FORMALIZED MATHEMATICS
Vol.1, No.4, September–October 1990
Université Catholique de Louvain

Topological Properties of Subsets in Real Numbers [1]

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Summary. The following notions for real subsets are defined: open set, closed set, compact set, intervals and neighbourhoods. In the sequel some theorems involving above mentioned notions are proved.

MML Identifier: `RCOMP_1`.

The notation and terminology used in this paper have been introduced in the following articles: [9], [3], [10], [1], [2], [7], [5], [6], [4], and [8]. For simplicity we adopt the following convention: n, m are natural numbers, x is arbitrary, s, g, g_1, g_2, r, p, q are real numbers, s_1, s_2 are sequences of real numbers, and X, Y, Y_1 are subsets of $\mathbb{R}$. In this article we present several logical schemes. The scheme *SeqChoice* concerns a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that for every element t of $\mathbb{N}$ holds $\mathcal{P}[t, f(t)]$

provided the following requirement is met:

- for every element t of $\mathbb{N}$ there exists an element ff of $\mathcal{A}$ such that $\mathcal{P}[t, ff]$.

The scheme *RealSeqChoice* concerns a binary predicate $\mathcal{P}$, and states that:

there exists s_1 such that for every n holds $\mathcal{P}[n, s_1(n)]$

provided the parameter meets the following requirement:

- for every n there exists r such that $\mathcal{P}[n, r]$.

We now state several propositions:

(1) $X \subseteq Y$ if and only if for every r such that $r \in X$ holds $r \in Y$.

(2) $r \in X$ if and only if $r \notin X^{\rm c}$.

[1] Supported by RPBP.III-24.C8.

ISSN 0777–4028

(3) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is lower bounded, then Y_1 is lower bounded.

(4) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is upper bounded, then Y_1 is upper bounded.

(5) If there exists x such that $x \in Y_1$ and $Y_1 \subseteq Y$ and Y is bounded, then Y_1 is bounded.

Let us consider g, s. The functor $[g,s]$ yields a subset of $\mathbb{R}$ and is defined by: $[g,s] = \{r : g \leq r \wedge r \leq s\}$.

Next we state the proposition

(6) $[g,s] = \{r : g \leq r \wedge r \leq s\}$.

Let us consider g, s. The functor $]g,s[$ yields a subset of $\mathbb{R}$ and is defined as follows:

$]g,s[= \{r : g < r \wedge r < s\}$.

Next we state a number of propositions:

(7) $]g,s[= \{r : g < r \wedge r < s\}$.

(8) $r \in]p-g, p+g[$ if and only if $|r-p| < g$.

(9) $r \in [p,g]$ if and only if $|(p+g) - 2 \cdot r| \leq g - p$.

(10) $r \in]p,g[$ if and only if $|(p+g) - 2 \cdot r| < g - p$.

(11) For all g, s such that $g \leq s$ holds $[g,s] =]g,s[\cup \{g,s\}$.

(12) If $p \leq g$, then $]g,p[= \emptyset$.

(13) If $p < g$, then $[g,p] = \emptyset$.

(14) If $p = g$, then $[p,g] = \{p\}$ and $[g,p] = \{p\}$ and $]p,g[= \emptyset$.

(15) If $p < g$, then $]p,g[\neq \emptyset$ but if $p \leq g$, then $p \in [p,g]$ and $g \in [p,g]$ and $[p,g] \neq \emptyset$ and $]p,g[\subseteq [p,g]$.

(16) If $r \in [p,g]$ and $s \in [p,g]$, then $[r,s] \subseteq [p,g]$.

(17) If $r \in]p,g[$ and $s \in]p,g[$, then $[r,s] \subseteq]p,g[$.

(18) If $p \leq g$, then $[p,g] = [p,g] \cup [g,p]$.

Let us consider X. We say that X is compact if and only if:

for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and s_2 is convergent and $\lim s_2 \in X$.

Next we state the proposition

(19) X is compact if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ there exists s_2 such that s_2 is a subsequence of s_1 and s_2 is convergent and $\lim s_2 \in X$.

Let us consider X. We say that X is closed if and only if:

for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

The following proposition is true

(20) X is closed if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent holds $\lim s_1 \in X$.

Let A be a non-empty set, and let X be a subset of A. Then X^{c} is a subset of A.

Let us consider X. We say that X is open if and only if:
X^{c} is closed.

One can prove the following propositions:

(21) X is open if and only if X^{c} is closed.

(22) For all s, g such that $s \leq g$ for every s_1 such that $\operatorname{rng} s_1 \subseteq [s, g]$ holds s_1 is bounded.

(23) For all s, g such that $s \leq g$ holds $[s, g]$ is closed.

(24) For all s, g such that $s \leq g$ holds $[s, g]$ is compact.

(25) For all p, q such that $p < q$ holds $]p, q[$ is open.

(26) If X is compact, then X is closed.

(27) Given X, s_1. Suppose $X \neq \emptyset$ and $\operatorname{rng} s_1 \subseteq X$ and for every p such that $p \in X$ there exist r, n such that $0 < r$ and for every m such that $n < m$ holds $r < |s_1(m) - p|$. Then for every s_2 such that s_2 is a subsequence of s_1 holds it is not true that: s_2 is convergent and $\lim s_2 \in X$.

(28) If there exists r such that $r \in X$ and X is compact, then X is bounded.

(29) If there exists r such that $r \in X$, then X is compact if and only if X is bounded and X is closed.

(30) For every X such that $X \neq \emptyset$ and X is closed and X is upper bounded holds $\sup X \in X$.

(31) For every X such that $X \neq \emptyset$ and X is closed and X is lower bounded holds $\inf X \in X$.

(32) For every X such that $X \neq \emptyset$ and X is compact holds $\sup X \in X$ and $\inf X \in X$.

(33) If X is compact and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X = [p, g]$.

A subset of $\mathbb{R}$ is called a real open subset if:
it is open.

We now state the proposition

(34) For every subset X of $\mathbb{R}$ holds X is a real open subset if and only if X is open.

Let us consider r. A real open subset is said to be a neighbourhood of r if:
there exists g such that $0 < g$ and it $=]r - g, r + g[$.

One can prove the following propositions:

(35) For every r and for every real open subset X holds X is a neighbourhood of r if and only if there exists g such that $0 < g$ and $X =]r - g, r + g[$.

(36) For all r, X holds X is a neighbourhood of r if and only if there exists g such that $0 < g$ and $X =]r - g, r + g[$.

(37) For every r and for every neighbourhood N of r holds $r \in N$.

(38) For every r and for every neighbourhoods N_1, N_2 of r there exists a neighbourhood N of r such that $N \subseteq N_1$ and $N \subseteq N_2$.

(39) For every real open subset X and for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$.

(40) For every real open subset X and for every r such that $r \in X$ there exists g such that $0 < g$ and $]r - g, r + g[\subseteq X$.

(41) For every X such that for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ holds X is open.

(42) For every X holds for every r such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ if and only if X is open.

(43) If $X \neq \emptyset$ and X is open and X is upper bounded, then $\sup X \notin X$.

(44) If $X \neq \emptyset$ and X is open and X is lower bounded, then $\inf X \notin X$.

(45) If X is open and X is bounded and for all g_1, g_2 such that $g_1 \in X$ and $g_2 \in X$ holds $[g_1, g_2] \subseteq X$, then there exist p, g such that $X =]p, g[$.

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Received June 18, 1990