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Projective Spaces - Part I

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Summary. In the class of all collinearity structures a subclass of (dimension free) projective spaces, defined by means of a suitable axiom system, is singled out. Whenever a real vector space V is at least 3-dimensional, the structure ProjectiveSpace(V) is a projective space in the above meaning. Some narrower classes of projective spaces are defined: Fano projective spaces, projective planes, and Fano projective planes. For any of the above classes an explicit axiom system is given, as well as an analytical example. There is also a construction a of 3-dimensional and a 4-dimensional real vector space; these are needed to show appropriate examples of projective spaces.

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The notation and terminology used here are introduced in the following papers: [1], [5], [7], [6], [3], [4], and [2]. For simplicity we adopt the following rules: V will denote a real linear space, p, q, r, u, v, w, y, u_1, v_1 will denote vectors of V, a, b, c, d, a_1, b_1 will denote real numbers, and z will be arbitrary. We now state three propositions:

(1) Suppose for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then u is a proper vector and v is a proper vector and w is a proper vector and u, v and w are not lineary dependent and u and v are not proportional.

(2) Given u, v, u_1, v_1. Suppose for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then u is a proper vector and v is a proper vector and u and v are not proportional and u_1 is a proper vector and v_1 is a proper vector and u_1 and v_1 are not proportional and u, v and u_1 are not lineary dependent and u_1, v_1 and u are not lineary dependent.

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(3) Suppose for every w there exist a, b, c such that $w = (a \cdot p + b \cdot q) + c \cdot r$ and for all a, b, c such that $(a \cdot p + b \cdot q) + c \cdot r = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then for every u, u_1 there exists y such that p, q and y are lineary dependent and u, u_1 and y are lineary dependent and y is a proper vector.

We follow a convention: A is a non-empty set, f, g, h, f_1 are elements of $\mathbb{R}^A$, and x_1, x_2, x_3, x_4 are elements of A. We now state a number of propositions:

(4) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

(5) Suppose that

(i) $x_1 \in A$,

(ii) $x_2 \in A$,

(iii) $x_3 \in A$,

(iv) $x_1 \neq x_2$,

(v) $x_1 \neq x_3$,

(vi) $x_2 \neq x_3$,

(vii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,

(viii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,

(ix) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

Then for all a, b, c such that
$+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$
holds $a = 0$ and $b = 0$ and $c = 0$.

(6) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for all a, b, c such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$.

(7) Suppose that

(i) $A = \{x_1, x_2, x_3\}$,

(ii) $x_1 \neq x_2$,

(iii) $x_1 \neq x_3$,

(iv) $x_2 \neq x_3$,

(v) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,

(vi) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,

(vii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$.

Then for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(8) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(9) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for all a, b, c such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(10) There exists a non-trivial real linear space V and there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

(11) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$ and for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

(12) Suppose that

(i) $x_1 \in A$,
(ii) $x_2 \in A$,
(iii) $x_3 \in A$,
(iv) $x_4 \in A$,
(v) $x_1 \neq x_2$,
(vi) $x_1 \neq x_3$,
(vii) $x_1 \neq x_4$,
(viii) $x_2 \neq x_3$,
(ix) $x_2 \neq x_4$,
(x) $x_3 \neq x_4$,
(xi) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
(xii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,
(xiii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$,
(xiv) for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

Given a, b, c, d. Suppose $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$. Then $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(13) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for all a, b, c, d such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(14) Suppose that

(i) $A = \{x_1, x_2, x_3, x_4\}$,

(ii) $x_1 \neq x_2$,

(iii) $x_1 \neq x_3$,

(iv) $x_1 \neq x_4$,

(v) $x_2 \neq x_3$,

(vi) $x_2 \neq x_4$,

(vii) $x_3 \neq x_4$,

(viii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,

(ix) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ but if $z \neq x_2$, then $g(z) = 0$,

(x) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ but if $z \neq x_3$, then $h(z) = 0$,

(xi) for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ but if $z \neq x_4$, then $f_1(z) = 0$.

Then for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(15) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(16) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for all a, b, c, d such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(17) There exists a non-trivial real linear space V and there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

We follow the rules: V is a non-trivial real linear space, u, v, w, y, w_1 are vectors of V, and p, p_1, p_2, q, q_1, r, r_1, r_2 are elements of the points of the projective space over V. The following propositions are true:

(18) p, q and r are collinear if and only if there exist u, v, w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is a proper vector and v is a proper vector and w is a proper vector and u, v and w are lineary dependent.

(19) p, q and p are collinear and p, p and q are collinear and p, q and q are collinear.

(20) If $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear, then r, r_1 and r_2 are collinear.

(21) If p, q and r are collinear, then p, r and q are collinear and q, p and r are collinear and r, q and p are collinear and r, p and q are collinear and q, r and p are collinear.

(22) If p, p_1 and p_2 are collinear and p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(23) If p, p_1 and p_2 are not collinear and p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(24) If p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear, then there exists r_2 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

(25) Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(26) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then there exist p, q, r such that p, q and r are not collinear.

(27) Suppose there exist u, v, w_1 such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear.

(28) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then there exist elements x_1, x_2 of the points of the projective space over V such that $x_1 \neq x_2$ and for every r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear.

(29) If there exist elements x_1, x_2 of the points of the projective space over V such that $x_1 \neq x_2$ and for every r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear, then for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(30) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(31) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a\cdot u+b\cdot v)+c\cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then there exist p, q, r such that p, q and r are not collinear and for every p, p_1, q, q_1 there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A collinearity structure is said to be a projective space defined in terms of collinearity if:

(i) for all elements p, q, r, r_1, r_2 of the points of it such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of it holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of it such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of it such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of it there exists an element r of the points of it such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of it such that p, q and r are not collinear.

Next we state three propositions:

(32) Let CS be a collinearity structure. Then CS is a projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of CS such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of CS such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of CS such that p, q and r are not collinear.

(33) For every projective space CS defined in terms of collinearity holds CS is a proper collinearity space.

(34) If there exist u, v, w such that for all a, b, c such that $(a\cdot u+b\cdot v)+c\cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a

projective space defined in terms of collinearity.

A projective space defined in terms of collinearity is called a Fanoian projective space defined in terms of collinearity if:

Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

The following propositions are true:

(35) Let CS be a projective space defined in terms of collinearity. Then CS is a Fanoian projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(36) Let CS be a collinearity structure. Then CS is a Fanoian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of CS such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of CS such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(37) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fanoian projective space defined in terms of collinearity.

A projective space defined in terms of collinearity is called a projective plane defined in terms of collinearity if:

for every elements p, p_1, q, q_1 of the points of it there exists an element r of the points of it such that p, p_1 and r are collinear and q, q_1 and r are collinear.

We now state three propositions:

(38) For every projective space CPS defined in terms of collinearity holds CPS is a projective plane defined in terms of collinearity if and only if for every elements p, p_1, q, q_1 of the points of CPS there exists an element r of the points of CPS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(39) Let CS be a collinearity structure. Then CS is a projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(40) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a projective plane defined in terms of collinearity.

A projective plane defined in terms of collinearity is said to be a Fanoian projective plane defined in terms of collinearity if:

Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

Next we state four propositions:

(41) Let CS be a projective plane defined in terms of collinearity. Then CS is a Fanoian projective plane defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1

are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(42) Let CS be a collinearity structure. Then CS is a Fanoian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of CS such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of CS holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of CS there exists an element r of the points of CS such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of CS such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of CS such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(43) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fanoian projective plane defined in terms of collinearity.

(44) For every CS being a collinearity structure holds CS is a Fanoian projective plane defined in terms of collinearity if and only if CS is a Fanoian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of CS there exists an element r of the points of CS such that p, p_1 and r are collinear and q, q_1 and r are collinear.

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