} seq = \operatorname{rng}(h \cdot seq)$.

(39) If $\operatorname{rng} seq \subseteq \operatorname{dom}(h_2 \cdot h_1)$, then $h_2 \cdot (h_1 \cdot seq) = (h_2 \cdot h_1) \cdot seq$.

(40) If $h$ is one-to-one, then $(h \restriction X)^{-1} = h^{-1} \restriction (h^\circ X)$.

(41) If $\operatorname{rng} h$ is bounded and $\sup(\operatorname{rng} h) = \inf(\operatorname{rng} h)$, then $h$ is a constant on $\operatorname{dom} h$.

(42) If $Y \subseteq \operatorname{dom} h$ and $h^\circ Y$ is bounded and $\sup(h^\circ Y) = \inf(h^\circ Y)$, then $h$ is a constant on $Y$.

We now define four new predicates. Let us consider $h$, $Y$. We say that $h$ is increasing on $Y$ if and only if:

for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

We say that $h$ is decreasing on $Y$ if and only if:

for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

We say that $h$ is non-decreasing on $Y$ if and only if:

for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

We say that $h$ is non-increasing on $Y$ if and only if:

for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

Let us consider $h$, $Y$. We say that $h$ is monotone on $Y$ if and only if:

$h$ is non-decreasing on $Y$ or $h$ is non-increasing on $Y$.

Next we state a number of propositions:

(43) $h$ is increasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) < h(r_2)$.

(44) $h$ is decreasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) < h(r_1)$.

(45) $h$ is non-decreasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_1) \leq h(r_2)$.

(46) $h$ is non-increasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 < r_2$ holds $h(r_2) \leq h(r_1)$.

(47) $h$ is monotone on $Y$ if and only if $h$ is non-decreasing on $Y$ or $h$ is non-increasing on $Y$.

(48) $h$ is non-decreasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_1) \leq h(r_2)$.

(49) $h$ is non-increasing on $Y$ if and only if for all $r_1$, $r_2$ such that $r_1 \in Y \cap \operatorname{dom} h$ and $r_2 \in Y \cap \operatorname{dom} h$ and $r_1 \leq r_2$ holds $h(r_2) \leq h(r_1)$.

(50) $h$ is increasing on $X$ if and only if $h \restriction X$ is increasing on $X$.

(51) $h$ is decreasing on $X$ if and only if $h \restriction X$ is decreasing on $X$.

(52) $h$ is non-decreasing on $X$ if and only if $h \restriction X$ is non-decreasing on $X$.

(53) $h$ is non-increasing on $X$ if and only if $h \restriction X$ is non-increasing on $X$.

(54) If $Y \cap \operatorname{dom} h = \emptyset$, then $h$ is increasing on $Y$ and $h$ is decreasing on $Y$ and $h$ is non-decreasing on $Y$ and $h$ is non-increasing on $Y$ and $h$ is monotone on $Y$.

(55) If $h$ is increasing on $Y$, then $h$ is non-decreasing on $Y$.

(56) If $h$ is decreasing on $Y$, then $h$ is non-increasing on $Y$.

(57) If $h$ is a constant on $Y$, then $h$ is non-decreasing on $Y$.

(58) If $h$ is a constant on $Y$, then $h$ is non-increasing on $Y$.

(59) If $h$ is non-decreasing on $Y$ and $h$ is non-increasing on $X$, then $h$ is a constant on $Y \cap X$.

(60) If $X \subseteq Y$ and $h$ is increasing on $Y$, then $h$ is increasing on $X$.

(61) If $X \subseteq Y$ and $h$ is decreasing on $Y$, then $h$ is decreasing on $X$.

(62) If $X \subseteq Y$ and $h$ is non-decreasing on $Y$, then $h$ is non-decreasing on $X$.

(63) If $X \subseteq Y$ and $h$ is non-increasing on $Y$, then $h$ is non-increasing on $X$.

(64) If $h$ is increasing on $Y$ and $0 < r$, then $r \diamond h$ is increasing on $Y$ but if $r = 0$, then $r \diamond h$ is a constant on $Y$ but if $h$ is increasing on $Y$ and $r < 0$, then $r \diamond h$ is decreasing on $Y$.

(65) If $h$ is decreasing on $Y$ and $0 < r$, then $r \diamond h$ is decreasing on $Y$ but if $h$ is decreasing on $Y$ and $r < 0$, then $r \diamond h$ is increasing on $Y$.

(66) If $h$ is non-decreasing on $Y$ and $0 \leq r$, then $r \diamond h$ is non-decreasing on $Y$ but if $h$ is non-decreasing on $Y$ and $r \leq 0$, then $r \diamond h$ is non-increasing on $Y$.

(67) If $h$ is non-increasing on $Y$ and $0 \leq r$, then $r \diamond h$ is non-increasing on $Y$ but if $h$ is non-increasing on $Y$ and $r \leq 0$, then $r \diamond h$ is non-decreasing on $Y$.

(68) If $r \in (X \cap Y) \cap \operatorname{dom}(h_1 + h_2)$, then $r \in X \cap \operatorname{dom} h_1$ and $r \in Y \cap \operatorname{dom} h_2$.

(69) (i) If $h_1$ is increasing on $X$ and $h_2$ is increasing on $Y$, then $h_1 + h_2$ is increasing on $X \cap Y$,

(ii) if $h_1$ is decreasing on $X$ and $h_2$ is decreasing on $Y$, then $h_1 + h_2$ is decreasing on $X \cap Y$,

(iii) if $h_1$ is non-decreasing on $X$ and $h_2$ is non-decreasing on $Y$, then $h_1 + h_2$ is non-decreasing on $X \cap Y$,

(iv) if $h_1$ is non-increasing on $X$ and $h_2$ is non-increasing on $Y$, then $h_1 + h_2$ is non-increasing on $X \cap Y$.

(70) If $h_1$ is increasing on $X$ and $h_2$ is a constant on $Y$, then $h_1 + h_2$ is increasing on $X \cap Y$ but if $h_1$ is decreasing on $X$ and $h_2$ is a constant on $Y$, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(71) If $h_1$ is increasing on $X$ and $h_2$ is non-decreasing on $Y$, then $h_1 + h_2$ is increasing on $X \cap Y$.

(72) If $h_1$ is non-increasing on $X$ and $h_2$ is a constant on $Y$, then $h_1 + h_2$ is non-increasing on $X \cap Y$.

(73) If $h_1$ is decreasing on $X$ and $h_2$ is non-increasing on $Y$, then $h_1 + h_2$ is decreasing on $X \cap Y$.

(74) If $h_1$ is non-decreasing on $X$ and $h_2$ is a constant on $Y$, then $h_1 + h_2$ is non-decreasing on $X \cap Y$.

(75) $h$ is increasing on $\{x\}$.

(76) $h$ is decreasing on $\{x\}$.

(77) $h$ is non-decreasing on $\{x\}$.

(78) $h$ is non-increasing on $\{x\}$.

(79) $\mathrm{id}_R$ is increasing on $R$.

(80) If $h$ is increasing on $X$, then $-h$ is decreasing on $X$.

(81) If $h$ is non-decreasing on $X$, then $-h$ is non-increasing on $X$.

(82) If $h$ is increasing on $[p, g]$ or $h$ is decreasing on $[p, g]$, then $h \restriction [p, g]$ is one-to-one.

(83) If $h$ is increasing on $[p, g]$, then $(h \restriction [p, g])^{-1}$ is increasing on $h\,^\circ\,[p, g]$.

(84) If $h$ is decreasing on $[p, g]$, then $(h \restriction [p, g])^{-1}$ is decreasing on $h\,^\circ\,[p, g]$.

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